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Research paper

Strong convergence of a fractional exponential integrator scheme for finite element discretization of time-fractional SPDE driven by fractional and standard Brownian motions

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ABSTRACT

The aim of this work is to provide the first strong convergence result of a numerical approximation of a general time-fractional second order stochastic partial differential equation involving a Caputo derivative in time of order $\alpha \in (\frac{1}{2}, 1)$ and driven simultaneously by a multiplicative standard Brownian motion and additive fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, more realistic to model the random effects on transport of particles in medium with thermal memory. We prove the existence and uniqueness results, and perform the spatial discretization using the standard finite element and the temporal discretization based on a generalized exponential time differencing method (GETD). We provide the temporal and spatial convergence proofs for our fully discrete scheme and the result shows that the convergence orders depend on the regularity of the initial data, the power of the fractional derivative, and the Hurst parameter H. Numerical results are provided to illustrate our theoretical results.

1. Introduction

Let $\Lambda \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a bounded with smooth boundary. On the Hilbert space $\mathcal{H} = L^2(\Lambda)$, we analyze the strong numerical approximation of the following time fractional SPDE

$$\begin{cases} {}^C\partial_t^\alpha X(t) + AX(t) = F(X(t)) + I_t^{1-\alpha}\left[G(X)\frac{dW(t)}{dt} + \Phi\frac{dB^H(t)}{dt}\right], \\ X(0) = X_0, \quad t \in [0, T], \end{cases} \tag{1}$$

where ${}^C\partial_t^\alpha$ is the Caputo fractional derivative with $\alpha \in (\frac{1}{2}, 1)$ and $I_t^{1-\alpha}$ is the fractional integral operator which will be given in the next section, $T > 0$ is the final time and A is unbounded (not necessarly self-adjoint) operator which is assumed to generate a semigroup $S(t) := e^{-tA}$. The functions $F : \mathcal{H} \to \mathcal{H}$, $G : \mathcal{H} \to \mathcal{H}$ and $\Phi : \mathcal{H} \to \mathcal{H}$ are deterministic mappings that will be specified later, X_0 is the initial data which is random, $W(t) = W(x, t)$ is a $\mathcal{H}$-valued Q-Wiener process defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ and $B^H(t) = B^H(x, t)$ in (1) is a $\mathcal{H}$-valued fractional Q_1-Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where the

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covariances operators $Q : \mathcal{H} \to \mathcal{H}$ and $Q_1 : \mathcal{H} \to \mathcal{H}$ are positive and linear self-adjoint operators. The filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is assumed to fulfill the usual assumptions (see [1, Definition 2.1.11]). It is well known [2,3] that the noises can be represented as follows

$$W(x,t) = \sum_{i=0}^{\infty} \beta_i(t) Q^{\frac{1}{2}} e_i(x) = \sum_{i=0}^{\infty} \sqrt{q_i}\beta_i(t) e_i(x), \tag{2}$$

$$B^H(x,t) = \sum_{i=0}^{\infty} \beta_i^H(t) Q_1^{\frac{1}{2}} e_i^1(x) = \sum_{i=0}^{\infty} \sqrt{q_i^1}\beta_i^H(t) e_i^1(x), \tag{3}$$

where q_i, e_i, $i \in \mathbb{N}$ are respectively the eigenvalues and eigenfunctions of the covariance operator Q, q_i^1, e_i^1, $i \in \mathbb{N}$ are respectively the eigenvalues and eigenfunctions of the covariance operator Q_1, β_i are mutually independent and identically distributed standard normal distributions and β_i^H are mutually independent and identically distributed fractional Brownian motion (fBm). The noises W and B^H are supposed to be independent. Precise assumptions on the nonlinear mappings G and Φ to ensure the existence of the mild solution of (1) will be given in the following section.

Equation of type (1) with $\Phi = 0$ might be used to model random effects on transport of particles in medium with thermal memory [4]. So due to the self-similar and long-range dependence properties of the fBm, when modeling such phenomena, it is recommended to incorporate the fBm process in order to obtain a more realistic model. During the last few decades, the theory of fractional partial differential equations has gained considerable interest over time. From the point of view of computations, several numerical methods have been proposed for solving time fractional partial differential equations (for details, see [5–10] and the reference therein). Note that the time stepping methods used in all the works mentioned until now are based on finite difference methods. However these schemes are explicit, but unstable, unless the time stepsize is very small. To solve that drawback, numerical method based on exponential integrators of Adams type have been proposed in [11]. The price to pay is the computation of Mittag–Leffler (ML) matrix functions. As ML matrix function is the generalized form of the exponential of matrix function, works in [12–14] have extended some exponential computational techniques to ML. Note that up to now all the numerical algorithms presented are for time fractional deterministic PDEs with self-adjoint linear operators.

Actually, few works have been done for numerical methods for Gaussian noise for time fractional stochastic partial differential equation (see [4,15–17]). Note that this above works have been done for self-adjoint linear operators, so numerical study for (1) with $\Phi \neq 0$ and non self-adjoint operator is still an open problem in the field, to the best of our knowledge. However, it is important to mention that if $H \neq \frac{1}{2}$ the process B^H is not a semi-martingale and the standard stochastic calculus techniques are therefore obsolete while studying SPDEs of type (1). Alternative approaches to the standard Itô calculus are therefore required in order to build a stochastic calculus framework for such fBm. In recent years, there have been various developments of stochastic calculus and stochastic differential equations with respect to the fBm especially for $H \in (\frac{1}{2}, 1)$ (see, for example [2,18,19]) and theory of SPDEs driven by fractional Brownian motion has been also studied. For example, linear and semilinear stochastic equations in a Hilbert space with an infinite dimensional fractional Brownian motion are considered in [20,21]. However numerical schemes for time fractional SPDEs (1) driven both by fractional Brownian motion and standard Brownian motion have been lacked in the scientific literature to the best of our knowledge. Our goal in this work is to build the first numerical method to approximate the time fractional stochastic partial differential Eq. (1) driven simultaneously by a multiplicative standard Brownian motion and an additive fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ using finite element method for spatial approximation and a fractional version of exponential Euler scheme for time discretization [22,23]. Since the ML function is more challenging than the exponential function, our main result is based on novel preliminary results of ML functions. The analysis is complicated and is very different to that of a standard exponential integrator scheme [24][1] since the fractional derivative is not local and therefore a numerical solution at a given time depends to all previous numerical solutions up to that time. This is in contrast to the standard exponential numerical scheme where the numerical solution at a given time depends only on that of the previous nearest solution. We provided the strong convergence of our fully discrete scheme for (1). Our strong convergence results examine how the convergence orders depend on the regularity of the initial data, the power of the fractional derivative, and the Hurst parameter.

The rest of the paper is structured as follows. In Section 2, Mathematical settings for standard and fractional calculus are presented, along with the existence, uniqueness, and regularities results of the mild solution of SPDE (1). In Section 3, numerical schemes for SPDE (1) are presented, we discuss about space and time regularity of the mild solution $X(t)$ of (1) given by (21). The spatial error analysis is done in Section 4. In Section 5, the strong convergence proof of the our novel numerical scheme is provided. We end the paper in Section 6 by providing numerical results to illustrate our theoretical results.

[1] Where $\alpha = 1$.

2. Mathematical settings, main assumptions and well posedness

In this section, we review briefly some useful results on standard and fractional calculus, introduce notations, definitions, preliminary results which will be needed throughout this paper and the proof of existence and uniqueness of the mild solution of (1).

Definition 1 (*Fractional Brownian Motion*). [19,24] An $\mathcal{H}$-valued Gaussian process $\{B^H(t), t \in [0,T]\}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is called a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if

- $\mathbb{E}[B^H(t)] = 0$ for all $t \in \mathbb{R}$,
- $\text{Cov}(B^H(t), B^H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$ for all $t, s \in \mathbb{R}$,
- $\{B^H(t), t \in [0,T]\}$ has continuous sample paths $\mathbb{P}$ a.s.,

where $\text{Cov}(X, Y)$ denotes the covariance operator for the Gaussian random variables X and Y and $\mathbb{E}$ stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$.

Notice that if $H = \frac{1}{2}$, the fractional Brownian motion coincides with the standard Brownian motion. Throughout this paper the Hurst parameter H is assumed to belong to $(\frac{1}{2}, 1)$.

Let $(K, \langle ., . \rangle_K, \|.\|)$ be a separable Hilbert space. For $p \geq 2$ and for a Banach space U, we denote by $L^p(\Omega, U)$ the Banach space of p-integrable U-valued random variables. We denote by $L(U, K)$ the space of bounded linear mapping from U to K endowed with the usual operator norm $\|.\|_{L(U,K)}$ and $\mathcal{L}_2(U, K) = HS(U, K)$ the space of Hilbert–Schmidt operators from U to K equipped with the following norm

$$\|l\|_{\mathcal{L}_2(U,K)} := \left(\sum_{i=0}^{\infty} \|l\psi_i\|^2\right)^{\frac{1}{2}}, \quad l \in \mathcal{L}_2(U, K), \tag{4}$$

where $(\psi_i)_{i\in\mathbb{N}}$ is an orthonormal basis on U. The sum in (4) is independent of the choice of the orthonormal basis of U. We use the notation $L(U, U) =: L(U)$ and $\mathcal{L}_2(U, U) =: \mathcal{L}_2(U)$. It is well known that for all $l \in L(U, K)$ and $l_1 \in \mathcal{L}_2(U)$, $ll_1 \in \mathcal{L}_2(U, K)$ and

$$\|ll_1\|_{\mathcal{L}_2(U,K)} \leq \|l\|_{L(U,K)} \|l_1\|_{\mathcal{L}_2(U)}.$$

We denote by $L_2^0 := HS(Q^{\frac{1}{2}}(\mathcal{H}), \mathcal{H})$ the space of Hilbert–Schmidt operators from $Q^{\frac{1}{2}}(\mathcal{H})$ to $\mathcal{H}$ with corresponding norm $\|.\|_{L_2^0}$ defined by

$$\|l\|_{L_2^0} := \left\|lQ^{\frac{1}{2}}\right\|_{HS} = \left(\sum_{i\in\mathbb{N}} \|lQ^{\frac{1}{2}}e_i\|^2\right)^{\frac{1}{2}}, \quad l \in L_2^0, \tag{5}$$

where $(e_i)_{i=0}^{\infty}$ are orthonormal basis of $\mathcal{H}$. The following lemma will be very important throughout this paper.

Lemma 1 (*Itô Isometry: [3, (4.30)], [24, (12)]*).

(i) Let $\theta \in L^2([0,T]; L_2^0)$, then the following holds

$$\mathbb{E}\left[\left\|\int_0^T \theta(s)dW(s)\right\|^2\right] = \mathbb{E}\left[\int_0^T \|\theta(s)\|_{L_2^0}^2 ds\right]. \tag{6}$$

(ii) Let $\Phi : \mathcal{H} \to \mathcal{H}$, then the following holds

$$\mathbb{E}\left[\left\|\int_0^T \Phi dB^H(s)\right\|^2\right] \leq C(H)\sum_{i=0}^{\infty}\left(\int_0^T \left\|\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}} ds\right)^{2H}. \tag{7}$$

Remark 1. Note that in the case $H = \frac{1}{2}$, the constant $C(H)$ in (7) is 1 and the inequality becomes the equality. In this case, the result (7) is then identically to (6).

More details on the definition of stochastic integral with respect to fractional Q-Brownian motion and their property are given in e.g [18–21].

Definition 2 (*[25, (2.1.1), (2.4.17)]*). The Caputo-type derivative of order α with respect to t is defined for all $t > 0$ by

$${}^C\partial_t^{\alpha} X(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial X(s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, & 0 < \alpha < 1 \\ \frac{\partial X}{\partial t}, & \alpha = 1, \end{cases} \tag{8}$$

and the Riemann–Liouville fractional integral operator I_t^α is defined for all $t > 0$ by

$$I_t^\alpha X(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} X(s) ds, & 0 < \alpha < 1 \\ X(t), & \alpha = 0. \end{cases} \tag{9}$$

where $\Gamma(\cdot)$ is the gamma function.

