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Closed Range Integral Operators on Fock Spaces

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Abstract

We study the closed range problem for generalized Volterra-type integral operators on Fock spaces. We first answer the problem using the notions of sampling sets, reverse Fock–Carleson measures, Berezin type integral transforms, and essential boundedness from below of some functions of the symbols of the operators. The answer is further analyzed to show that the operators have closed ranges only when the derivative of the composition symbol belongs to the unit circle. It turns out that there exists no nontrivial closed range integral operator acting between two different Fock spaces. The main results equivalently describe when the operators are bounded below. Explicit expressions for the range of the operators are also provided, namely that the closed ranges contain only elements of the space which vanish at the origin. We further describe conditions under which the operators admit order bounded structures.

Keywords Fock space · Closed range · Reverse Calreson measures · Volterra-type integral · Order bounded · Bounded below · Sampling set · Essentially bounded

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1 Introduction and Preliminary Results

We denote by $\mathbb{C}$ the complex plane and $\mathcal{H}(\mathbb{C})$ the set of entire functions on $\mathbb{C}$. For $1 \le p \le \infty$, consider the Fock spaces $\mathcal{F}_p$ consisting of all f in $\mathcal{H}(\mathbb{C})$ for which

$$\|f\|_p := \begin{cases} \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z)\right)^{\frac{1}{p}} < \infty, & p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{1}{2}|z|^2} < \infty, & p = \infty, \end{cases}$$

where A denotes the Lebesgue area measure on $\mathbb{C}$.

For f and g in $\mathcal{H}(\mathbb{C})$, we define the Volterra-type integral operator V_g and its companion J_g by

$$V_g f(z) := \int_0^z f(w)g'(w)dw \quad \text{and} \quad J_g f := V_f g. \tag{1.1}$$

The two operators are related to the pointwise multiplication operator, $M_g f = gf$, by $f(0)g(0) + V_g f + J_g f = M_g f$. The boundedness and compactness properties of the operators on Fock spaces were characterized in [5, 12, 14]. It was shown that V_g is bounded on $\mathcal{F}_p$ if and only if g is a polynomial of at most degree two while compactness is characterized by degree of g being at most one. Similarly, J_g or M_g is bounded if and only if g is a constant function, and compact only when g identically vanishes. Inspired by all these, the question whether generalizing the operators to $V_{(g,\psi)}$ and $J_{(g,\psi)}$, where

$$V_{(g,\psi)} f(z) := \int_0^z f(\psi(w))g'(w)dw \text{ and } J_{(g,\psi)} f(z) := \int_0^z f'(\psi(w))g(w)dw, \tag{1.2}$$

improve their boundedness and compactness properties were investigated in [12, 13, 15]. Here, we remind that the initial motivation to study $V_{(g,\psi)}$ and $J_{(g,\psi)}$ came from their applications in the study of isometry; see for example [6]. Setting,

$$\begin{aligned} M_{(g,\psi)}(z) &= |g'(z)|(1+|z|)^{-1} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \quad \text{and} \\ \widetilde{M}_{(g,\psi)}(z) &= |g(z)|(1+|\psi(z)|)(1+|z|)^{-1} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \end{aligned} \tag{1.3}$$

for all z in $\mathbb{C}$, the following result was proved in [15].

Theorem 1.1 *Let $1 \le p, q \le \infty$ and $g, \psi \in \mathcal{H}(\mathbb{C})$.*

(i) *If $p \le q$, then*

(a) *$V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if* $\sup_{z\in\mathbb{C}} M_{(g,\psi)}(z) =: M < \infty$, *and compact if and only if $M_{(g,\psi)}(z) \to 0$ as $|z| \to \infty$.*

(b) *$J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if* $\sup_{z\in\mathbb{C}} \widetilde{M}_{(g,\psi)}(z) =: \widetilde{M} < \infty$, *and compact if and only if $\widetilde{M}_{(g,\psi)}(z) \to 0$ as $|z| \to \infty$.*

(ii) *If $p > q$, then $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded (compact) if and only if $M_{(g,\psi)} \in L^r(\mathbb{C}, dA)$. Similarly, $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded (compact) if and only if $\widetilde{M}_{(g,\psi)} \in L^r(\mathbb{C}, dA)$, where $r = \frac{pq}{p-q}$ for $p < \infty$ and $r = q$ for $p = \infty$.*

The result shows the compactness and boundedness structures of V_g and J_g are indeed significantly improved under such generalizations. For example, by setting $\psi(z) = z+b$ and $g(z) = e^{-\overline{b}z}$, we observe that the boundedness statements in (i) and (ii) hold for any b in $\mathbb{C}$ while both V_g and J_g fail to be bounded. Similarly, for $\psi(z) = \frac{1}{2}z + b$ and $g(z) = e^{-\frac{1}{2}\overline{b}z}$, the compactness statements in (i) and (ii) hold for all b in $\mathbb{C}$ while neither V_g nor J_g is compact.

The following two useful lemmas follow from Theorem 1.1. The lemmas provide some explicit expressions for the function g or its derivative.

Lemma 1.2 *Let $1 \le p \le \infty$ and $g, \psi \in \mathcal{H}(\mathbb{C})$. Then*

(i) *if $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded, then $\psi(z) = az + b$ with $0 \le |a| \le 1$. If $|a| = 1$, then*

$$g(z) = \begin{cases} K_{-\overline{a}b}(z)\big(a_1 z + a_2\big) + a_3, & b \neq 0 \\ b_1 z^2 + b_2 z + b_3, & b = 0, \end{cases} \tag{1.4}$$

for some $a_i, b_i \in \mathbb{C}$, $i = 1, 2, 3$ and all z in $\mathbb{C}$

(ii) *if $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded, then $\psi(z) = az+b$ with $0 \le |a| \le 1$. If $|a| = 1$, then*

$$g = g(0)K_{-\overline{a}b}. \tag{1.5}$$

Similarly, for $|a| < 1$, we get the following representations.

Lemma 1.3 *Let $1 \le p \le \infty$ and $g, \psi \in \mathcal{H}(\mathbb{C})$.*

(i) *Let $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$ and hence $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$, and $0 \le |a| < 1$. If g' is non-vanishing, then $V_{(g,\psi)}$ is*

 (a) *compact on $\mathcal{F}_p$ if and only if*

$$g'(z) = e^{a_0 + a_1 z + a_2 z^2} \tag{1.6}$$

 for some constants a_0, a_1, a_2 in $\mathbb{C}$ such that $|a_2| < \frac{1-|a|^2}{2}$.

 (b) *not compact on $\mathcal{F}_p$ if and only if g' has the form in (1.6) with $|a_2| = \frac{1-|a|^2}{2}$ and either $a_1 + a\overline{b} = 0$ or $a_1 + a\overline{b} \neq 0$ and*

$$a_2 = -\frac{(1-|a|^2)(a_1 + a\overline{b})^2}{2|a_1 + a\overline{b}|^2}.$$

(ii) *Let $J_{(g,\psi)}$ be bounded on $\mathcal{F}_p$ and hence $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $0 \le |a| < 1$. If g is non-vanishing, then $J_{(g,\psi)}$ is*

(a) *compact on $\mathcal{F}_p$ if and only if*

$$g(z) = e^{b_0+b_1z+b_2z^2} \tag{1.7}$$

for some constants $b_0, b_1, b_2 \in \mathbb{C}$ such that $|b_2| < \frac{1-|a|^2}{2}$.

(b) *not compact on $\mathcal{F}_p$ if and only if g has the form in (1.7) with $|b_2| = \frac{1-|a|^2}{2}$ and either $b_1 + a\overline{b} = 0$ or $b_1 + a\overline{b} \neq 0$ and*

$$b_2 = -\frac{(1-|a|^2)(a_1 + a\overline{b})^2}{2|b_1 + a\overline{b}|^2}.$$

The proof of both Lemmas 1.2 and 1.3 will be given later in Sect. 3.

1.1 The Closed Range Problem

The closed range problem has been one of the basic problems in operator theory which finds lots of connections in various parts of mathematics, especially, in the theory of Fredholm operators, generalized inverses and dynamical sampling structures [10, 11]. The problem has been studied by many authors on various spaces of analytic functions; see for example [7, 16, 19] and the reference therein. Recently, the second author studied the problem for V_g on Fock spaces, and proved that a bounded V_g has a non-trivial closed range if and only if g has a non-zero degree two term in its polynomial expansion [11]. Clearly, J_g has closed range if and only if g is a constant. Similarly, in [10] it was proved that a bounded composition operator $C_\psi : f \to f \circ \psi$ has a non-trivial closed range on the spaces if and only if ψ is a first degree polynomial. One of the main goals of this work is to consider the generalized operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$, and answer the question whether such generalizations improve the closed range structures of V_g, J_g, and C_ψ as it already does for boundedness and compactness. We plan to answer the question first using various notions including sampling sets, reproducing kernel thesis, reverse Fock–Carleson measures and essential boundedness from below of the functions $M_{(g,\psi)}$ and $\widetilde{M}_{(g,\psi)}$. Then, we analyze further such notions and describe the integral operators with closed ranges using simple conditions in Theorem 2.6.

The rest of the paper is organized as follows. In the next section we present the main results of the paper and answer the closed range problem for the integral operators in Theorems 2.3, 2.4, 2.5, and 2.6. As a consequence, we characterize conditions under which the Volterra-type integral operator V_g and composition operator C_ψ admit closed range structures in Corollaries 2.7 and 2.8 respectively. Next, we consider the question of when the integral operators have order bounded structures and provide a complete answer in Theorem 2.9. This result is applied in particular for the operators V_g, J_g and C_ψ and obtained interesting results in Corollaries 2.10 and 2.11.
Sect. 3 deals with the proofs of all the results obtained while the last section contains some further discussions on the main results.

We conclude this section with a word on nations that will be used in the rest of the manuscript. The notion $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2 Main Results

We may first note that if $\psi = b$ is a constant, then the operators have clearly closed ranges given by $\mathcal{R}(V_{(g,\psi)}) = \{f(b)(g - g(0)) : f \in \mathcal{F}_p\}$ and

$$\mathcal{R}(J_{(g,\psi)}) = \Big\{f'(b)G : f \in \mathcal{F}_p,\ G(z) = \int_0^z g(w)dw\Big\}.$$

Similarly, if $g = c$ is a constant, then $V_{(g,\psi)}$ reduces to the zero operator which obviously is closed. The next result shows $J_{(g,\psi)}$ has also closed range as well.

Proposition 2.1 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ and $1 \le p, q \le \infty$. Let $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ be bounded and hence $\psi(z) = az + b,\ 0 \le |a| \le 1$. If $g = c$ is a constant, then $J_{(g,\psi)}$ has closed range given by*

$$\mathcal{R}(J_{(g,\psi)}) = \{cf \circ \psi - cf(b) : f \in \mathcal{F}_p\}. \tag{2.1}$$

We may assume now that both the symbols ψ and g are nonconstant, and provide the next key necessary condition.

Theorem 2.2 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ be nonconstant and $1 \le p, q \le \infty$. Then a bounded*

(i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *has closed range only if $p = q$.*
(ii) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *has closed range only if $p = q$.*

The result makes it clear that the action of the integral operators between two different Fock spaces results no non-trivial closed range property. Therefore, we will restrict ourselves to study their actions only on the space $\mathcal{F}_p$.

2.1 Sampling Sets, and Closed Range $V_{(g,\psi)}$ and $J_{(g,\psi)}$

Ghatage et al. [7] introduced the notion of sampling set for a space to study bounded below composition operators on Block spaces. Since then, the notion has been used to investigate the closed range problem in various Banach spaces. Now, we generalize the notion and for $1 \le p \le \infty$, a subset S of $\mathbb{C}$ is a (p, n) sampling set (dominating set) for $\mathcal{F}_p$ if there exists an n in $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$ and a positive constant δ_n such that for all f in $\mathcal{F}_p$

$$\delta_n \|f\|_p \le \begin{cases} \sup_{z\in S} \frac{|f^{(n)}(z)|}{(1+|z|)^n} e^{-\frac{|z|^2}{2}}, & p = \infty \\ \left(\int_S \frac{|f^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z)\right)^{\frac{1}{p}}, & p < \infty, \end{cases}$$

where $f^{(n)}$ denotes the n^{th} order derivative of f and $f^0 = f$. Here, an interesting question is to ask for examples of sets which satisfy the sampling set condition for

$\mathcal{F}_p$. Due to the norm estimates in (3.2), the whole complex plane minus any set of measure zero is a prototype example of a (p, n) sampling set for $n = 0$ or $n = 1$. In fact, applying the n^{th} order derivative Littelwood–Paley type estimates in [8, 17], it is easy to observe that the prototype example works for all n in $\mathbb{N}_0$.

For each $\epsilon > 0$, we define two associated sets

$$\Omega^{\epsilon}_{(g,\psi)} := \{z \in \mathbb{C} : M_{(g,\psi)}(z) \geq \epsilon\} \text{ and } G^{\epsilon}_{(g,\psi)} := \psi\big(\Omega^{\epsilon}_{(g,\psi)}\big),$$

where $M_{(g,\psi)}$ is the function as in (1.3). With this, we may now state the next main result which provides a number of equivalent conditions for $V_{(g,\psi)}$ to have closed range on Fock spaces.