Proposition 1 (*See [17]*). *Let us consider the generalized Mittag–Leffler function $E_{\alpha,\beta}(t)$ [26] and the Mainardi's Wright-type function $M_\alpha(\theta)$ [27] defined as*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \qquad \text{and} \qquad M_\alpha(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n! \Gamma(1 - \alpha(1+\theta))}, \quad 0 < \alpha < 1, \ \ \theta > 0.$$

Then the following results hold

$$M_\alpha(\theta) \geq 0, \qquad \int_0^\infty \theta^\mu M_\alpha(\theta) d\theta = \frac{\Gamma(1+\mu)}{\Gamma(1+\alpha\mu)}, \qquad -1 < \mu < \infty, \qquad \theta > 0, \tag{10}$$

and

$$E_{\alpha,1}(t) = \int_0^\infty M_\alpha(\theta) e^{t\theta} d\theta, \quad E_{\alpha,\alpha}(t) = \int_0^\infty \alpha\theta M_\alpha(\theta) e^{t\theta} d\theta. \tag{11}$$

In the rest of this paper to simplify the presentation, we assume the SPDE (1) to be second order of the following type.

$$\begin{aligned} &{}^C\partial_t^\alpha X(t,x) + [-\nabla \cdot (D\nabla X(t,x)) + q \cdot \nabla X(t,x)] \\ &= f(x, X(t,x)) + I_t^{1-\alpha} \left[g(x, X(t,x)) \frac{dW(t,x)}{dt} + \phi(x) \frac{dB^H(t,x)}{dt} \right] \end{aligned} \tag{12}$$

where $f, g : \Lambda \times \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, $\phi : \mathbb{R} \to \mathbb{R}$ is bounded. In the abstract framework (12), the linear operator A takes the form

$$\begin{aligned} Au &= -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(D_{i,j}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{d} q_i(x) \frac{\partial u}{\partial x_i} \\ D &= \left(D_{i,j}\right)_{1 \leq i,j \leq d}, \qquad q = (q_i)_{1 \leq i \leq d}, \end{aligned}$$

where $D_{i,j} \in L^\infty(\Lambda)$, $q_i \in L^\infty(\Lambda)$. We assume that there exists a positive constant $c_1 > 0$ such that

$$\sum_{i,j=1}^{d} D_{i,j}(x) \xi_i \xi_j \geq c_1 |\xi|^2, \qquad \xi \in \mathbb{R}^d, \qquad x \in \bar{\Omega},$$

The functions $F : \mathcal{H} \to \mathcal{H}$, $G : \mathcal{H} \to L_2^0$ and $\Phi \in L_2^0$ are defined by

$$(F(v))(x) = f(x, v(x)), \quad (G(v)u)(x) = g(x, v(x)) \cdot u(x), \quad (\Phi w)(x) = \phi(x) \cdot w(x)$$

for all $x \in \Lambda$, $v \in \mathcal{H}$, u, $w \in Q^{1/2}(\mathcal{H})$. As in [22,28], we introduce two spaces $\mathbb{H}$, and V such that $\mathbb{H} \subset V$, the two spaces depend on the boundary conditions of Λ and the domain of the operator A. For Dirichlet (or first-type) boundary conditions, we take

$$V = \mathbb{H} = H_0^1(\Lambda) = \{v \in H^1(\Lambda) : v = 0 \text{ on } \partial\Lambda\}.$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition, we take $V = H^1(\Omega)$

$$\mathbb{H} = \{v \in H^2(\Lambda) : \partial v / \partial v_{\mathcal{A}} + \alpha_0 v = 0, \quad \text{on} \quad \partial\Lambda\}, \qquad \alpha_0 \in \mathbb{R},$$

where $\partial v / \partial v_{\mathcal{A}}$ is the normal derivative of v and $v_{\mathcal{A}}$ is the exterior pointing normal $n = (n_i)$ to the boundary of $\mathcal{A}$ given by

$$\partial v / \partial v_{\mathcal{A}} = \sum_{i,j=1}^{d} n_i(x) D_{i,j}(x) \frac{\partial v}{\partial x_j}, \quad x \in \partial\Lambda.$$

Using Gårding's inequality (see e.g. [29]), it holds that there exist two constants c_0 and $\lambda_0 > 0$ such that the bilinear form $a(.,.)$ associated to A satisfies

$$a(v,v) \geq \lambda_0 \|v\|_{H^1(\Lambda)}^2 - c_0 \|v\|^2, \quad v \in V. \tag{13}$$

By adding and subtracting c_0Xdt on both sides of (12), we have a new linear operator still denoted by A, and the corresponding bilinear form is also still denoted by a. Therefore, the following coercivity property holds

$$a(v,v) \geq \lambda_0 \|v\|^2_{H^1(\Lambda)}, \quad v \in V. \tag{14}$$

Note that we have created a new linear term $-c_0X$ in the right-hand side of (12). Thus we obtain a new equivalent form to (12) as

$$\begin{aligned} &{}^C\partial_t^\alpha X(t,x) + [-\nabla\cdot(D\nabla X(t,x)) + q\cdot\nabla X(t,x) - c_0X(t,x)] \\ &= f(x, X(t,x)) - c_0X(t,x) + I_t^{1-\alpha}\left[g(x,X(t,x))\frac{dW(t,x)}{dt} + \phi(x)\frac{dB^H(t,x)}{dt}\right]. \end{aligned} \tag{15}$$

We rewrite it in its contracted form as follows

$${}^C\partial_t^\alpha X(t) + AX(t) = F(X(t)) + I_t^{1-\alpha}\left[G(X)\frac{dW(t)}{dt} + \Phi\frac{dB^H(t)}{dt}\right]. \tag{16}$$

Note that the expression of nonlinear term F has changed as we included the term $-c_0X$ in the new nonlinear term that we still denote by F. The coercivity property (14) implies that A is the infinitesimal generator of a *contraction* semigroup $S(t) = e^{-tA}$ on $L^2(\Lambda)$ [28]. Note that this is due to the fact that the real part of the eigenvalues of A is positive. Note also that the coercivity property (14) also implies that A is a positive operator and its fractional powers are well defined and for any $\alpha > 0$ we have

$$\begin{cases} A^{-\alpha} &= \frac{1}{\Gamma(\alpha)}\displaystyle\int_0^\infty t^{\alpha-1}\mathrm{e}^{-tA}dt, \\ A^\alpha &= (A^{-\alpha})^{-1}, \end{cases} \tag{17}$$

where $\Gamma(\alpha)$ is the Gamma function.

Remark 2.1. As we have mentioned, the linear operator A is the infinitesimal generator of a contraction semigroup $S(t) = e^{-tA}$, and therefore

$$\|S(t)\|_{L(\mathcal{H})} \leq 1. \tag{18}$$

Let

$$\mathcal{S}_1(t) := E_{\alpha,1}(-t^\alpha A) \qquad \text{and} \qquad \mathcal{S}_2(t) := E_{\alpha,\alpha}(-t^\alpha A), \tag{19}$$

be the fractional semigroups from Proposition 1 and (18), we also have

$$\|\mathcal{S}_1(t)\|_{L(\mathcal{H})} \leq 1 \qquad \text{and} \qquad \|\mathcal{S}_2(t)\|_{L(\mathcal{H})} \leq \frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}. \tag{20}$$

Note that (18)–(20) are also held is $S(t)$ and $S_1(t)$ are replaced respectively by their semi discrete form $S_h(t)$ and $S_{1h}(t)$ obtained after the finite element method space discretization. In the sequel of this paper, the contraction properties (18), (20) for $S_{1h}(t)$ will play a key role in the analysis of our numerical scheme.

Following the same lines as in [4,16, (2.2)-(2.5)] and using the equivalent model (16), we represent the mild solution of (1) as:

Proposition 2. *For any $0 < \alpha < 1$, the $\mathcal{F}_t$-adapted stochastic process $\{X(t), t \in [0,T]\}$ is called mild solution to (1), if there holds*

$$\begin{aligned} X(t) &= \mathcal{S}_1(t)X_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_2(t-s)F(X(s))ds \\ &\quad + \int_0^t \mathcal{S}_1(t-s)G(X(s))dW(s) + \int_0^t \mathcal{S}_1(t-s)\Phi dB^H(s). \end{aligned} \tag{21}$$

$\mathbb{P}$ a.s. Where $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ are defined by (19).

Proof. We define the Laplace transform of the function X with respect to t as

$$\widehat{X}(z) = \mathcal{L}\{X(t)\} = \int_0^{+\infty} e^{-zt}X(t)dt,$$

then the Laplace transform of the Caputo derivative ${}^C\partial_t^\alpha$ and the Riemann–Liouville fractional integral operator I_t^α are given by (see [16, (1.5)])

$$\mathcal{L}\{{}^C\partial_t^\alpha X(t)\} = z^\alpha\widehat{X}(z) - z^{\alpha-1}X(0), \qquad \mathcal{L}\{I_t^\alpha X(t)\} = z^{-\alpha}\widehat{X}(z). \tag{22}$$

Applying the Laplace transform to both sides of (16) and using (22) we deduce that

$$\begin{aligned} & z^{\alpha}\widehat{X}(z) - z^{\alpha-1}X(0) + A\widehat{X}(z) \\ =\ & \mathcal{L}\{F(X(t))\} + z^{\alpha-1}\left[\mathcal{L}\{G(X(t))\frac{dW(t)}{dt}\} + \mathcal{L}\{\Phi\frac{dB^H(t)}{dt}\}\right], \end{aligned}$$

thus we obtain

$$\begin{aligned} \widehat{X}(z) =\ & \frac{z^{\alpha-1}}{z^{\alpha}+A}X(0) + \frac{1}{z^{\alpha}+A}\mathcal{L}\{F(X(t))\} \\ & + \frac{z^{\alpha-1}}{z^{\alpha}+A}\mathcal{L}\{G(X(t))\frac{dW(t)}{dt}\} + \frac{z^{\alpha-1}}{z^{\alpha}+A}\mathcal{L}\{\Phi\frac{dB^H(t)}{dt}\}. \end{aligned} \tag{23}$$

Recall that the Laplace transform of the Mittag–Leffler function (see [16, (2.4)], $\beta = 1$ and $\beta = \alpha$) is given by

$$\mathcal{L}\{E_{\alpha,1}(-\lambda t^{\alpha})\} = \frac{z^{\alpha-1}}{z^{\alpha}+\lambda} \quad \text{and} \quad \mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\} = \frac{1}{z^{\alpha}+\lambda}.$$

The Eq. (23) now yields

$$\begin{aligned} \mathcal{L}\{X(t)\} =\ & \mathcal{L}\{E_{\alpha,1}(-At^{\alpha})\}X(0) + \mathcal{L}\{E_{\alpha,\alpha}(-At^{\alpha})\}\mathcal{L}\{F(X(t))\} \\ & + \mathcal{L}\{E_{\alpha,1}(-At^{\alpha})\}\mathcal{L}\left(G(X(t))\frac{dW(t)}{dt}\right) + \mathcal{L}\{E_{\alpha,1}(-At^{\alpha})\}\mathcal{L}\left(\Phi\frac{dB^H(t)}{dt}\right). \end{aligned} \tag{24}$$

Applying the inverse Laplace transform to (19) and (24), we obtain

$$\begin{aligned} X(t) =\ & E_{\alpha,1}(-At^{\alpha})X(0) + \left(t^{\alpha-1}E_{\alpha,\alpha}(-At^{\alpha})\right) * F(X(t)) \\ & + \left(E_{\alpha,1}(-At^{\alpha})\right) * \left(G(X(t))\frac{dW(t)}{dt}\right) + \left(E_{\alpha,1}(-At^{\alpha})\right) * \left(\Phi\frac{dB^H(t)}{dt}\right) \\ =\ & \mathcal{S}_1(t)X(0) + \left(t^{\alpha-1}\mathcal{S}_2(t)\right) * F(X(t)) + \mathcal{S}_1(t) * \left(G(X(t))\frac{dW(t)}{dt}\right) \\ & + \mathcal{S}_1(t) * \left(\Phi\frac{dB^H(t)}{dt}\right), \end{aligned}$$

and so

$$\begin{aligned} X(t) =\ & \mathcal{S}_1(t)X_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_2(t-s)F(X(s))ds \\ & + \int_0^t \mathcal{S}_1(t-s)G(X(s))dW(s) + \int_0^t \mathcal{S}_1(t-s)\Phi dB^H(s). \end{aligned}$$

The proof of Proposition 2 is thus completed. ■

Thanks to (11), the fractional semigroup operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ can be rewritten as

$$\mathcal{S}_1(t) := E_{\alpha,1}(-At^{\alpha}) = \int_0^{\infty} M_{\alpha}(\theta)e^{-A\theta t^{\alpha}}d\theta = \int_0^{\infty} M_{\alpha}(\theta)S(\theta t^{\alpha})d\theta, \tag{25}$$

and

$$\mathcal{S}_2(t) := E_{\alpha,\alpha}(-At^{\alpha}) = \int_0^{\infty} \alpha\theta M_{\alpha}(\theta)e^{-A\theta t^{\alpha}}d\theta = \int_0^{\infty} \alpha\theta M_{\alpha}(\theta)S(\theta t^{\alpha})d\theta. \tag{26}$$

We obtain the following properties of the fractional semigroup $(\mathcal{S}_i(t))_{t\in(0,T)}$, $i = 1, 2$.