Theorem 2.3 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ be nonconstant, $1 \leq p \leq \infty$, and $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. Then the following statements are equivalent.*

(i) *$V_{(g,\psi)}$ has closed range on $\mathcal{F}_p$;*
(ii) *There exists $\epsilon > 0$ such that $G^{\epsilon}_{(g,\psi)}$ is a $(p, 0)$ sampling set for $\mathcal{F}_p$;*
(iii) *For all z in $\mathbb{C}$, there exists a positive constant C such that*

$$\|V_{(g,\psi)}K_z\|_p \geq C\|K_z\|_p, \tag{2.2}$$

where $K_z(w) = e^{\bar{z}w}$ is the reproducing kernel function in $\mathcal{F}_p$;
(iv) *There exist positive numbers ϵ, r, and σ such that*

$$A(G^{\epsilon}_{(g,\psi)} \cap D(z, r)) \geq \sigma r^2$$

for all z in $\mathbb{C}$ and $D(z, r)$ is a disc of radius r and center z in $\mathbb{C}$;
(v) *The function $M_{(g,\psi)}$ is essentially bounded away from zero on $\mathbb{C}$. The rang of $V_{(g,\psi)}$ is given by*

$$\mathcal{R}(V_{(g,\psi)}) = \big\{f \in \mathcal{F}_p : f(0) = 0\big\}. \tag{2.3}$$

Let us now give some examples that illustrate the improvement of the closedness property for the integral operator V_g. Set $\psi(z) = az+b$, $|a| = 1$ and $g(z) = zK_{-\overline{a}b}(z)$. Then by Theorem 2.3, the operator $V_{(g,\psi)}$ has closed range for all b in $\mathbb{C}$ while V_g fails to have.

Next, we consider the integral operator $J_{(g,\psi)}$. For $\epsilon > 0$, we may set again

$$\Lambda^{\epsilon}_{(g,\psi)} := \{z \in \mathbb{C} : \widetilde{M}_{(g,\psi)}(z) \geq \epsilon\} \text{ and } \Gamma^{\epsilon}_{(g,\psi)} := \psi\big(\Lambda^{\epsilon}_{(g,\psi)}\big),$$

where the function $\widetilde{M}_{(g,\psi)}$ is as in (1.3), and state the next main result.

Theorem 2.4 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ be nonconstant, $1 \leq p \leq \infty$, and $J_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. Then the following statements are equivalent.*

(i) *$J_{(g,\psi)}$ has closed range on $\mathcal{F}_p$;*
(ii) *There exists $\epsilon > 0$ such that $\Gamma^{\epsilon}_{(g,\psi)}$ is a $(p, 1)$ sampling set for $\mathcal{F}_p$;*

(iii) *For all z in $\mathbb{C}$, there exists a positive constant C such that*

$$\|J_{(g,\psi)}K_z\|_p \geq C\|K_z\|_p;$$

(iv) *There exist positive numbers ϵ, r, and σ such that*

$$A(\Gamma^{\epsilon}_{(g,\psi)} \cap D(z,r)) \geq \sigma r^2$$

for all z in $\mathbb{C}$;

(v) *The function $\widetilde{M}_{(g,\psi)}$ is essentially bounded away from zero on $\mathbb{C}$. The range of $J_{(g,\psi)}$ is given by*

$$\mathcal{R}(J_{(g,\psi)}) = \big\{f \in \mathcal{F}_p : f(0) = 0\big\}. \tag{2.4}$$

The operator J_g is bounded if and only if g is a constant [12]. A nonzero constant g obviously generates a closed range operator J_g but the property gets improved significantly under $J_{(g,\psi)}$. For example, set $\psi(z) = az + b$, $|a| = 1$ and $g(z) = K_{-\overline{a}b}(z)$. Then $J_{(g,\psi)}$ has closed range for all b in $\mathbb{C}$ while J_g is not even bounded.

2.2 Reverse Fock–Carleson Measures, and Closed Range $V_{(g,\psi)}$ and $J_{(g,\psi)}$

The notion of Carleson measure has been well studied and used in various contexts since its introduction by Carleson [2] as a tool to solve the corona problem. In this section, we generalize the notion and for each $1 \leq p < \infty$ and $n \in \mathbb{N}_0$, define a (p,n) Fock–Carleson measure on Fock spaces. We call a positive Borel measure μ is a (p,n) Fock–Carleson measure for $\mathcal{F}_p$ if there exists a positive constant C_n such that

$$\int_{\mathbb{C}} |f^{(n)}(z)|^p d\mu(z) \leq C_n\|f\|_p^p \tag{2.5}$$

for all f in $\mathcal{F}_p$, where as before $f^{(n)}$ is the n^{th} order derivative of f and $f^{(0)} = f$. The measure μ is called a (p,n) reverse Fock–Carleson measure if the inequality in (2.5) is reversed. When $n = 0$, the definition reduces to the classical version.

We may now state the following result, which gives two more equivalent conditions to the lists in Theorems 2.3 and 2.4 whenever $p < \infty$.

Theorem 2.5 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ be nonconstant and $1 \leq p < \infty$.*

(i) *Let $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. Then the following statements are equivalent.*

  (a) *$V_{(g,\psi)}$ has closed range on $\mathcal{F}_p$;*
  (b) *There exists $\epsilon > 0$ such that $\mu^{\epsilon}_{(g,\psi)}$ is a $(p,0)$ reverse Fock–Carleson measure, where*

$$d\mu^{\epsilon}_{(g,\psi)}(z) = \chi_{G^{\epsilon}_{(g,\psi)}}(z)e^{-\frac{p}{2}|z|^2}dA(z);$$

(c) $\mu_{(g,\psi,p)}$ *is a* $(p,0)$ *reverse Fock–Carleson measure, where*

$$d\mu_{(g,\psi,p)}(z) = |g'(\psi^{-1}(z))|^p(1+|\psi^{-1}(z)|)^{-p}e^{-\frac{p|\psi^{-1}(z)|^2}{2}}dA(z).$$

(ii) *Let* $J_{(g,\psi)}$ *be bounded on* $\mathcal{F}_p$*. Then the following statements are equivalent.*

(a) $J_{(g,\psi)}$ *has closed range on* $\mathcal{F}_p$*;*
(b) *There exists* $\epsilon > 0$ *such that* $\theta^{\epsilon}_{(g,\psi)}$ *is a* $(p,1)$ *reverse Fock–Carleson measure, where*

$$d\theta^{\epsilon}_{(g,\psi)}(z) = \chi_{\Gamma^{\epsilon}_{(g,\psi)}}(z)(1+|z|)^{-p}e^{-\frac{p}{2}|z|^2}dA(z);$$

(c) $\theta_{(g,\psi,p)}$ *is a* $(p,1)$ *reverse Fock–Carleson measure, where*

$$d\theta_{(g,\psi,p)}(z) = |g(\psi^{-1}(z))|^p(1+|\psi^{-1}(z)|)^{-p}e^{-\frac{p|\psi^{-1}(z)|^2}{2}}dA(z).$$

2.3 Some Consequences of the Results above

Now we consider some concrete cases and simplify some of the main results obtained. More specifically, an application of Lemmas 1.2, 1.3, Theorems 2.3, and 2.4 provides the following interesting and easy to apply condition.

Theorem 2.6 *Let* $1 \leq p \leq \infty$ *and* $g, \psi \in \mathcal{H}(\mathbb{C})$ *be nonconstant.*

(i) *If* $V_{(g,\psi)}$ *is bounded on* $\mathcal{F}_p$ *and hence* $\psi(z) = az + b,\ |a| \leq 1$*, then its range is closed if and only if* $|a| = 1$ *and* $g'(z) = (cz+d)K_{-\overline{a}b}(z)$ *for some* $c, d \in \mathbb{C}$ *and* $c \neq 0$*.*
(ii) *If* $J_{(g,\psi)}$ *is bounded on* $\mathcal{F}_p$ *and hence* $\psi(z) = az + b,\ |a| \leq 1$*, then its range is closed if and only if* $|a| = 1$*.*

Note that while the closed range properties for both $V_{(g,\psi)}$ and $J_{(g,\psi)}$ require that $|a| = 1$, the operator $V_{(g,\psi)}$ requires in addition a non zero c in the explicit expression of the function g'.

An immediate consequence of Theorems 2.3 and 2.6 is the following.

Corollary 2.7 *Let* $1 \leq p \leq \infty$ *and* $g \in \mathcal{H}(\mathbb{C})$ *be nonconstant. Let* V_g *be bounded on* $\mathcal{F}_p$*, and hence* $g(z) = az^2 + bz + c$ *for some* $a, b, c \in \mathbb{C}$*. Then, the following statements are equivalent.*

(i) V_g *has a closed range on* $\mathcal{F}_p$*;*
(ii) $a \neq 0$*;*
(iii) $\mathbb{C}$ *is a* $(p,0)$ *sampling set for* $\mathcal{F}_p$*.*

The composition operator is related to the integral operators in (1.2) by $C_\psi f = J_{(1,\psi)}f + f(\psi(0))$. Then an application of Theorems 2.4 and 2.6 gives the following equivalent conditions.

Corollary 2.8 *Let* $\psi \in \mathcal{H}(\mathbb{C})$ *be nonconstant and* $1 \leq p \leq \infty$*. Let* C_ψ *be bounded on* $\mathcal{F}_p$ *and hence* $\psi(z) = az + b$ *where* $|a| \leq 1$ *and* $b = 0$ *when* $|a| = 1$*. Then the following statements are equivalent.*

(i) C_ψ *has closed range on* $\mathcal{F}_p$*;*
(ii) $|a| = 1$*;*
(iii) $\mathbb{C}$ *is a* $(p, 1)$ *sampling set for* $\mathcal{F}_p$*;*
(iv) $\|C_\psi K_z\|_p = \|K_z\|_p$ *for all* z *in* $\mathbb{C}$*;*
(v) C_ψ *is surjective on* $\mathcal{F}_p$*.*

2.4 Order Bounded Integral Operators

Another important question is as to when the generalized operators admit ordered bounded structures in their actions between Fock spaces. Recall that an operator $T : \mathcal{F}_p \to \mathcal{F}_q$ is order bounded if there exists a positive function h in $L^q\big(\mathbb{C}, dA_q\big)$ such that for all f in $\mathcal{F}_p$ with $\|f\|_p \leq 1$,

$$|T(f(z))| \leq h(z)$$

almost everywhere with respect to the measure A, where $dA_q(z) = e^{-\frac{q}{2}|z|^2} dA(z)$ for $q < \infty$ and for $q = \infty$ we take the supremum of the function against the weight $e^{-|z|^2/2}$. For the integral operators, we prove that such property holds if and only if the respective functions in (1.3) are in L^q. We state the result below.

Theorem 2.9 *Let* $1 \leq p, q \leq \infty$ *and* $g, \psi \in \mathcal{H}(\mathbb{C})$*. Then the operator*

(i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded if and only if* $M_{(g,\psi)} \in L^q(\mathbb{C}, dA)$.
(ii) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded if and only if* $\widetilde{M}_{(g,\psi)} \in L^q(\mathbb{C}, dA)$.

In particular for the operators V_g and J_g, we get the following more simplified and interesting versions.

Corollary 2.10 *Let* $1 \leq p, q \leq \infty$ *and* $g \in \mathcal{H}(\mathbb{C})$*. Then*

(i) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded if and only if* g *is a complex polynomial of at most degree one and* $q > 2$.
(ii) $J_g : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded if and only if* g *is identically zero.*

The generalized operators in (1.2) improved again the order bounded structures of V_g and J_g. Indeed, setting $\psi(z) = \frac{1}{2}z + b$ and $g(z) = e^{-\overline{b}z}$, we observe that both $V_{(g,\psi)}$ and $J_{(g,\psi)}$ are order bounded while neither V_g nor J_g is.

From $C_\psi f = J_{(1,\psi)} f + f(\psi(0))$ and Theorem 2.9, we record the following characterization of composition operators on Fock spaces.

Corollary 2.11 *Let* $\psi \in \mathcal{H}(\mathbb{C})$ *and* $1 \leq p, q \leq \infty$*. Then* $C_\psi : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded if and only if* $\psi(z) = az + b$ *for some* $a, b \in \mathbb{C}$ *and* $|a| < 1$.

3 Proofs of the Results

In this section we present the proofs of the results. We may begin with Lemmas 1.2 and 1.3.

Proof of Lemma 1.2 The proof of the linear form for ψ under a bounded $V_{(g,\psi)}$ or $J_{(g,\psi)}$ is given in [15]. Thus, we proceed to show the representations in (1.4) and (1.5) whenever $|a| = 1$.
(i) Since the operator is bounded, using the notations in Theorem 1.1,

$$\begin{aligned} M_{(g,\psi)}(z) &= |g'(z)|(1+|z|)^{-1} e^{\frac{1}{2}(|az+b|^2-|z|^2)} \\ &= e^{\frac{|b|^2}{2}} |g'(z) K_{\overline{a}b}(z)|(1+|z|)^{-1} \leq M \end{aligned}$$

for all z in $\mathbb{C}$ and hence

$$|g'(z) K_{\overline{a}b}(z)| \leq M e^{-\frac{|b|^2}{2}}(1+|z|). \tag{3.1}$$

For simplicity, let $h = g' K_{\overline{a}b}$. Since h is an entire function, we may consider its Laurent series expansion and write

$$h(z) = \sum_{j=0}^{\infty} \alpha_j z^j,$$

where the $\alpha_j' s$ are constants in $\mathbb{C}$. Setting $z = re^{i\theta}$, an integration gives

$$\int_{-\pi}^{\pi} |h(re^{i\theta})|^2 \frac{d\theta}{2\pi} = |\alpha_0|^2 + \sum_{j\geq 1} |\alpha_j|^2 r^{2j}$$

from which and (3.1)

$$|\alpha_0|^2 + \sum_{j=1}^{\infty} |\alpha_j|^2 r^{2j} \leq M^2 e^{-|b|^2}(1+r)^2.$$

This holds for sufficiently large r only when $\alpha_j = 0$ for all $j \geq 2$. Therefore,

$$g'(z) = \frac{h(z)}{K_{\overline{a}b}(z)} = (\alpha_0 + \alpha_1 z) e^{-a\overline{b}(z)} = (\alpha_0 + \alpha_1 z) K_{-\overline{a}b}(z).$$

We arrive at the form of g after integration by parts.