Lemma 2 ([23, *Lemma 4*]). *Let $t \in (0,T)$, $0 < t_1 < t_2 \leq T$, $T < \infty$, $\frac{1}{2} < \alpha < 1$, $\rho \geq 0$, $0 \leq \eta < 1$, $0 \leq \kappa < \varpi \leq 1$, $0 \leq \sigma \leq \nu \leq 1$ and $\delta \geq 0$, there exists a constant $C > 0$ such that for all $i = 1, 2$ and $u \in \mathcal{H}$*

$$\|A^{\rho}\mathcal{S}_i(t)\|_{L(\mathcal{H})} \leq Ct^{-\alpha\rho}, \qquad \|A^{-\eta}(\mathcal{S}_1(t_2) - \mathcal{S}_1(t_1))\|_{L(\mathcal{H})} \leq C(t_2-t_1)^{\alpha\eta}, \tag{27}$$

$$\|A^{\kappa}(\mathcal{S}_1(t_2) - \mathcal{S}_1(t_1))\|_{L(\mathcal{H})} \leq C(t_2-t_1)^{\alpha(\varpi-\kappa)}t_1^{-\alpha\varpi}, \tag{28}$$

$$\left\|A^{\nu}\left[t_1^{\alpha-1}\mathcal{S}_2(t_1) - t_2^{\alpha-1}\mathcal{S}_2(t_2)\right]u\right\|_{L(\mathcal{H})} \leq C(t_2-t_1)^{1-[1-(\nu-\sigma)]\alpha}\left\|A^{\sigma}u\right\|, \tag{29}$$

and

$$A^{\delta}\mathcal{S}_i(t) = \mathcal{S}_i(t)A^{\delta} \quad \text{on} \quad D(A^{\delta}). \tag{30}$$

Proof. See [23, Lemma 4] for the proof of (27), (30) and [16, Lemma 3.3] for that of (29). The proof of (28) is similar to that of [23, (29)], we have hence

$$\begin{aligned}
&\|A^{\kappa}\left(\mathcal{S}_1(t_2)-\mathcal{S}_1(t_1)\right)\|_{L(\mathcal{H})}\\
&=\left\|\int_0^{\infty}A^{\kappa}M_{\alpha}(\theta)\left(S(\theta t_2^{\alpha})-S(\theta t_1^{\alpha})\right)d\theta\right\|_{L(\mathcal{H})}\\
&\leq\int_0^{\infty}M_{\alpha}(\theta)\left\|A^{-\varpi}S(\theta t_1^{\alpha})\right\|_{L(\mathcal{H})}\left\|A^{\kappa-\varpi}\left(e^{-A\theta(t_2^{\alpha}-t_1^{\alpha})}-I\right)\right\|_{L(\mathcal{H})}d\theta\\
&\leq C\int_0^{\infty}\left[\theta\left(t_2^{\alpha}-t_1^{\alpha}\right)\right]^{\varpi-\kappa}\left(\theta t_1^{\alpha}\right)^{-\varpi}M_{\alpha}(\theta)d\theta\\
&\leq C(t_2^{\alpha}-t_1^{\alpha})^{\varpi-\kappa}t_1^{-\alpha\varpi}\int_0^{\infty}\theta^{-\kappa}M_{\alpha}(\theta)d\theta\\
&\leq C\frac{\Gamma(1-\kappa)}{\Gamma(1-\alpha\kappa)}(t_2-t_1)^{\alpha(\varpi-\kappa)}t_1^{-\alpha\varpi}\\
&\leq C\,(t_2-t_1)^{\alpha(\varpi-\kappa)}t_1^{-\alpha\varpi}. \quad\blacksquare
\end{aligned}$$

Remark 2. Lemma 2 also holds with a uniform constant C (independent of h) when A and $\mathcal{S}_i(t)$, $i=1,2$ are replaced respectively by their discrete versions A_h and $\mathcal{S}_{ih}(t)$ defined in Section 4.

In order to ensure the existence and the uniqueness of mild solution for SPDE (1) and for the purpose of convergence analysis, we make the following assumptions.

Assumption 1 (*Initial Value*)**.** We assume that $X_0:\Omega\to\mathcal{H}$ is $\mathcal{F}_0/\mathcal{B}(\mathcal{H})$-measurable mapping and $X_0\in L^2(\Omega, D(A^{\frac{2H+\beta-1}{2}}))$ with $\beta\in[0,1]$.

Assumption 2 (*Non Linearity Term F*)**.** We assume the non-linear mapping $F:\mathcal{H}\to\mathcal{H}$, to be linear growth and Lipschitz continuous, that is for $\kappa\in[0,1]$, there exists positive constant $L>0$ such that

$$\|F(u)-F(v)\|^2\leq L\|u-v\|^2,\quad \|A^{\kappa}F(u)\|^2\leq L\left(1+\|A^{\kappa}u\|^2\right),\quad u,v\in\mathcal{H}. \tag{31}$$

Assumption 3 (*Standard Noise Term*)**.** We assume that the diffusion coefficient $G:\mathcal{H}\to L_2^0$ satisfies the global Lipschitz condition and the linear growth, that is for $\tau\in[0,1]$, there exists a positive constant $L>0$ such that

$$\|G(u)-G(v)\|^2_{L_2^0}\leq L\|u-v\|^2,\qquad \|A^{\tau}G(u)\|^2_{L_2^0}\leq L\left(1+\|A^{\tau}u\|^2\right),\qquad u,v\in\mathcal{H}, \tag{32}$$

Assumption 4 (*Fractional Noise Term*)**.** The deterministic mapping $\Phi:\mathcal{H}\to\mathcal{H}$ satisfies

$$\|A^{\frac{\beta-1}{2}}\Phi\|^2_{L_2^0}<\infty, \tag{33}$$

where β is defined as in Assumption 1.

We are now ready to present the result of existence and uniqueness of mild solution of SPDE (1) in the following theorem, which is one of our main result.

Theorem 1. *Under the Assumptions 1–4, the SPDEs (1) admits a unique mild solution $X(t)\in L^2(\Omega\times[0,T],\mathcal{H})$ asymptotic stable in mean square, that is*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|X(t)\|^2\right]<\infty, \tag{34}$$

where $L^2(\Omega\times[0,T],\mathcal{H})$ denotes the space of squared integrable $\mathcal{H}$-valued random variables.

Proof. We define the operator $\xi:L^2(\Omega\times[0,T],\mathcal{H})\to L^2(\Omega\times[0,T],\mathcal{H})$ by,

$$\begin{aligned}
\xi x(t)=\mathcal{S}_1(t)x_0&+\int_0^t(t-s)^{\alpha-1}\mathcal{S}_2(t-s)F(x(s))ds+\int_0^t\mathcal{S}_1(t-s)G(x(s))dW(s)\\
&+\int_0^t\mathcal{S}_1(t-s)\Phi dB^H(s).
\end{aligned}\tag{35}$$

In order to obtain our result, we use the Banach fixed point to prove that the mapping ξ has a unique fixed point in $L^2(\Omega\times[0,T],\mathcal{H})$. The proof will be split into two steps.

Step 1: First, we show that $\xi(L^2(\Omega\times[0,T],\mathcal{H}))\subset L^2(\Omega\times[0,T],\mathcal{H})$.
Let $x\in L^2(\Omega\times[0,T],\mathcal{H})$, using (35), triangle inequality and the estimate

$$\left(\sum_{i=1}^{n}a_i\right)^2\leq n\sum_{i=1}^{n}a_i^2,$$

we have

$$\begin{aligned}
&\mathbb{E}[\|\xi x(t)\|^2]\\
\leq\;& 4\mathbb{E}\,\|\mathcal{S}_1(t)x_0\|^2+4\mathbb{E}\left\|\int_0^t(t-s)^{\alpha-1}\mathcal{S}_2(t-s)F(x(s))ds\right\|^2\\
+\;& 4\mathbb{E}\left\|\int_0^t\mathcal{S}_1(t-s)G(x(s))dW(s)\right\|^2+4\mathbb{E}\left\|\int_0^t\mathcal{S}_1(t-s)\Phi dB^H(s)\right\|^2\\
=:\;& 4\sum_{i=1}^{4}I_i.
\end{aligned}\tag{36}$$

Using the fact that fractional semigroup $\mathcal{S}_1(t)$ is a contraction (20) yields

$$I_1:=\mathbb{E}\,\|\mathcal{S}_1(t)x_0\|^2\leq\mathbb{E}\,\|x_0\|^2<\infty.\tag{37}$$

Using Cauchy–Schwarz inequality, the stability property of fractional semigroup $\mathcal{S}_2(t)$ (20) and Assumption 2 with $\kappa=0$ yields

$$\begin{aligned}
I_2:=\;&\mathbb{E}\left\|\int_0^t(t-s)^{\alpha-1}\mathcal{S}_2(t-s)F(x(s))ds\right\|^2\\
\leq\;& L\left(\frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\right)^2\left(\int_0^t(t-s)^{2\alpha-2}ds\right)\int_0^t\mathbb{E}(1+\|x(s)\|^2)ds\\
\leq\;&\frac{c_0^2\,t^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\right)^2\left(1+\|x\|^2_{L^2(\Omega\times[0,T],\mathcal{H})}\right)<\infty.
\end{aligned}\tag{38}$$

Using also the contraction argument of the semigroup $\mathcal{S}_1(t)$ (20), Ito isometry (6) and Assumption 3 with $\tau=0$, we have

$$\begin{aligned}
I_3:=\;&\mathbb{E}\left\|\int_0^t\mathcal{S}_1(t-s)G(x(s))dW(s)\right\|^2\\
=\;&\mathbb{E}\left[\int_0^t\|\mathcal{S}_1(t-s)G(x(s))\|^2_{L_2^0}\,ds\right]\\
\leq\;& L\int_0^t\mathbb{E}(1+\|x(s)\|^2)ds\\
\leq\;& Lt+L\,\|x\|^2_{L^2(\Omega\times[0,T],\mathcal{H})}<\infty.
\end{aligned}\tag{39}$$

Applying (7), inserting an appropriate power of A and Lemma 2 with $\rho=\delta=\frac{1-\beta}{2}$, we have

$$\begin{aligned}
I_4:=\;&\mathbb{E}\left\|\int_0^t\mathcal{S}_1(t-s)\Phi dB^H(s)\right\|^2\\
\leq\;& C(H)\sum_{i=0}^{\infty}\left(\int_0^t\left\|\mathcal{S}_1(t-s)\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\
\leq\;& C(H)\sum_{i=0}^{\infty}\left(\int_0^t\left\|A^{\frac{1-\beta}{2}}\mathcal{S}_1(t-s)\right\|^{\frac{1}{H}}_{L(\mathcal{H})}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\
\leq\;& C(H)\left(\int_0^t(t-s)^{\frac{-\alpha(1-\beta)}{2H}}ds\right)^{2H}\left(\sum_{i=0}^{\infty}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^2\right)\\
\leq\;& C(H)\left(1-\frac{\alpha(1-\beta)}{2H}\right)^{-2H}t^{2H-\alpha(1-\beta)}\left\|A^{\frac{\beta-1}{2}}\Phi\right\|^2_{L_2^0}<\infty.
\end{aligned}\tag{40}$$

Inserting (37), (38), (39) and (40) in (36) implies that $\mathbb{E}\,\|\xi x(t)\|^2<\infty$ for all $t\in[0,T]$. Thus we conclude that $\xi x\in L^2(\Omega\times[0,T],\mathcal{H})$.

Step 2: Next, we show that the mapping ξ is contractive.
For this, we follows the same lines as [30, P31-33] and for $u \in \mathbb{R}$, we introduce the following norm

$$\|x\|_{L^2(\Omega\times[0,T],\mathcal{H}),u} := \sup_{0\leq t\leq T} e^{-ut}\|x(t)\|_{L^2(\Omega,\mathcal{H})}$$

on $L^2(\Omega\times[0,T],\mathcal{H})$ which is equivalent to $\|\cdot\|_{L^2(\Omega\times[0,T],\mathcal{H})}$. Let $x,y \in L^2(\Omega\times[0,T],\mathcal{H})$, then from (35) we have

$$\begin{aligned}
&\|(\xi x)(t)-(\xi y)(t)\|_{L^2(\Omega,\mathcal{H})}\\
\leq\ & \left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}_2(t-s)(F(x(s))-F(y(s)))ds\right\|_{L^2(\Omega,\mathcal{H})}\\
+\ & \left\|\int_0^t \mathcal{S}_1(t-s)(G(x(s))-G(y(s)))dW(s)\right\|_{L^2(\Omega,\mathcal{H})}\\
=:\ & J_1+J_2.
\end{aligned} \tag{41}$$

The stability property of fractional semigroup $\mathcal{S}_2(t)$ (20) and Assumption 2 allow to have

$$\begin{aligned}
J_1 &:= \left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}_2(t-s)(F(x(s))-F(y(s)))ds\right\|_{L^2(\Omega,\mathcal{H})}\\
&\leq \int_0^t \left\|(t-s)^{\alpha-1}\mathcal{S}_2(t-s)(F(x(s))-F(y(s)))\right\|_{L^2(\Omega,\mathcal{H})ds}\\
&\leq \frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|F(x(s))-F(y(s))\|_{L^2(\Omega,\mathcal{H})}ds\\
&\leq \frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\sqrt{L}\int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|_{L^2(\Omega,\mathcal{H})}ds\\
&\leq \frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\sqrt{L}\left(\int_0^t (t-s)^{\alpha-1}e^{us}ds\right)\|x-y\|_{L^2(\Omega\times[0,T],\mathcal{H}),u}.
\end{aligned} \tag{42}$$