(ii) For this part, we use Theorem 1.1 again and argue as in the preceding one,

$$\begin{aligned} \widetilde{M}_{(g,\psi)}(z) &= |g(z)|(1+|\psi(z)|)(1+|z|)^{-1} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \\ &= |g(z) K_{\overline{a}b}(z)|(1+|az+b|)(1+|z|)^{-1} \leq \widetilde{M} \end{aligned}$$

for all z in $\mathbb{C}$. This shows the function $g K_{\overline{a}b}$ is bounded and hence a constant C by Liouville's theorem. Therefore,

$$g = C K_{-\overline{a}b} = g(0) K_{-\overline{a}b}.$$

□

Proof of Lemma 1.3 (i) By Theorem 1.1, the function $M_{(g,\psi)}$ is bounded

$$|g'(z)| \leq M(1+|z|)e^{\frac{1}{2}(|z|^2-|az+b|^2)}$$

for all z in $\mathbb{C}$. It follows that g' has order at most 2. The rest of the proof and part (ii) follow from a simple variant of the proof of [3, Theorem 3.2].

Before proceeding to the proofs of the main results, we recall the following Littelwood–Paley type estimate on Fock spaces. For each f in $\mathcal{F}_p$, it holds that

$$\|f\|_p \simeq \begin{cases} \Big(|f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p(1+|z|)^{-p}e^{-\frac{p}{2}|z|^2}dA(z)\Big)^{\frac{1}{p}}, & p<\infty \\ |f(0)| + \sup_{z\in\mathbb{C}} |f'(z)|(1+|z|)^{-1}e^{-\frac{1}{2}|z|^2}, & p=\infty. \end{cases} \tag{3.2}$$

The estimates are proved in [5, 14], and we will appeal to them several times in the sequel. □

Proof of Proposition 2.1 For each f in $\mathcal{F}_p$,

$$J_{(g,\psi)}f(z) = c\int_0^z f'(\psi(w))dw = cf(\psi(z)) - cf(b),$$

and hence equality of the sets in (2.1) holds. We proceed to show that the range set is closed. Let $c(f_n(\psi) - f_n(b))$ be a sequence in $\mathcal{R}(J_{(g,\psi)})$ which converges to cf in the space $\mathcal{F}_q$. We need to show that cf belongs to $\mathcal{R}(J_{(g,\psi)})$. If $c=0$, the assertion obviously follows. Thus, assuming $c\neq 0$ and applying (3.2)

$$\big\|c(f_n(\psi) - f_n(b) - f)\big\|_q^q \simeq \big|cf(0)\big|^q + |c|^q\int_{\mathbb{C}} \frac{\big|(f_n(\psi(z)) - f(z))'\big|^q}{(1+|z|)^q}e^{-\frac{q}{2}|z|^2}dA(z) \to 0$$

then the result follows from the previous lemma. Thus, assume ψ is nonconstant and consider the function

$$h_f = c^{-1}f\circ\psi^{-1} \in \mathcal{F}_p.$$

It follows that for all z in $\mathbb{C}$

$$J_{(g,\psi)}h_f(z) = \int_0^z f'(w)dw = f(z) - f(0) = f(z),$$

and hence f belongs to $\mathcal{R}(J_{(g,\psi)})$.

Our next lemma identifies conditions under which $V_{(g,\psi)}$ and $J_{(g,\psi)}$ become injective maps. □

Lemma 3.1 *Let $g,\psi\in\mathcal{H}(\mathbb{C})$ and $1\leq p,q\leq\infty$. Then a bounded*

(i) $V_{(g,\psi)}:\mathcal{F}_p\to\mathcal{F}_q$ *is injective if and only if both g and ψ are nonconstant.*

(ii) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *cannot be injective. On the other hand,* $J_{(g,\psi)}$ *restricted to the subspace* $\mathcal{F}_p^0 = \{f \in \mathcal{F}_p : f(0) = 0\}$ *is injective if and only if both* g *and* ψ *are nonconstant.*

Proof Suppose $V_{(g,\psi)}$ is injective. Then g is obviously not a constant, if not $V_{(g,\psi)}$ reduces to the zero operator. On the other hand, by Lemma 1.2, $\psi(z) = az + b$, $0 \leq |a| \leq 1$. If $a = 0$, we consider the functions $f_1(z) = z - b$ and $f_2(z) = 2(z - b)$, and observe that $V_{(g,\psi)} f_1 = V_{(g,\psi)} f_2$ while $f_1 \neq f_2$. Therefore, ψ is not a constant either.

Conversely, assume both g and ψ are not constants and $V_{(g,\psi)} f_1(z) = V_{(g,\psi)} f_2(z)$ for some $f_1, f_2 \in \mathcal{F}_p$. Then taking derivatives on both sides,

$$g'(z)(f_1(az + b) - f_2(az + b)) = 0$$

for all z in $\mathbb{C}$. This shows that $f_1 = f_2$ at all points z except possibly on the zero set of g'. Since both functions are entire, by uniqueness we deduce $f_1 = f_2$.

(ii) We observe that $J_{(g,\psi)}$ maps all constant functions to the zero function and fails to be injective. The proof for $J_{(g,\psi)}$ restricted to $\mathcal{F}_p^0$ or the space modulo $\mathbb{C}$ follows as in part (i).

Next, we recall the connection between the closed range problem and bounded below of linear operators on Banach spaces. An operator T is said to be bounded below if there exits a positive constant c such that $\|Tf\| \geq c\|f\|$ for every f in the underlying space. As known from an application of the Open Mapping Theorem, an injective bounded linear operator on Banach spaces has a closed range if and only if it is bounded below; see for example [1, Theorem 2.5].

By Lemma 3.1 and the discussion preceding it, if both g and ψ are nonconstant, then $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$, $1 \leq p \leq q$ has a closed range if and only if it is bounded below. Note that if either ψ or g is not a constant, then boundedness from below fails. For example for $\psi = b$, the estimate $\|V_{(g,\psi)} f\|_q \simeq \|g\|_q \geq \delta \|f\|_p$ does not necessarily hold for each f. Consider the sequence $f_n = z^n$ and observe that when $n \to \infty$, $\|f_n\|_p \to \infty$ and cannot be bounded by $\|g\|_q$ for all $n \in \mathbb{N}$.

On the other hand, since $J_{(g,\psi)}$ is not injective, we consider its restriction on the space $\mathcal{F}_p^0$. Note also that since for each f in $\mathcal{F}_p$, the function $f - f(0)$ belongs to $\mathcal{F}_p^0$ and $J_{(g,\psi)} f = J_{(g,\psi)}(f - f(0))$, using the equivalent norm in (3.2), the range of $J_{(g,\psi)}$ coincides with its range when it acts from $\mathcal{F}_p^0$. We may now record the following. □

Lemma 3.2 *Let* $g, \psi \in \mathcal{H}(\mathbb{C})$ *and* $1 \leq p, q \leq \infty$. *Suppose both* g *and* ψ *are nonconstant. Then a bounded*

(i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *has a closed range if and only if it is bounded below.*
(ii) $J_{(g,\psi)} : \mathcal{F}_p^0 \to \mathcal{F}_q$ *has a closed range if and only if it is bounded below.*

Because of this lemma, the closed range problem for the integral operators reduces now to finding conditions under which the operators are bounded from below. Thus, in the rest of the manuscript we focus on such conditions.

3.1 Proof of Theorem 2.2

(i) Let us consider the case $p < q < \infty$ first and assume $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ has closed range. By Lemma 3.1, the operator is bounded below. We consider the sequence of the monomials $f_n(z) = z^n$, $n = 1, 2, \ldots$ in $\mathcal{F}_p$ and by [18, p. 40],

$$\|f_n\|_p^p \simeq \left(\frac{n}{e}\right)^{\frac{np}{2}} \sqrt{n}. \tag{3.3}$$

Applying (3.2), Lemma 1.2, and Theorem 1.1

$$\begin{aligned}\|V_{(g,\psi)} f_n\|_q^q &\simeq \int_{\mathbb{C}} |g'(z)(1+|z|)^{-1}|^q |f_n(\psi(z))|^q e^{-\frac{q}{2}|z|^2} dA(z)\\ &= \int_{\mathbb{C}} M_{(g,\psi)}^q(z) |f_n(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z)\\ &\leq M^q \int_{\mathbb{C}} |f_n(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \lesssim \|f_n\|_q^q.\end{aligned}$$

This and boundedness below imply

$$\|f_n\|_q \geq \epsilon \|f_n\|_p \tag{3.4}$$

for some $\epsilon > 0$. By (3.3), the estimate in (3.4) holds only if

$$n^{\frac{1}{2q} - \frac{1}{2p}} \gtrsim \epsilon$$

for all n in $\mathbb{N}$, which gives a contradiction when $n \to \infty$.
Similarly, for $p < q = \infty$, we have

$$\|f_n\|_\infty = (n/e)^{n/2}, \tag{3.5}$$

and by (3.2), Lemma 1.2, and Theorem 1.1 again

$$\begin{aligned}\|V_{(g,\psi)} f_n\|_\infty &\simeq \sup_{z\in\mathbb{C}} M_{(g,\psi)}(z) |f_n(\psi(z))| e^{-\frac{|\psi(z)|^2}{2}}\\ &\leq M \sup_{z\in\mathbb{C}} |f_n(\psi(z))| e^{-\frac{|\psi(z)|^2}{2}} \lesssim \|f_n\|_\infty.\end{aligned}$$

Therefore, with (3.5) and bounded below

$$(n/e)^{n/2} = \|f_n\|_\infty \geq \epsilon \|f_n\|_p = \epsilon \left(\frac{n}{e}\right)^{\frac{n}{2}} n^{\frac{1}{2p}}$$

for some $\epsilon > 0$. This gives a contradiction when $n \to \infty$ again.

If $p > q$, then by Theorem 1.1, $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact. It is known that a compact operator has a closed range if and only if the range is finite dimensional. On the other hand, by Lemma 3.1, the linear operator is injective on $\mathcal{F}_p$ and hence its range is infinite dimensional.

The proof of part (ii) goes by following similar arguments as in part (i).

3.2 Proofs of Theorem 2.3 and Theorem 2.4

The proofs of Theorem 2.3 and Theorem 2.4 will be rather long. Thus, we will split them and reformulate various statements to make them more accessible. To this plan, the equivalencies of the statements in (i) and (ii) are proved in Proposition 3.3. The assertion (i) implies (iii) is simply a special case. Thus, we prove (iii) implies (iv) in Proposition 3.4 and (iv) implies (v) in Lemma 3.5. We conclude the proofs of the theorems after showing (iv) implies (i) in Lemma 3.6.

Proposition 3.3 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ be nonconstant and $1 \le p \le \infty$.*

(i) *Let $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. Then $V_{(g,\psi)}$ is bounded below if and only if there exists $\epsilon > 0$ such that $G^{\epsilon}_{(g,\psi)}$ is a $(p, 0)$ sampling set for $\mathcal{F}_p$.*

(ii) *Let $J_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. Then $J_{(g,\psi)}$ is bounded below on $\mathcal{F}_p^0$ if and only if there exists $\epsilon > 0$ such that $\Gamma^{\epsilon}_{(g,\psi)}$ is a $(p, 1)$ sampling set for $\mathcal{F}_p^0$.*

(i) Assume $p < \infty$ and suppose that $G^{\epsilon}_{(g,\psi)}$ is a $(p, 0)$ sampling set. Then there exists a $\delta > 0$ such that for each f in $\mathcal{F}_p$

$$\delta \|f\|_p^p \le \int_{G^{\epsilon}_{(g,\psi)}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z).$$

Applying (3.2),

$$\begin{aligned} \|V_{(g,\psi)} f\|_p^p &\simeq \int_{\mathbb{C}} |g'(z)(1+|z|)^{-1}|^p |f(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z) \\ &\ge \int_{\Omega^{\epsilon}_{(g,\psi)}} M^p_{(g,\psi)}(z) |f(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \\ &\ge \epsilon^p \int_{G^{\epsilon}_{(g,\psi)}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \ge \epsilon^p \delta \|f\|_p^p \end{aligned}$$

from which boundedness from below follows.