Using the contraction argument of the semigroup, Ito isometry (6) and Assumption 3, we have

$$\begin{aligned}
J_2 &:= \left(\mathbb{E}\left\|\int_0^t \mathcal{S}_1(t-s)(G(x(s))-G(y(s)))dW(s)\right\|^2\right)^{\frac{1}{2}}\\
&= \left(\mathbb{E}\left[\int_0^t \|\mathcal{S}_1(t-s)(G(x(s))-G(y(s)))\|^2_{L_2^0}ds\right]\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^t \|G(x(s))-G(y(s))\|_{L^2(\Omega,L_2^0)}ds\right)^{\frac{1}{2}}\\
&\leq \sqrt{L}\left(\int_0^t e^{2us}ds\right)^{\frac{1}{2}}\|x-y\|_{L^2(\Omega\times[0,T],\mathcal{H}),u}.
\end{aligned} \tag{43}$$

Hence putting (42) and (43) in (41) yields

$$\begin{aligned}
&\|(\xi x)(t)-(\xi y)(t)\|_{L^2(\Omega,\mathcal{H})}\\
&\leq \sqrt{L}\left[\frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}e^{us}ds+\left(\frac{1}{2u}\left(e^{2ut}-1\right)\right)^{\frac{1}{2}}\right]\\
&\|x-y\|_{L^2(\Omega\times[0,T],\mathcal{H}),u}.
\end{aligned} \tag{44}$$

By Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\int_0^t (t-s)^{\alpha-1}e^{us}ds &\leq \left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{\frac{1}{2}}\left(\int_0^t e^{2us}ds\right)^{\frac{1}{2}}\\
&\leq \frac{1}{\sqrt{2\alpha-1}}T^{2\alpha-1}\left(\frac{1}{2u}\left(e^{2ut}-1\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{45}$$

Inserting (45) in (44) and multiplying the obtained inequality by e^{-ut}, hence for all $u > 0$,

$$
\begin{aligned}
& e^{-ut}\,\|(\xi x)(t)-(\xi y)(t)\|_{L^2(\Omega,\mathcal{H})} \\
& \leq \sqrt{L}\left[\frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\frac{1}{\sqrt{2\alpha-1}}T^{2\alpha-1}\left(\frac{1}{2u}\left(1-e^{-2ut}\right)\right)^{\frac{1}{2}}+\left(\frac{1}{2u}\left(1-e^{-2ut}\right)\right)^{\frac{1}{2}}\right] \\
& \|x-y\|_{L^2(\Omega\times[0,T],\mathcal{H}),u} \\
& \leq \sqrt{L}\left[\frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\frac{1}{\sqrt{2\alpha-1}}T^{2\alpha-1}+1\right](2u)^{-\frac{1}{2}}\,\|x-y\|_{L^2(\Omega\times[0,T],\mathcal{H}),u}\,.
\end{aligned}
$$

Therefore, for $u > 0$ sufficiently large $\xi : L^2(\Omega\times[0,T],\mathcal{H}) \to L^2(\Omega\times[0,T],\mathcal{H})$ is a contraction with respect to the norm $\|\cdot\|_{L^2(\Omega,\mathcal{H}),u}$ and there exists a unique fixed point $x \in L^2(\Omega\times[0;T];\mathcal{H})$ which is a unique mild solution to (1) and $x(t)$ is asymptotic stable in mean square.

The proof is thus completed. ■

In all that follows, C denotes a positive constant that may change from line to line. In the Banach space $D(A^{\frac{\gamma}{2}})$, $\gamma \in \mathbb{R}$, we use the notation $\|A^{\frac{\gamma}{2}}\cdot\| = \|\cdot\|_\gamma$. We are now ready to present some regularity results.

3. Regularity of the mild solution

We discuss the space and time regularities of the mild solution $X(t)$ of (1) given by (21) in this section. The following Theorem presents the spatial and temporal regularity results.

Theorem 2. *Under Assumptions 1–4, the unique mild solution $X(t)$ given by (21) satisfies the following space regularity*

$$
\left\|A^{\frac{2H+\beta-1}{2}}X(t)\right\|_{L^2(\Omega;\mathcal{H})} \leq C\left(1+\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|_{L^2(\Omega;\mathcal{H})}\right), \qquad t\in[0,T]. \tag{46}
$$

Furthermore for $0 \leq t_1 < t_2 \leq T$, the following optimal time regularity holds

$$
\begin{aligned}
& \|X(t_2)-X(t_1)\|_{L^2(\Omega;\mathcal{H})} \\
& \leq C(t_2-t_1)^{\frac{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}{2}}\left(1+\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|_{L^2(\Omega;\mathcal{H})}\right).
\end{aligned} \tag{47}
$$

Moreover (46) and (47) *hold when A and X are replaced by their semidiscrete version A_h and X^h defined in Section 4.*

Proof. We begin by proving (46). Premultiplying (21) by $A^{\frac{2H+\beta-1}{2}}$, taking the squared-norm and the estimate $\left(\sum_{i=1}^n a_i\right)^2 \leq n\sum_{i=1}^n a_i^2$ yields

$$
\begin{aligned}
\left\|A^{\frac{2H+\beta-1}{2}}X(t)\right\|^2_{L^2(\Omega;\mathcal{H})} \leq\; & 4\left\|A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t)X_0\right\|^2_{L^2(\Omega;\mathcal{H})} \\
& + 4\left\|\int_0^t (t-s)^{\alpha-1}A^{\frac{2H+\beta-1}{2}}\mathcal{S}_2(t-s)F(X(s))ds\right\|^2_{L^2(\Omega;\mathcal{H})} \\
& + 4\left\|\int_0^t A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t-s)G(X(s))dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\
& + 4\left\|\int_0^t A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t-s)\Phi dB^H(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\
=:\; & 4\sum_{i=1}^4 II_i.
\end{aligned} \tag{48}
$$

We bound II_i, $i = 1, 2, 3$ one by one.

Firstly, using the fact that $\mathcal{S}_1(t)$ is a contraction (20) and (30) with $\delta = \frac{2H+\beta-1}{2}$ yields

$$
II_1 := \left\|A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t)X_0\right\|^2_{L^2(\Omega;\mathcal{H})} \leq \left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|^2_{L^2(\Omega;\mathcal{H})}. \tag{49}
$$

Secondly, using Cauchy–Schwarz inequality, the stability property of fractional semigroup $\mathcal{S}_2(t)$ (20), (30) and Assumption 2 with $\delta = \kappa = \frac{2H+\beta-1}{2}$ yields

$$
II_2 := \mathbb{E}\left[\left\|\int_0^t (t-s)^{\alpha-1}A^{\frac{2H+\beta-1}{2}}\mathcal{S}_2(t-s)F(X(s))ds\right\|^2\right]
$$

$$\leq L\left(\int_0^t (t-s)^{2\alpha-2}\|\mathcal{S}_2(t-s)\|^2_{L(\mathcal{H})}ds\right)\int_0^t\left(1+\mathbb{E}\left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2\right)ds$$
$$\leq \frac{Lt^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha\Gamma(2)}{\Gamma(1+\alpha)}\right)^2\left(t+\int_0^t\mathbb{E}\left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2ds\right)$$
$$\leq C+C\int_0^t\left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2_{L^2(\Omega,\mathcal{H})}ds. \tag{50}$$

Thirdly, using the Itô isometry (6), the contraction of $\mathcal{S}_1(t)$ (20), (30) and Assumption 3 with $\delta=\tau=\frac{2H+\beta-1}{2}$ allow to have

$$II_3 := \mathbb{E}\left[\left\|\int_0^t A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t-s)G(X(s))dW(s)\right\|^2\right]$$
$$= \mathbb{E}\left[\int_0^t\left\|A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t-s)G(X(s))\right\|^2_{L_2^0}ds\right]$$
$$\leq \mathbb{E}\left[\int_0^t\left\|A^{\frac{2H+\beta-1}{2}}G(X(s))\right\|^2_{L_2^0}ds\right]$$
$$\leq C\mathbb{E}\left[\int_0^t 1+\left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2ds\right]$$
$$\leq C+C\int_0^t\left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2_{L^2(\Omega;\mathcal{H})}ds. \tag{51}$$

Fourthly, using (7), Lemma 2 (27) with $\rho=H$, (30) with $\delta=1-\beta$ and Assumption 4, we have

$$II_4 := \mathbb{E}\left[\left\|\int_0^t A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t-s)\Phi dB^H(s)\right\|^2\right]$$
$$\leq C(H)\sum_{i=0}^{\infty}\left(\int_0^t\left\|A^{\frac{2H+\beta-1}{2}}\mathcal{S}_1(t-s)\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}$$
$$\leq C(H)\sum_{i=0}^{\infty}\left(\int_0^t\|A^H\mathcal{S}_1(t-s)\|^{\frac{1}{H}}_{L(\mathcal{H})}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}$$
$$\leq C(H)\left(\int_0^t(t-s)^{-\alpha}ds\right)^{2H}\left(\sum_{i=0}^{\infty}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^2\right)$$
$$\leq C(H)\frac{t^{2H(1-\alpha)}}{(1-\alpha)^{2H}}\left\|A^{\frac{\beta-1}{2}}\Phi\right\|^2_{L_2^0}\leq C. \tag{52}$$

Inserting (49)–(52) in (48) hence yields

$$\left\|A^{\frac{2H+\beta-1}{2}}X(t)\right\|^2_{L^2(\Omega;\mathcal{H})}$$
$$\leq C\left(1+\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|^2_{L^2(\Omega;\mathcal{H})}\right)+C\int_0^t\left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2_{L^2(\Omega;\mathcal{H})}ds.$$

Applying the continuous Gronwall's lemma proves (46).

Now for the proof of (47), we rewrite the mild solution (21) at times $t=t_2$ and $t=t_1$ and we subtract $X(t_2)$ by $X(t_1)$ as

$$X(t_2)-X(t_1)$$
$$=(\mathcal{S}_1(t_2)-\mathcal{S}_1(t_1))X_0+\int_0^{t_1}\left[(t_2-s)^{\alpha-1}\mathcal{S}_2(t_2-s)-(t_1-s)^{\alpha-1}\mathcal{S}_2(t_1-s)\right]F(X(s))ds$$
$$+\int_0^{t_1}\left[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)\right]G(X(s))dW(s)+\int_0^{t_1}\left[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)\right]\Phi\, dB^H(s)$$
$$+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{S}_2(t_2-s)F(X(s))\,ds+\int_{t_1}^{t_2}\mathcal{S}_1(t_2-s)G(X(s))\,dW(s)$$
$$+\int_{t_1}^{t_2}\mathcal{S}_1(t_2-s)\Phi\, dB^H(s). \tag{53}$$

Taking the L^2-norm on both sides and using triangle inequality yields

$$\|X(t_2)-X(t_1)\|_{L^2(\Omega;\mathcal{H})} \leq \sum_{i=1}^{7} III_i. \tag{54}$$

Inserting an appropriate power of A, using Lemma 2 more precisely (27) and (30) with $\eta=\delta=\frac{2H+\beta-1}{2}$, Assumption 1 implies

$$\begin{aligned} III_1 &:= \|(\mathcal{S}_1(t_2)-\mathcal{S}_1(t_1))X_0\|_{L^2(\Omega;\mathcal{H})} \\ &\leq \left\|A^{-\frac{2H+\beta-1}{2}}(\mathcal{S}_1(t_2)-\mathcal{S}_1(t_1))A^{\frac{2H+\beta-1}{2}}X_0\right\|_{L^2(\Omega;\mathcal{H})} \\ &\leq C(t_2-t_1)^{\frac{\alpha(2H+\beta-1)}{2}}\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|_{L^2(\Omega;\mathcal{H})}. \end{aligned} \tag{55}$$

For the estimate of III_2, by triangle inequality, inserting an appropriate power of A, using (30), (29) with $\nu=\frac{\beta}{2}$ and $\sigma=0$, Assumption 2 and (34), we get

$$\begin{aligned} III_2 &:= \left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}S_2(t_2-s)-(t_1-s)^{\alpha-1}S_2(t_1-s)\right]F(X(s))ds\right\|_{L^2(\Omega;\mathcal{H})} \\ &\leq \int_0^{t_1}\|A^{\frac{\beta}{2}}\left[(t_2-s)^{\alpha-1}S_2(t_2-s)-(t_1-s)^{\alpha-1}S_2(t_1-s)\right]A^{-\frac{\beta}{2}}F(X(s))\|_{L^2(\Omega;\mathcal{H})}ds \\ &\leq C\int_0^{t_1}(t_2-t_1)^{1-(1-\frac{\beta}{2})\alpha}\|A^{-\frac{\beta}{2}}F(X(s))\|_{L^2(\Omega;\mathcal{H})}ds \\ &\leq C(t_2-t_1)^{1-(1-\frac{\beta}{2})\alpha}\int_0^{t_1}\left(1+\mathbb{E}[\sup_{0\leq s\leq T}\|X(s)\|^2]\right)^{\frac{1}{2}}ds \\ &\leq C(t_2-t_1)^{1-(1-\frac{\beta}{2})\alpha}t_1 \leq C(t_2-t_1)^{\frac{2-(2-\beta)\alpha}{2}}. \end{aligned} \tag{56}$$

The estimate of III_5 is already obtained in [23, (45)] and we have

$$III_5 \leq C(t_2-t_1)^{\alpha}. \tag{57}$$

Using the Itô isometry property (6), (27), (30), Assumption 3 with $\eta=\delta=\tau=\frac{2H+\beta-1}{2}$ and (46), we obtain

$$\begin{aligned} III_3^2 &= \left\|\int_0^{t_1}[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)]G(X(s))dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\ &= \mathbb{E}\left[\int_0^{t_1}\left\|A^{-\frac{2H+\beta-1}{2}}[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)]A^{\frac{2H+\beta-1}{2}}G(X(s))\right\|^2 ds\right] \\ &\leq C\int_0^{t_1}(t_2-t_1)^{\alpha(2H+\beta-1)}\mathbb{E}\left[\left\|A^{\frac{2H+\beta-1}{2}}G(X(s))\right\|^2_{L_2^0}\right]ds \\ &\leq C\int_0^{t_1}(t_2-t_1)^{\alpha(2H+\beta-1)}\left(1+\mathbb{E}\left[\|A^{\frac{2H+\beta-1}{2}}X(s)\|^2\right]\right)ds \\ &\leq C(t_2-t_1)^{\alpha(2H+\beta-1)}t_1\left(1+\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|^2_{L^2(\Omega;\mathcal{H})}\right) \\ &\leq C(t_2-t_1)^{\alpha(2H+\beta-1)}\left(1+\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|^2_{L^2(\Omega;\mathcal{H})}\right). \end{aligned} \tag{58}$$

Let us estimate now III_4^2. Using (7), inserting an appropriate power of A, (28) with $\kappa=\frac{1-\beta}{2}$ and $\varpi=H$, Assumption 4 yields

$$\begin{aligned} III_4^2 &:= \left\|\int_0^{t_1}[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)]\,\Phi\,dB^H(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\ &= \mathbb{E}\left\|\int_0^{t_1}[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)]\,\Phi\,dB^H(s)\right\|^2 \\ &\leq C(H)\sum_{i=0}^{\infty}\left(\int_0^{t_1}\left\|A^{\frac{1-\beta}{2}}[\mathcal{S}_1(t_2-s)-\mathcal{S}_1(t_1-s)]A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H} \end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{i=0}^{\infty}\left(\int_0^{t_1}\left[(t_2-t_1)^{\alpha\left(H-\frac{1-\beta}{2}\right)}(t_1-s)^{-\alpha H}\right]^{\frac{1}{H}}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\
&\leq C\sum_{i=0}^{\infty}\left(\int_0^{t_1}\left[(t_2-t_1)^{\alpha\left(H-\frac{1-\beta}{2}\right)}(t_1-s)^{-\alpha H}\right]^{\frac{1}{H}}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\
&\leq C(t_2-t_1)^{\alpha(2H+\beta-1)}\left(\int_0^{t_1}(t_1-s)^{-\alpha}ds\right)^{2H}\left\|A^{\frac{\beta-1}{2}}\Phi\right\|^2_{L_2^0}\\
&\leq C(t_2-t_1)^{\alpha(2H+\beta-1)}t_1^{2H(1-\alpha)}\\
&\leq C(t_2-t_1)^{\alpha(2H+\beta-1)}.
\end{aligned}\tag{59}
$$

Now for the sixth term, we use Itô isometry property (6), the contraction of $\mathcal{S}_1(t)$, Assumption 3 with $\tau=0$ and Theorem 1 to obtain

$$
\begin{aligned}
III_6^2 &:= \left\|\int_{t_1}^{t_2}\mathcal{S}_1(t_2-s)G(X(s))\,dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\\
&= \mathbb{E}\left[\int_{t_1}^{t_2}\|\mathcal{S}_1(t_2-s)G(X(s))\|^2_{L_2^0}\,ds\right]\\
&\leq \int_{t_1}^{t_2}\|\mathcal{S}_1(t_2-s)\|^2_{L(\mathcal{H})}\,\mathbb{E}\,\|G(X(s))\|^2_{L_2^0}\,ds\\
&\leq \left(\int_{t_1}^{t_2}ds\right)\left(1+\mathbb{E}\left[\sup_{0\leq s\leq T}\|X(s)\|^2\right]\right)\\
&\leq C(t_2-t_1).
\end{aligned}\tag{60}
$$

Finally, using (7), inserting an appropriate power of A, Lemma 2 with $\rho=\delta=\frac{1-\beta}{2}$ and Assumption 4 yields

$$
\begin{aligned}
III_7^2 &:= \left\|\int_{t_1}^{t_2}\mathcal{S}_1(t_2-s)\Phi\,dB^H(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\\
&= \mathbb{E}\left\|\int_{t_1}^{t_2}\mathcal{S}_1(t_2-s)\Phi\,dB^H(s)\right\|^2\\
&\leq C(H)\sum_{i=0}^{\infty}\left(\int_{t_1}^{t_2}\left\|\mathcal{S}_1(t_2-s)\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\
&\leq C(H)\sum_{i=0}^{\infty}\left(\int_{t_1}^{t_2}\left\|A^{\frac{1-\beta}{2}}\mathcal{S}_1(t_2-s)\right\|^{\frac{1}{H}}_{L(\mathcal{H})}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\
&\leq C(H)\left(\int_{t_1}^{t_2}(t_2-s)^{-\frac{\alpha(1-\beta)}{2H}}ds\right)^{2H}\left\|A^{\frac{\beta-1}{2}}\Phi\right\|^2_{L_2^0}\\
&\leq C(t_2-t_1)^{2H+\alpha(\beta-1)}\leq C(t_2-t_1)^{\alpha(2H+\beta-1)}.
\end{aligned}\tag{61}
$$

Substituting (55)–(61) in (54) yields

$$
\begin{aligned}
&\|X(t_2)-X(t_1)\|_{L^2(\Omega;\mathcal{H})}\\
&\leq C(t_2-t_1)^{\frac{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}{2}}\left(1+\left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|_{L^2(\Omega;\mathcal{H})}\right).
\end{aligned}
$$

The proof of Theorem 2 is thus completed. ■

4. Space approximation and error estimates

We consider the discretization of the spatial domain by a finite element triangulation with maximal length h satisfying the usual regularity assumptions. Let $V_h\subset\mathcal{H}$ denote the space of continuous functions that are piecewise linear over triangulation $\mathcal{J}_h$. To discretize in space, we introduce P_h from $L^2(\Omega)$ to V_h define for $u\in L^2(\Omega)$ by

$$
(P_hu,\xi)=(u,\xi),\qquad \forall\xi\in V_h.\tag{62}
$$

The discrete operator $A_h:V_h\to V_h$ is defined by

$$
(A_h\rho,\xi)=-a(\rho,\xi),\qquad \forall\rho,\xi\in V_h,\tag{63}
$$

where a is the corresponding bilinear form of A. Like the operator A, the discrete operator A_h is also the generator of a contraction semigroup $S_h(t) := e^{-tA_h}$. The semidiscrete space version of problem (16) is to find $X^h(t) = X^h(\cdot, t)$ such that for $t \in (0, T]$

$$\begin{cases} {}^C\partial_t^\alpha X^h(t) + A^h X^h(t) = P_h F(X^h(t)) + I_t^{1-\alpha}\left[P_h G(X^h(t))\frac{dW(t)}{dt} + P_h\Phi\frac{dB^H(t)}{dt}\right], \\ X^h(0) = P_h X_0. \end{cases} \tag{64}$$

Note that A_h, P_hG and $P_h\Phi$ satisfy the same assumptions as A, G and Φ respectively. The mild solution of (64) can be represented as follows

$$\begin{aligned} X^h(t) &= \mathcal{S}_{1h}(t)X_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{2h}(t-s)P_hF(X^h(s))ds \\ &\quad + \int_0^t \mathcal{S}_{1h}(t-s)P_hG(X^h(s))dW(s) + \int_0^t \mathcal{S}_{1h}(t-s)P_h\Phi dB^H(s). \end{aligned} \tag{65}$$

Where $\mathcal{S}_{1h}$ and $\mathcal{S}_{2h}$ are the semidiscrete version of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively defined by (25) and (26).

Let us define the error operators

$$\mathcal{T}_{1h}(t) := \mathcal{S}_1(t) - \mathcal{S}_{1h}(t)P_h, \qquad \mathcal{T}_{2h}(t) := \mathcal{S}_2(t) - \mathcal{S}_{2h}(t)P_h.$$

Then we have the following lemma.

Lemma 3 (*[23, Lemma 3]*). *There exists a positive constant C such that:*

(i) For $r \in [0, 2]$, $\rho \le r$, $t \in (0, T]$, $v \in D(A^\rho)$

$$\|\mathcal{T}_{1h}(t)v\| \le Ch^r t^{-\alpha(r-\rho)/2}\|v\|_\rho, \qquad \|\mathcal{T}_{2h}(t)v\| \le Ch^r t^{-\alpha(r-\rho)/2}\|v\|_\rho. \tag{66}$$

(ii) For $0 \le \gamma \le 1$,

$$\left\|\int_0^t s^{\alpha-1}\mathcal{T}_{2h}(s)v ds\right\| \le Ch^{2-\gamma}\|v\|_{-\gamma}, \qquad v \in D(A^{-\gamma}),\ t > 0. \tag{67}$$

The following lemma provides an estimate in the mean square sense for the error between the solution of SPDE (16) and the spatially semidiscrete approximation (65).

Lemma 4 (*Space Error*). *Let X and X^h be the mild solution of* (16) and (64), *respectively. Let Assumptions 1–4 be fulfilled then there exists a constant C independent of h, such that*

$$\left\|X(t) - X^h(t)\right\|_{L^2(\Omega;\mathcal{H})} \le Ch^{2H+\beta-1}, \qquad 0 \le t \le T. \tag{68}$$

Proof. Define $e(t) := X(t) - X^h(t)$. By (21) and (65), taking the norm, using triangle inequality and the estimate $\left(\sum_{i=1}^n a_i\right)^2 \le n\sum_{i=1}^n a_i^2$ we deduce that

$$\begin{aligned} &\|e(t)\|^2_{L^2(\Omega;\mathcal{H})} \\ \le\ & 4\|\mathcal{S}_1(t)X_0 - \mathcal{S}_{1h}(t)P_hX_0\|^2_{L^2(\Omega;\mathcal{H})} \\ +\ & 4\left\|\int_0^t \left[(t-s)^{\alpha-1}\mathcal{S}_2(t-s)X(s) - (t-s)^{\alpha-1}\mathcal{S}_{2h}(t-s)P_hF(X^h(s))\right] ds\right\|^2_{L^2(\Omega;\mathcal{H})} \\ +\ & 4\left\|\int_0^t \left[\mathcal{S}_1(t-s)G(X(s)) - \mathcal{S}_{1h}(t-s)P_hG(X^h(s))\right] dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\ +\ & 4\left\|\int_0^t \left[\mathcal{S}_1(t-s) - \mathcal{S}_{1h}(t-s)P_h\right]\Phi\, dB^H(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\ =:\ & 4\sum_{i=1}^4 IV_i. \end{aligned} \tag{69}$$

We will analyze the above terms IV_i, $i = 1, 2, 3, 4$ one by one.

For the first term IV_1, using Lemma 3 with $r = \gamma = 2H + \beta - 1$ and Assumption 1 yields

$$\begin{aligned} IV_1 &:= \|\mathcal{S}_1(t)X_0 - \mathcal{S}_{1h}(t)P_hX_0\|^2_{L^2(\Omega;\mathcal{H})} \\ &= \|[\mathcal{S}_1(t) - \mathcal{S}_{1h}(t)P_h]X_0\|^2_{L^2(\Omega;\mathcal{H})} \\ &= \|\mathcal{T}_{1h}(t)X_0\|^2_{L^2(\Omega;\mathcal{H})} \end{aligned}$$

$$\begin{aligned}
&\leq C\, h^{2(2H+\beta-1)} \left\| A^{\frac{2H+\beta-1}{2}} X_0 \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq C\, h^{2(2H+\beta-1)}.
\end{aligned} \tag{70}$$

For the second term IV_2, by adding and subtracting a term, applying triangle inequality and the estimate $(a+b)^2 \leq 2a^2+2b^2$, we split it in two terms as follows

$$\begin{aligned}
IV_2 &:= \left\| \int_0^t \left[(t-s)^{\alpha-1}\mathcal{S}_2(t-s)F(X(s)) - (t-s)^{\alpha-1}\mathcal{S}_{2h}(t-s)P_hF(X^h(s)) \right] ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq 2 \left\| \int_0^t (t-s)^{\alpha-1} \left[\mathcal{S}_2(t-s) - \mathcal{S}_{2h}(t-s)P_h \right] F(X(s))\, ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&+ 2 \left\| \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{2h}(t-s)P_h(F(X(s)) - F(X^h(s)))\, ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&=: 2IV_{21} + 2IV_{22}.
\end{aligned} \tag{71}$$