Conversely, we argue towards contradiction and assume $G^{\epsilon}_{(g,\psi)}$ is not a $(p, 0)$ sampling set for each $\epsilon > 0$. Then, there exists a unit norm sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_p$ such that

$$\int_{G^{1/k}_{(g,\psi)}} |f_k(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \to 0 \text{ as } \ k \to \infty. \tag{3.6}$$

We may apply (3.2) again and write

$$\|V_{(g,\psi)} f_k\|_p^p \simeq \left(\int_{\Omega^{1/k}_{(g,\psi)}} + \int_{\mathbb{C}\backslash \Omega^{1/k}_{(g,\psi)}} \right) \frac{|g'(z)|^p}{(1+|z|)^p} |f_k(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z),$$

and proceed to estimate the two pieces of integrals separately. For the first,

$$\begin{aligned}
&\int_{\Omega^{1/k}_{(g,\psi)}} |g'(z)(1+|z|)^{-1}|^p |f_k(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z) \\
&= \int_{\Omega^{1/k}_{(g,\psi)}} M^p_{(g,\psi)}(z) |f_k(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \\
&\le M^p \int_{\Omega^{1/k}_{(g,\psi)}} |f_k(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \\
&\simeq \int_{G^{1/k}_{(g,\psi)}} |f_k(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \to 0, \quad \text{as} \quad k \to \infty
\end{aligned}$$

which follows by (3.6). We estimate the remaining integral as

$$\begin{aligned}
&\int_{\mathbb{C}\backslash\Omega^{1/k}_{(g,\psi)}} |g'(z)(1+|z|)^{-1}|^p |f_k(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z) \\
&= \int_{\mathbb{C}\backslash\Omega^{1/k}_{(g,\psi)}} M^p_{(g,\psi)}(z) |f_k(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \\
&\le \frac{1}{k^p} \int_{\mathbb{C}\backslash\Omega^{1/k}_{(g,\psi)}} |f_k(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \le \frac{2\pi}{pk^p} \to 0
\end{aligned}$$

as $k \to \infty$. Now, both integral converge to zero when $k \to \infty$ and contradicts boundedness from below.

Next, we show for $p = \infty$ and consider first the sufficiency. For f in $\mathcal{F}_\infty$,

$$\delta \|f\|_\infty \le \sup_{z \in G^\epsilon_{(g,\psi)}} |f(z)| e^{-\frac{|z|^2}{2}} = \sup_{z \in \Omega^\epsilon_{(g,\psi)}} |f(\psi(z))| e^{-\frac{|\psi(z)|^2}{2}}. \tag{3.7}$$

Applying eventually the estimate in (3.2), we have

$$\begin{aligned}
\sup_{z \in \Omega^\epsilon_{(g,\psi)}} |f(\psi(z))| e^{-\frac{|\psi(z)|^2}{2}} &= \sup_{z \in \Omega^\epsilon_{(g,\psi)}} \frac{|g'(z)|}{M_{(g,\psi)}(z)(1+|z|)} |f(\psi(z))| e^{-\frac{|z|^2}{2}} \\
&\le \frac{1}{\epsilon} \sup_{z \in \Omega^\epsilon_{(g,\psi)}} \frac{|g'(z)|}{1+|z|} |f(\psi(z))| e^{-\frac{|z|^2}{2}} \simeq \frac{1}{\epsilon} \|V_{(g,\psi)} f\|_\infty
\end{aligned}$$

from which and (3.7), $V_{(g,\psi)}$ is bounded below.

Conversely, suppose $V_{(g,\psi)}$ is bounded below. Then there exists a positive constant α such that for each f in $\mathcal{F}_\infty$,

$$\|V_{(g,\psi)}f\|_\infty \simeq \sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+|z|}|f(\psi(z))|e^{-\frac{|z|^2}{2}} \geq \alpha\|f\|_\infty.$$

It follows that for each f, there exists w_f in $\mathbb{C}$ such that

$$\frac{|g'(w_f)|}{1+|w_f|}|f(\psi(w_f))|e^{-\frac{|w_f|^2}{2}} \geq \frac{\alpha}{2}\|f\|_\infty.$$

On the other hand,

$$\begin{aligned}\frac{|g'(w_f)|}{1+|w_f|}|f(\psi(w_f))|e^{-\frac{|w_f|^2}{2}} &= M_{(g,\psi)}(w_f)|f(\psi(w_f))|e^{-\frac{|\psi(w_f)|^2}{2}}\\ &\leq M_{(g,\psi)}(w_f)\|f\|_\infty\end{aligned}$$

and hence $M_{(g,\psi)}(w_f) \geq \alpha/2$. Setting $\epsilon = \alpha/2$, we observe that $w_f \in \Omega^\epsilon_{(g,\psi)}$. Furthermore,

$$\begin{aligned}M\frac{|g'(w_f)|}{1+|w_f|}|f(\psi(w_f))|e^{-\frac{|w_f|^2}{2}} &\geq \left(M_{(g,\psi)}(w_f)\right)^2|f(\psi(w_f))|e^{-\frac{|\psi(w_f)|^2}{2}}\\ &\geq (\alpha/2)^2\|f\|_\infty\end{aligned}$$

and from which

$$\begin{aligned}\|f\|_\infty &\leq \frac{4}{\alpha^2}\frac{M|g'(w_f)|}{1+|w_f|}|f(\psi(w_f))|e^{-\frac{|w_f|^2}{2}}\\ &= \frac{4}{\alpha^2}MM_{(g,\psi)}(w_f)|f(\psi(w_f))|e^{-\frac{|\psi(w_f)|^2}{2}}\\ &\leq \frac{4M^2}{\alpha^2}|f(\psi(w_f))|e^{-\frac{|\psi(w_f)|^2}{2}}\\ &\leq \frac{4M^2}{\alpha^2}\sup_{z\in G^\epsilon_{(g,\psi)}}|f(z)|e^{-\frac{|z|^2}{2}}.\end{aligned}$$

Hence, $G^\epsilon_{(g,\psi)}$ is a $(p,0)$ sampling set.

(ii) We argue as in the proof of part (i). If $\Gamma^{\epsilon}_{(g,\psi)}$ is a $(p,1)$ sampling set for $\mathcal{F}_p^0$ for some $\epsilon > 0$, then for each f in $\mathcal{F}_p^0$

$$\begin{aligned}\int_{\mathbb{C}} \frac{|g(z)|^p}{(1+|z|)^p}|f'(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z) &\geq \int_{\Lambda^{\epsilon}_{(g,\psi)}} \widetilde{M}^p_{(g,\psi)}(z) \frac{|f'(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\geq \frac{\epsilon^p}{|a|^2}\int_{\Gamma^{\epsilon}_{(g,\psi)}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2} dA(z)\\ &\geq \frac{\delta\epsilon^p}{|a|^2}\int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2} dA(z) \gtrsim \frac{\delta\epsilon^p}{|a|^2}\|f\|_p^p,\end{aligned}$$

where the last estimate follows by (3.2). This proves the sufficiency.