Firstly, adding and subtracting a term, applying Cauchy-Schwartz inequality, Lemma 3(i) with $r = 2H+\beta-1$, $\rho = 0$, Lemma 2 more precisely (47) for the first term and Lemma 3(ii) with $\gamma = 0$, Theorem 1 and Assumption 2 with $\kappa = 0$ yields

$$\begin{aligned}
IV_{21} &:= \left\| \int_0^t (t-s)^{\alpha-1} \left[\mathcal{S}_2(t-s) - \mathcal{S}_{2h}(t-s)P_h \right] F(X(s))\, ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq 2 \left\| \int_0^t (t-s)^{\alpha-1}\mathcal{T}_{2h}(t-s)(F(X(s)) - F(X(t)))\, ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&+ 2 \left\| \int_0^t (t-s)^{\alpha-1}\mathcal{T}_{2h}(t-s)F(X(t))\, ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq 2 \left(\int_0^t (t-s)^{2\alpha-2}\, ds \right) \left(\int_0^t \mathbb{E}\left\| \mathcal{T}_{2h}(t-s)(F(X(s)) - F(X(t))) \right\|^2 \right) ds \\
&+ 2 \left\| \int_0^t s^{\alpha-1}\mathcal{T}_{2h}(s)F(X(t))\, ds \right\|^2_{L^2(\Omega;\mathcal{H})}.
\end{aligned} \tag{72}$$

Furthermore, we have

$$\begin{aligned}
IV_{21} &\leq C\, h^{2(2H+\beta-1)} \int_0^t (t-s)^{-\alpha(2H+\beta-1)} \left\| F(X(s)) - F(X(t)) \right\|^2_{L^2(\Omega;\mathcal{H})} ds \\
&+ C\, h^4 \left\| F(X(t)) \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq C\, h^{2(2H+\beta-1)} \int_0^t (t-s)^{-\alpha(2H+\beta-1)} \left\| X(t) - X(s) \right\|^2_{L^2(\Omega;\mathcal{H})} ds \\
&+ C\, h^4 \left(1 + \left\| X(t) \right\|^2_{L^2(\Omega;\mathcal{H})} \right) \\
&\leq C\, h^{2(2H+\beta-1)} \int_0^t (t-s)^{\min(0;1-\alpha(2H+\beta-1))}\, ds + C\, h^4 \mathbb{E}\left[\sup_{0\leq t\leq T} \left\| X(t) \right\|^2 \right] \\
&\leq C\, h^{2(2H+\beta-1)}.
\end{aligned} \tag{73}$$

Secondly, applying the Cauchy-Schwartz inequality, boundedness of P_h, $\mathcal{S}_{2h}(t)$ and Assumption 2, it holds that

$$\begin{aligned}
IV_{22} &:= \left\| \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{2h}(t-s)P_h \left(F(X(s)) - F(X^h(s)) \right) ds \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq \left(\int_0^t (t-s)^{2\alpha-2}\, ds \right) \left(\int_0^t \mathbb{E}\left\| \mathcal{S}_{2h}(t-s)P_h \left(F(X(s)) - F(X^h(s)) \right) \right\|^2 ds \right) \\
&\leq C \int_0^t \left\| e(s) \right\|^2_{L^2(\Omega;\mathcal{H})} ds.
\end{aligned} \tag{74}$$

Putting (72) and (74) in (71), we obtain

$$IV_2 \leq C\, h^{2(2H+\beta-1)} + C \int_0^t \left\| e(s) \right\|^2_{L^2(\Omega;\mathcal{H})}. \tag{75}$$

For the third term IV_3, by adding and subtracting a term, using the triangle inequality and the estimate $(a+b)^2 \leq 2a^2+2b^2$, we have

$$
\begin{aligned}
IV_3 &= \left\| \int_0^t \left[\mathcal{S}_1(t-s)G(X(s)) - \mathcal{S}_{1h}(t-s)P_hG(X^h(s))\right] dW(s) \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq 2 \left\| \int_0^t \left[\mathcal{S}_1(t-s) - \mathcal{S}_{1h}(t-s)P_h\right] G(X(s))\, dW(s) \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&+ 2 \left\| \int_0^t \mathcal{S}_{1h}(t-s)P_h(G(X(s)) - G(X^h(s)))\, dW(s) \right\|^2_{L^2(\Omega;\mathcal{H})}.
\end{aligned}
$$

Applying Itô isometry (6), Lemma 3(i) with $r = \gamma = 2H+\beta-1$ for the first term, Assumption 3 with $\tau = 2H+\beta-1$, Theorem 2, contraction argument of $\mathcal{S}_{1,h}(t)$ (20) and boundedness of P_h it holds that

$$
\begin{aligned}
IV_3 &\leq 2\int_0^t \mathbb{E}\,\|\mathcal{T}_{1h}G(X(s))\|^2_{L_2^0}\, ds + \int_0^t \mathbb{E}\left\|\mathcal{S}_{1h}(t-s)P_h(G(X(s)) - G(X^h(s)))\right\|^2_{L_2^0} ds \\
&\leq C\,h^{2(2H+\beta-1)} \int_0^t \mathbb{E}\left\|A^{\frac{2H+\beta-1}{2}}G(X(s))\right\|^2_{L_2^0} ds \\
&+ C\int_0^t \mathbb{E}\left\|G(X(s)) - G(X^h(s))\right\|^2_{L_2^0} ds \\
&\leq C\,h^{2(2H+\beta-1)} \int_0^t \left(1 + \left\|A^{\frac{2H+\beta-1}{2}}X(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\right) ds + C\int_0^t \left\|X(s) - X^h(s)\right\|^2_{L^2(\Omega;\mathcal{H})} ds \\
&\leq C\,h^{2(2H+\beta-1)} \left(1 + \left\|A^{\frac{2H+\beta-1}{2}}X_0\right\|^2_{L^2(\Omega;\mathcal{H})}\right) + C\int_0^t \|e(s)\|^2_{L^2(\Omega;\mathcal{H})} ds \\
&\leq C\,h^{2(2H+\beta-1)} + C\int_0^t \|e(s)\|^2_{L^2(\Omega;\mathcal{H})}\, ds.
\end{aligned} \tag{76}
$$

For the estimation of IV_4, (7), Lemma 3 with $r = 2H+\beta-1$ and $\gamma = \beta-1$, Assumption 4 yields

$$
\begin{aligned}
IV_4 &= \left\| \int_0^t \left[\mathcal{S}_1(t-s) - \mathcal{S}_{1h}(t-s)P_h\right] \Phi\, dB^H(s) \right\|^2_{L^2(\Omega;\mathcal{H})} \\
&\leq C(H)\sum_{i=0}^{\infty}\left(\int_0^t \left\|\mathcal{T}_{1h}(t-s)\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}} ds\right)^{2H} \\
&\leq C(H)\sum_{i=0}^{\infty}\left(\int_0^t C\,h^{\frac{2H+\beta-1}{H}}(t-s)^{-\alpha}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}} ds\right)^{2H} \\
&\leq C\,h^{2(2H+\beta-1)}\left(\int_0^t (t-s)^{-\alpha}\, ds\right)^{2H}\left(\sum_{i=0}^{\infty}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^2\right) \\
&\leq C\,h^{2(2H+\beta-1)}t^{2H(1-\alpha)}\left\|A^{\frac{\beta-1}{2}}\Phi\right\|^2_{L_2^0} \\
&\leq C\,h^{2(2H+\beta-1)}.
\end{aligned} \tag{77}
$$

Putting (70), (75), (76), (77) in (69) and applying Gronwall's inequality ends the proof. ■

5. Generalized exponential time differencing method (GETD) and its error estimate

In this section, we consider a fully discrete approximation of SPDE (16).

5.1. Generalized exponential time differencing method

Let Δt be the time step size and $M \in \mathbb{N}$ such that $T = M\Delta t$, hence for all $m \in \{0, 1, \ldots, M\}$, X_m^h denotes the numerical approximation of $X^h(t_m)$ with $t_m = m\Delta t$. Applying the generalized exponential time differencing method[2] (see [23]) to

[2] or the fractional exponential integrator scheme

(64) gives the following fully discrete scheme.

$$\begin{aligned} X_m^h &= \mathcal{S}_{1h}(t_m)P_hX_0 + \Delta t\sum_{j=0}^{m-1}(t_m-t_j)^{\alpha-1}\mathcal{S}_{2h}(t_m-t_j)P_hF(X_j^h) \\ &\quad + \sum_{j=0}^{m-1}\mathcal{S}_{1h}(t_m-t_j)P_hG(X_j^h)\,\Delta W_j + \sum_{j=0}^{m-1}\mathcal{S}_{1h}(t_m-t_j)P_h\Phi\,\Delta B_j^H \qquad (78)\\ &= E_{\alpha,1}(-t_m^\alpha A_h)P_hX_0 + \Delta t\sum_{j=0}^{m-1}(t_m-t_j)^{\alpha-1}E_{\alpha,\alpha}(-(t_m-t_j)^\alpha A_h)P_hF(X_j^h) \\ &\quad + \sum_{j=0}^{m-1}E_{\alpha,1}(-(t_m-t_j)^\alpha A_h)P_hG(X_j^h)\,\Delta W_j + \sum_{j=0}^{m-1}E_{\alpha,1}(-(t_m-t_j)^\alpha A_h)P_h\Phi\,\Delta B_j^H, \qquad (79)\end{aligned}$$

where

$$\Delta W_j := W(t_{j+1}) - W(t_j) = \sum_{i=0}^{\infty}\sqrt{q_i}\left(\beta_i(t_{j+1})-\beta_i(t_j)\right)e_i$$

and

$$\Delta B_j^H := B^H(t_{j+1}) - B^H(t_j) = \sum_{i=0}^{\infty}\sqrt{q_i}\left(\beta_i^H(t_{j+1})-\beta_i^H(t_j)\right)e_i.$$

Our key result is given in the following theorem

Theorem 3. *Let $X(t_m)$ be the mild solution of (16) at time $t_m = m\Delta t$ given by (21). Let X_m^h be the numerical approximation given at (78). Under Assumptions 1–4, the following holds*

$$\left\|X(t_m)-X_m^h\right\|_{L^2(\Omega;\mathcal{H})} \le C\left(h^{2H+\beta-1} + \Delta t^{\frac{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}{2}}\right), \qquad (80)$$

where $\beta\in[0,1]$, $H\in(\frac{1}{2},1)$ and $\alpha\in(\frac{1}{2},1)$.

Before giving the proof, let us present a preparatory result

Lemma 5 (*[31, (89)] And [32, (70)]*). *Let σ be such that $-1\le\sigma\le 1$, hence the following estimate holds*

$$\left\|A_h^\sigma P_h u\right\| \le C\left\|A^\sigma u\right\|, \qquad (81)$$

where C is a positive constant independent of h.

5.2. Proof of main result in Theorem 3

We are now ready to prove our main theorem. In fact using the standard technique in the error analysis, we split the fully discrete error in two terms as

$$\begin{aligned}\left\|X(t_m)-X_m^h\right\|_{L^2(\Omega;\mathcal{H})} &\le \left\|X(t_m)-X^h(t_m)\right\|_{L^2(\Omega;\mathcal{H})} + \left\|X^h(t_m)-X_m^h\right\|_{L^2(\Omega;\mathcal{H})} \\ &=: err_0 + err_1. \qquad (82)\end{aligned}$$

Note that the space error err_0 is estimated by Lemma 4. It remains to estimate the time error err_1. We recall that the mild solution at time $t_m = m\Delta t$ of the semidiscrete problem (64) is given by

$$\begin{aligned}X^h(t_m) &= \mathcal{S}_{1h}(t_m)X_0^h + \int_0^{t_m}(t_m-s)^{\alpha-1}\mathcal{S}_{2h}(t_m-s)P_hF(X^h(s))ds \\ &\quad + \int_0^{t_m}\mathcal{S}_{1h}(t_m-s)P_hG(X^h(s))dW(s) + \int_0^{t_m}\mathcal{S}_{1h}(t_m-s)P_h\Phi dB^H(s). \qquad (83)\end{aligned}$$

Iterating (83) yields

$$\begin{aligned}X^h(t_m) &= \mathcal{S}_{1h}(t_m)X_0^h + \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(t_m-s)^{\alpha-1}\mathcal{S}_{2h}(t_m-s)P_hF(X^h(s))ds \\ &\quad + \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-s)P_hG(X^h(s))dW(s) + \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-s)P_h\Phi dB^H(s). \qquad (84)\end{aligned}$$

Thanks to (78), we rewrite the numerical solution X_m^h in the integral form as

$$\begin{aligned} X_m^h &= \mathcal{S}_{1h}(t_m)P_hX_0 + \Delta t\sum_{j=0}^{m-1}(t_m-t_j)^{\alpha-1}\mathcal{S}_{2h}(t_m-t_j)P_hF(X_j^h) \\ &+ \sum_{j=0}^{m-1}\mathcal{S}_{1h}(t_m-t_j)P_hG(X_j^h)\,\Delta W_j + \sum_{j=0}^{m-1}\mathcal{S}_{1h}(t_m-t_j)P_h\Phi\,\Delta B_j^H \\ &= \mathcal{S}_{1h}(t_m)X_0^h + \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(t_m-t_j)^{\alpha-1}\mathcal{S}_{2h}(t_m-t_j)P_hF(X_j^h)ds \\ &+ \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-t_j)P_hG(X_j^h)dW(s) + \int_0^{t_m}\mathcal{S}_{1h}(t_m-\llcorner s\lrcorner)P_h\Phi dB^H(s), \end{aligned} \tag{85}$$

where the notation $\llcorner s\lrcorner$ is defined as in [24, (89)] by

$$\llcorner s\lrcorner := \left[\frac{s}{\Delta t}\right]\Delta t. \tag{86}$$

Subtracting (84) and (85), applying the triangle inequality, taking the square and using the estimate $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, it follows that

$$\begin{aligned} err_1^2 \ \leq\ & 3\left\| \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\Big((t_m-s)^{\alpha-1}\mathcal{S}_{2h}(t_m-s)P_hF(X^h(s)) - \right. \\ & \left. (t_m-t_j)^{\alpha-1}\mathcal{S}_{2h}(t_m-t_j)P_hF(X_j^h)\Big)\,ds\right\|^2_{L^2(\Omega;\mathcal{H})} \\ &+ 3\left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\Big(\mathcal{S}_{1h}(t_m-s)P_hG(X^h(s)) - \mathcal{S}_{1h}(t_m-t_j)P_hG(X_j^h)\Big)\,dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\ &+ \left\|\int_0^{t_m}\left(\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-\llcorner s\lrcorner)\right)P_h\Phi dB^H(s)\right\|^2_{L^2(\Omega;\mathcal{H})} \\ &=: 3(V_1^2+V_2^2+V_3^2). \end{aligned} \tag{87}$$

By adding and subtracting a term, we recast V_1 as follows

$$\begin{aligned} V_1 &= \sum_{j=0}^{m-1}\int_{t_k}^{t_{k+1}}\left[(t_m-s)^{\alpha-1}S_{2h}(t_m-s)-(t_m-t_j)^{\alpha-1}S_{2h}(t_m-t_j)\right]P_hF(X^h(s))ds \\ &+ \sum_{j=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-t_j)^{\alpha-1}S_{2h}(t_m-t_j)P_h\left[F(X^h(s))-F(X^h(t_k))\right]ds \\ &+ \sum_{j=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-t_j)^{\alpha-1}S_{2h}(t_m-t_j)P_h\left[F(X^h(t_k))-F(X_k^h)\right]ds \\ &=: V_{11}+V_{12}+V_{13}. \end{aligned} \tag{88}$$

Using triangle inequality, the discrete version of (29), Assumption 2 with the boundedness of P_h and the discrete version of (34) leads to

$$\begin{aligned} V_{11} &\leq \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left\|A^{\frac{\beta}{2}}\left[(t_m-s)^{\alpha-1}S_{2h}(t_m-s)-(t_m-t_j)^{\alpha-1}S_{2h}(t_m-t_j)\right]\right\|_{L(\mathcal{H})} \\ &\qquad \left\|A^{-\frac{\beta}{2}}P_hF(X^h(s))\right\|_{L^2(\Omega;\mathcal{H})}ds \\ &\leq C\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(s-t_j)^{1-(1-\frac{\beta}{2})\alpha}ds\right)\left(1+\mathbb{E}\left[\sup_{0\leq s\leq T}\left\|X^h(s)\right\|^2\right]\right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C\Delta t^{1-(1-\frac{\beta}{2})\alpha}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} ds\right)\\
&\leq C\Delta t^{1-(1-\frac{\beta}{2})\alpha} t_m \leq C\Delta t^{\frac{2-(2-\beta)\alpha}{2}}. \qquad (89)
\end{aligned}$$

Following closely [23, (80)-(82)], we have

$$V_{12}+V_{13} \leq C\Delta t^{\frac{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}{2}} + C\Delta t^{\alpha}\sum_{j=0}^{m-1}\left\|X^h(t_j)-X_j^h\right\|_{L^2(\Omega;\mathcal{H})}. \qquad (90)$$

Adding (89) and (90) we have

$$V_1 \leq C\Delta t^{\frac{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}{2}} + C\Delta t^{\alpha}\sum_{j=0}^{m-1}\left\|X^h(t_j)-X_j^h\right\|_{L^2(\Omega;\mathcal{H})}. \qquad (91)$$

Hence

$$V_1^2 \leq C\Delta t^{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)} + C\Delta t^{2\alpha-1}\sum_{j=0}^{m-1}\left\|X^h(t_j)-X_j^h\right\|^2_{L^2(\Omega;\mathcal{H})}. \qquad (92)$$

Adding and subtracting a term, using triangle inequality, we recast V_2 as follows

$$\begin{aligned}
V_2 &:= \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(\mathcal{S}_{1h}(t_m-s)P_hG(X^h(s))-\mathcal{S}_{1h}(t_m-t_j)P_hG(X_j^h)\right)dW(s)\right\|_{L^2(\Omega;\mathcal{H})}\\
&\leq \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left[\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-t_j)\right]P_hG(X^h(s))dW(s)\right\|_{L^2(\Omega;\mathcal{H})}\\
&+ \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-t_j)P_h\left[G(X^h(s))-G(X^h(t_j))\right]dW(s)\right\|_{L^2(\Omega;\mathcal{H})}\\
&+ \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-t_j)P_h(\Delta t)\left[G(X^h(t_j))-G(X_j^h)\right]dW(s)\right\|_{L^2(\Omega;\mathcal{H})}\\
&=: \sum_{i=1}^{3}V_{2i}. \qquad (93)
\end{aligned}$$

Using the martingale property of the stochastic integral, the Itô isometry (6), inserting an appropriate power of A_h, (81), the semidiscrete version of (27) and (30) with $\sigma=\eta=\delta=\frac{2H+\beta-1}{2}$ and $t_1=t_m-s$, $t_2=t_m-t_j$, Assumption 3 with $\tau=\frac{2H+\beta-1}{2}$ and the semidiscrete version of Theorem 2(more precisely (46)) yields

$$\begin{aligned}
V_{21}^2 &:= \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left[\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-t_j)\right]P_hG(X^h(s))dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\\
&= \sum_{j=0}^{m-1}\mathbb{E}\left[\int_{t_j}^{t_{j+1}}\left\|\left[\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-t_j)\right]A_h^{-\frac{2H+\beta-1}{2}}A_h^{\frac{2H+\beta-1}{2}}P_hG(X^h(s))\right\|^2_{L_2^0}ds\right]\\
&\leq \sum_{j=0}^{m-1}\mathbb{E}\left[\int_{t_j}^{t_{j+1}}\left\|A_h^{-\frac{2H+\beta-1}{2}}\left[\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-t_j)\right]\right\|^2_{L(\mathcal{H})}\left\|A_h^{\frac{2H+\beta-1}{2}}P_hG(X^h(s))\right\|^2_{L_2^0}ds\right]\\
&\leq C\sum_{j=0}^{m-1}\mathbb{E}\left[\int_{t_j}^{t_{j+1}}(s-t_j)^{\alpha(2H+\beta-1)}\left\|A^{\frac{2H+\beta-1}{2}}G(X^h(s))\right\|^2_{L_2^0}ds\right]\\
&\leq C\Delta t^{\alpha(2H+\beta-1)}\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(1+\mathbb{E}\left[\left\|A^{\frac{2H+\beta-1}{2}}X^h(s)\right\|^2\right]\right)ds
\end{aligned}$$

$$\leq C\Delta t^{\alpha(2H+\beta-1)}\int_0^{t_m}\left(1+\sup_{0\leq s\leq T}\left\|A_h^{\frac{2H+\beta-1}{2}}X^h(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\right)ds$$
$$\leq C\Delta t^{\alpha(2H+\beta-1)}. \tag{94}$$

To estimate the last two terms V_{22}^2 and V_{23}^2, using the martingale property of the stochastic integral, applying Itô isometry, boundedness of P_h and $\mathcal{S}_{1h}$, Assumption 3 and the semidiscrete version of Theorem 2 (more precisely (47)) holds

$$\begin{aligned}
V_{22}^2 &=: \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-t_j)P_h\left[G(X^h(s))-G(X^h(t_j))\right]dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\\
&= \sum_{j=0}^{m-1}\mathbb{E}\left[\int_{t_j}^{t_{j+1}}\left\|\mathcal{S}_{1h}(t_m-t_j)P_h\left[G(X^h(s))-G(X^h(t_j))\right]\right\|^2_{L_2^0}ds\right]\\
&\leq C\sum_{j=0}^{m-1}\left\|\mathcal{S}_{1h}(t_m-t_j)P_h\right\|^2_{L(\mathcal{H})}\int_{t_j}^{t_{j+1}}\mathbb{E}\left\|\left(G(X^h(s))-G(X^h(t_j))\right)\right\|^2_{L_2^0}ds\\
&\leq C\sum_{j=0}^{m-1}\int_{t_k}^{t_{k+1}}\left\|X^h(s)-X^h(t_k)\right\|^2_{L^2(\Omega;\mathcal{H})}ds\\
&\leq C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(s-t_j)^{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}ds\leq C\Delta t^{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}.
\end{aligned}\tag{95}$$

We also have

$$\begin{aligned}
V_{23}^2 &=: \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathcal{S}_{1h}(t_m-t_j)\left[P_hG(X^h(t_j))-P_hG(X_j^h)\right]dW(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\\
&= \sum_{j=0}^{m-1}\mathbb{E}\left[\int_{t_j}^{t_{j+1}}\left\|\mathcal{S}_{1h}(t_m-t_j)P_h\left(G(X^h(t_j))-G(X_j^h)\right)\right\|^2_{L_2^0}ds\right]\\
&\leq \sum_{j=0}^{m-1}\mathbb{E}\left[\int_{t_j}^{t_{j+1}}\left\|\mathcal{S}_{1h}(t_m-t_j)P_h\right\|^2_{L(\mathcal{H})}\left\|\left(G(X^h(t_j))-G(X_j^h)\right)\right\|^2_{L_2^0}ds\right]\\
&\leq C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\mathbb{E}\left\|X^h(t_j)-X_j^h\right\|^2ds\\
&\leq C\Delta t\sum_{j=0}^{m-1}\left\|X^h(t_j)-X_j^h\right\|^2_{L^2(\Omega;\mathcal{H})}.
\end{aligned}\tag{96}$$

Putting (94), (95) and (96) in (93) leads to

$$V_2^2\leq C\Delta t^{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}+C\Delta t\sum_{k=0}^{m-1}\left\|X^h(t_k)-X_k^h\right\|^2_{L^2(\Omega;\mathcal{H})}. \tag{97}$$

For the approximation of V_3^2, using (7), inserting an appropriate power of A_h, the semidiscrete version of (28) and (30) with $t_1=t_m-s$, $t_2=t_m-t_j$, $\kappa=\delta=\frac{1-\beta}{2}$ and $\varpi=H$, Lemma 5 with $\sigma=\frac{\beta-1}{2}$ and Assumption 4 yields

$$\begin{aligned}
V_3^2 &:= \left\|\int_0^{t_m}\left(\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-\llcorner s\lrcorner)\right)P_h\Phi dB^H(s)\right\|^2_{L^2(\Omega;\mathcal{H})}\\
&\leq C(H)\sum_{i=0}^{\infty}\left(\int_0^{t_m}\left\|\left(\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-\llcorner s\lrcorner)\right)A_h^{\frac{1-\beta}{2}}A_h^{\frac{\beta-1}{2}}P_h\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}
\end{aligned}$$

$$\leq C(H)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left\|A_h^{\frac{1-\beta}{2}}\left(\mathcal{S}_{1h}(t_m-s)-\mathcal{S}_{1h}(t_m-t_j)\right)\right\|_{L(\mathcal{H})}^{\frac{1}{H}}\right.$$
$$\left.\left\|A_h^{\frac{\beta-1}{2}}P_h\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}. \tag{98}$$

Furthermore, we also have

$$\begin{aligned}V_3^2 &\leq C\sum_{i=0}^{\infty}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(s-t_j)^{\frac{\alpha(2H+\beta-1)}{2H}}(t_m-s)^{-\alpha}\left\|A^{\frac{\beta-1}{2}}\Phi Q^{\frac{1}{2}}e_i\right\|^{\frac{1}{H}}ds\right)^{2H}\\ &\leq C\Delta t^{\alpha(2H+\beta-1)}\left(\int_0^{t_m}(t_m-s)^{-\alpha}ds\right)^{2H}\left\|A^{\frac{\beta-1}{2}}\Phi\right\|_{L_2^0}^2\\ &\leq C\Delta t^{\alpha(2H+\beta-1)}.\end{aligned} \tag{99}$$

Substituting (92), (97) and (98) in (87) yields

$$\begin{aligned}&\left\|X^h(t_m)-X_m^h\right\|_{L(\mathcal{H})}^2\\ &\leq C\Delta t^{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}+C\Delta t^{2\alpha-1}\sum_{k=0}^{m-1}\left\|X^h(t_k)-X_k^h\right\|_{L(\mathcal{H})}^2.\end{aligned} \tag{100}$$

Applying the discrete Gronwall's inequality to (100) and taking the squared-root leads

$$\left\|X^h(t_m)-X_m^h\right\|_{L(\mathcal{H})}\leq C\Delta t^{\frac{\min(\alpha(2H+\beta-1),2-(2-\beta)\alpha)}{2}}. \tag{101}$$

Adding (68) and (101) completes the proof. □

6. Numerical simulations

As we have mentioned in the introduction, the price to pay to simulate our scheme is the computation of Mittag–Leffler (ML) matrix functions, which is more challenging than the standard exponential matrix functions [22,29]. We have used the Matlab function built in [33,34] to compute the ML matrix functions. For our simulations, we consider the stochastic advection diffusion reaction SPDE (1)–(12) with constant diagonal diffusion tensor $\mathbf{D} = 10^{-2}\mathbf{I}_2 = (D_{i,j})$ in (12), and mixed Neumann–Dirichlet boundary conditions on $\Lambda = [0, L_1] \times [0, L_2]$. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x, y) : x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. The random solution X in (12) can represent the transport of the pollutants in porous media.

We take $Q = Q_1$ in (2)–(3) such that the eigenfunctions $\{e_{i,j}\} = \{e_i^{(1)} \otimes e_j^{(2)}\}_{i,j\geq 0}$ of the covariance operators $Q = Q_1$ are the same as for Laplace operator $-\Delta$ with homogeneous boundary condition given by

$$e_0^{(l)}(x)=\sqrt{\frac{1}{L_l}},\qquad e_i^{(l)}(x)=\sqrt{\frac{2}{L_l}}\cos\left(\frac{i\pi}{L_l}x\right),\ i\in\mathbb{N}$$

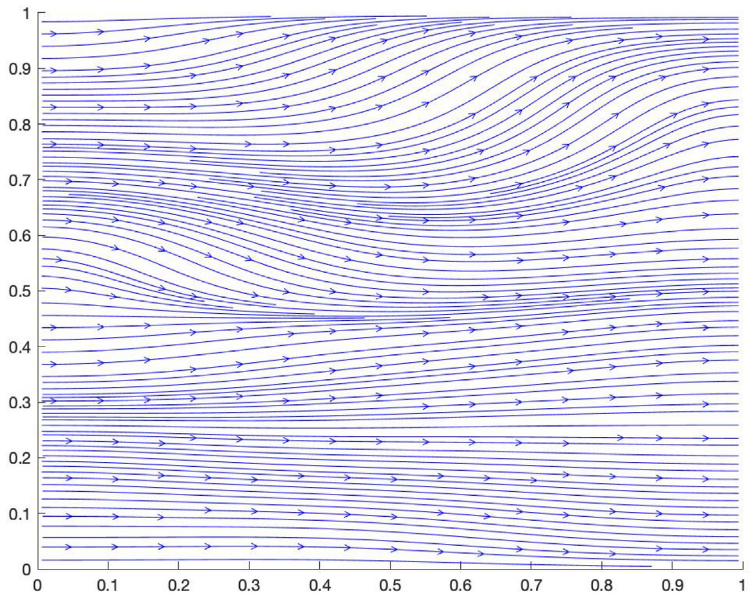
where $l \in \{1, 2\}$, $x \in \Lambda$. Here we take $L_1 = L_2 = 1$. In the noises representations (2)–(3), we have used

$$q_{i,j}^1 = q_{i,j} = \left(i^2+j^2\right)^{-(\beta+\delta)},\ \beta>0, \tag{102}$$

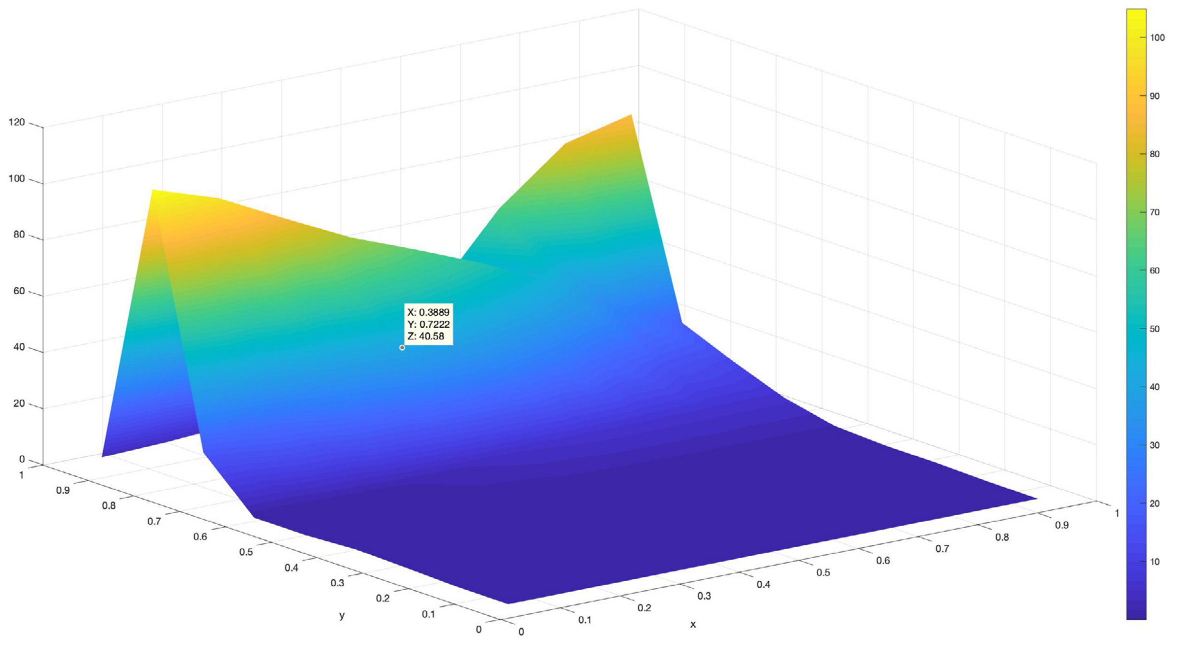
for some small $\delta > 0$. We have used $g(x, X) = X$ in (12) and $\phi(x) = 1$ and $\beta = 1$. The grid has been chosen such that the size of the matrix A_h is 100. The final time is $T = 0.1$ and the reference samples solution or "exact samples solution" are numerical samples solution with the smaller time step $\Delta t = 0.003$. In our simulations, we have used $\delta = 0.001$, we have taken the function f used in (12) to be $f(x, X) = \frac{X}{1+|X|}$[3] for all $(x, X) \in \Lambda \times \mathbb{R}$ and the initial solution $X_0 = 0.1$. Assumptions 1–4 are obviously satisfied for Nemytskii operators F and G defined by 1, the covariance operator Q_1 and the initial solution $X_0 = 0.1$ (see for example [22,24,29]). The velocity field $\mathbf{q} = (q_i)$ in (12) is obtained exactly as in [22,24]. The streamline of the velocity field $\mathbf{q}$ is given in Fig. 1(a) and the corresponding norm of the velocity field is given in Fig. 1(b). The mean of 50 samples of the solution is given in Fig. 2(a), as you can observe this figure is in agreement with Fig. 1(b) as area with high norms of the velocity field has low concentrations X, since the concentrations has been transported by the high speed velocity.

In Fig. 2(b), the errors graph for our fractional exponential scheme with three parameters (β, α, H) is shown. The orders of convergence is 0.5187 for $(\beta, \alpha, H) = (1, 0.51, 0.99)$, 0.5102 for $(\beta, \alpha, H) = (1, 0.51, 0.7)$ and 0.5130 for

[3] It is easy to prove that f is global Lipschitz respect to X, indeed $|f'(X)| \leq 1$

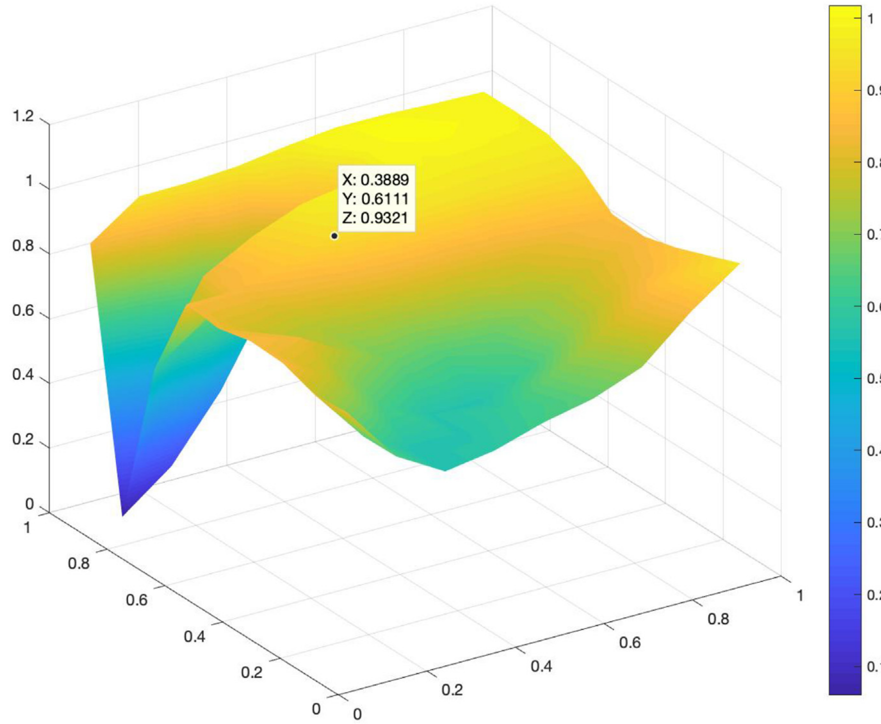


(a)

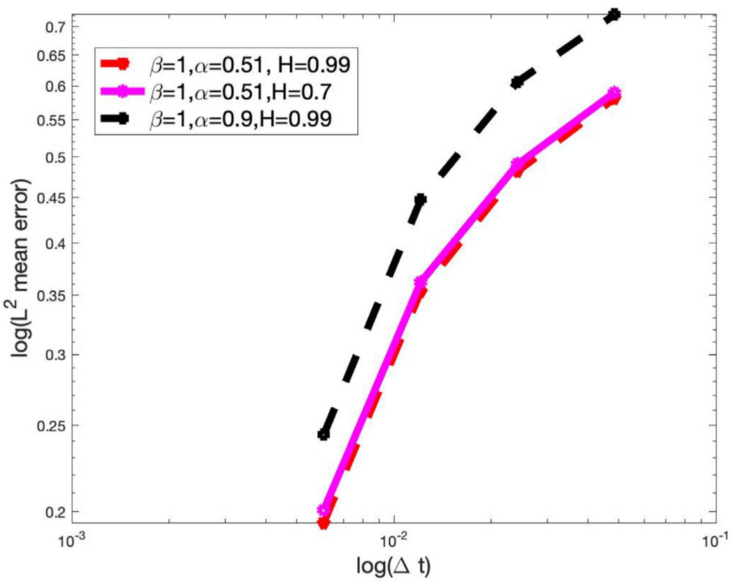


(b)

Fig. 1. The streamline of the velocity field **q** in (a) along with the corresponding norm of the velocity field in (b).



(a)



(b)

Fig. 2. Convergence in the root mean square L^2 norm at $T = 0.1$ as a function of Δt for our fractional exponential scheme in (b). We have used here 50 realizations and the mean of those 50 samples solution is given in (a).

Table 1
L^2 mean error for $(\beta, \alpha, H) = (1, 0.51, 0.99)$. The final time is $T = 0.1$ and the corresponding strong convergence order after fitting data in log scale is 0.5187 as we can observe in Fig. 2(b).

Parameters (β, α, H)	$(\beta, \alpha, H) = (1, 0.51, 0.99)$			
Time step size (Δt)	0.0061	0.0121	0.0242	0.0485
L^2 mean error	0.1945	0.3534	0.4822	0.5814

Table 2
L^2 mean error for $(\beta, \alpha, H) = (1, 0.51, 0.7)$. The final time is $T = 0.1$ and the corresponding strong convergence order after fitting data in log scale is 0.5102 as we can observe in Fig. 2(b).

Parameters (β, α, H)	$(\beta, \alpha, H) = (1, 0.51, 0.7)$			
Time step size (Δt)	0.0061	0.0121	0.0242	0.0485
L^2 mean error	0.2010	0.3618	0.4906	0.5904

Table 3
L^2 mean error for $(\beta, \alpha, H) = (1, 0.9, 0.99)$. The final time is $T = 0.1$ and the corresponding strong convergence order after fitting data in log scale is 0.5130 as we can observe in Fig. 2(b).

Parameters (β, α, H)	$(\beta, \alpha, H) = (1, 0.9, 0.99)$			
Time step size (Δt)	0.0061	0.0121	0.0242	0.0485
L^2 mean error	0.2442	0.4476	0.6071	0.7216

$(\beta, \alpha, H) = (1, 0.9, 0.99)$. As you can observe the error is lower with optimal order of convergence when α is near 0.5, this is in perfect agreement with our error estimate in Theorem 3 where the optimal order of convergence is predicted to be $\frac{1}{2} - \epsilon$, ϵ can be small enough (see Tables 1–3).

CRediT authorship contribution statement

Aurelien Junior Noupelah: Conceptualization, Methodology, Writing – review & editing, Investigation, Writing – original draft. **Antoine Tambue:** Conceptualization, Methodology, Software, Writing – review, Software, Supervision, Validation, Writing – original draft. **Jean Louis Woukeng:** Conceptualization, Methodology, Writing – review, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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