Conversely, suppose on the contrary that $\Gamma^{\epsilon}_{(g,\psi)}$ is not a $(p,1)$ sampling set for $\mathcal{F}_p$ for any $\epsilon > 0$. Then there exists a unit norm sequence $f_k, k \in \mathbb{N}$ in $\mathcal{F}_p$ such that

$$\int_{\Gamma^{1/k}_{(g,\psi)}} |f_k'(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \to 0 \text{ as } k \to \infty.$$

We now proceed to estimate $\|J_{(g,\psi)} f_k\|_p^p$ as

$$\begin{aligned}\|J_{(g,\psi)} f_k\|_p^p &\simeq \int_{\mathbb{C}} \frac{|g(z)|^p}{(1+|z|)^p}|f_k'(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \left(\int_{\Lambda^{1/k}_{(g,\psi)}} + \int_{\mathbb{C}\setminus\Lambda^{1/k}_{(g,\psi)}}\right)\frac{|g(z)|^p}{(1+|z|)^p}|f_k'(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z),\end{aligned}$$

where we denote the two integrals by I_1 and I_2. To estimate I_1, we observe

$$\begin{aligned}I_1 &\leq \widetilde{M}^p \int_{\Lambda^{1/k}_{(g,\psi)}} \frac{|f_k'(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\simeq \int_{\Gamma^{1/k}_{(g,\psi)}} \frac{|f_k'(z)|^p}{(1+|z|)^P} e^{-\frac{p}{2}|z|^2} dA(z) \to 0,\end{aligned}$$

as $k \to \infty$. Similarly, we estimate I_2 as

$$\begin{aligned}I_2 &= \int_{\mathbb{C}\setminus\Lambda^{1/k}_{(g,\psi)}} \frac{|g(z)|^P}{(1+|z|)^p}|f_k'(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &\leq \frac{1}{k^p}\int_{\mathbb{C}\setminus\Lambda^{1/k}_{(g,\psi)}} \frac{|f_k'(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\lesssim \frac{1}{k^p}\|f_k\|_p^p \simeq \frac{1}{k^p} \to 0,\ k \to \infty,\end{aligned}$$

which contradicts the assumption that the operator is bounded below.
The case $p = \infty$ follows in a similar way.

The following local estimate from [18, p. 37] may be needed in our consideration further. For each entire function f and $p \neq \infty$, there exists a positive constant C such that

$$|f(z)|^p e^{-\frac{p|z|^2}{2}} \leq \frac{C}{r^2} \int_{D(z,r)} |f(w)|^p e^{-\frac{p|w|^2}{2}} dA(w), \tag{3.8}$$

where $D(z, r)$ is a disc in $\mathbb{C}$ with center z and radius r.

Proposition 3.4 *Let $g, \psi \in \mathcal{H}(\mathbb{C})$ be nonconstant and $1 \leq p \leq \infty$.*

(i) *Let $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. If $V_{(g,\psi)}$ is bounded below on the kernel functions, then there exist positive numbers ϵ, r, and σ such that*

$$A\big(G^{\epsilon}_{(g,\psi)} \cap D(z, r)) \gtrsim A(D(z, r)\big) \geq \sigma r^2 \tag{3.9}$$

for all z in $\mathbb{C}$.

(ii) *Let $J_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. If $J_{(g,\psi)}$ is bounded below on the kernel functions, then there exist positive numbers ϵ, r, and σ such that*

$$A\big(\Gamma^{\epsilon}_{(g,\psi)} \cap D(z, r)) \gtrsim A(D(z, r)\big) \geq \sigma r^2 \tag{3.10}$$

for all z in $\mathbb{C}$.

Proof (i) Let $p < \infty$ and $\alpha > 0$ such that for all $w \in \mathbb{C}$

$$\begin{aligned}\alpha \leq \|V_{(g,\psi)} k_w\|_p^p &\leq \frac{cp}{2\pi} \int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} |k_w(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \frac{cp}{2\pi}\left(\int_{\Omega^{\epsilon}_{(g,\psi)}} + \int_{\mathbb{C}\backslash\Omega^{\epsilon}_{(g,\psi)}}\right) \frac{|g'(z)|^p}{(1+|z|)^p} |k_w(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z),\end{aligned}$$

where c is a positive constant from the relation in (3.2) and ϵ a small positive number which can be made a bite more specific later. It follows that

$$\begin{aligned}&\int_{\Omega^{\epsilon}_{(g,\psi)}} |g'(z)(1+|z|)^{-1}|^p |k_w(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &\geq \frac{2\pi\alpha}{cp} - \int_{\mathbb{C}\backslash\Omega^{\epsilon}_{(g,\psi)}} |g'(z)(1+|z|)^{-1}|^p |k_w(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \frac{2\pi\alpha}{cp} - \int_{\mathbb{C}\backslash\Omega^{\epsilon}_{(g,\psi)}} M^p_{(g,\psi)}(z) |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &> \frac{2\pi\alpha}{cp} - \epsilon^p \int_{\mathbb{C}\backslash\Omega^{\epsilon}_{(g,\psi)}} |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\geq \frac{2\pi\alpha}{cp} - \epsilon^p \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) = \frac{2\pi\alpha}{cp} - \frac{2\pi\epsilon^p}{p|a|^2}.\end{aligned}$$

By change of variables,

$$\int_{G^{\epsilon}_{(g,\psi)}} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|k_w(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z) > \frac{2\pi\alpha|a|^2}{cp} - \frac{2\pi\epsilon^p}{p}. \quad (3.11)$$

Next, for each $w \in \mathbb{C}$ and $R > 0$ we estimate from below the left-hand quantity in (3.11) on the set $G^{\epsilon}_{(g,\psi)} \cap D(w, R)$ as

$$\begin{aligned}
&\int_{G^{\epsilon}_{(g,\psi)}\cap D(w,R)} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|k_w(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z)\\
&= \int_{G^{\epsilon}_{(g,\psi)}} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|k_w(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z)\\
&\quad - \int_{G^{\epsilon}_{(g,\psi)}\setminus D(w,R)} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|k_w(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z).
\end{aligned}$$

The difference of the integrals above is bounded below by

$$\begin{aligned}
&\frac{2\pi\alpha|a|^2}{cp} - \frac{2\pi\epsilon^p}{p} - \int_{\mathbb{C}\setminus D(w,R)} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|k_w(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z)\\
&= \frac{2\pi\alpha|a|^2}{cp} - \frac{2\pi\epsilon^p}{p} - \int_{\mathbb{C}\setminus D(w,R)} M^p_{(g,\psi)}(\psi^{-1}(z))|k_w(z)|^p e^{-\frac{p}{2}|z|^2} dA(z)\\
&\geq \frac{2\pi\alpha|a|^2}{cp} - \frac{2\pi\epsilon^p}{p} - M^p \int_{|z-w|>R} |k_w(z)|^p e^{-\frac{p}{2}|z|^2} dA(z). \quad (3.12)
\end{aligned}$$

On the other hand, using the local estimate in (3.8), the exists a positive constant C_1 such that

$$\begin{aligned}
&\int_{G^{\epsilon}_{(g,\psi)}\cap D(w,R)} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|k_w(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z)\\
&\leq M^p \int_{G^{\epsilon}_{(g,\psi)}\cap D(w,R)} |k_w(z)|^p e^{-\frac{p}{2}|z|^2} dA(z)\\
&\leq \frac{C_1 M^p}{R^2} \int_{G^{\epsilon}_{(g,\psi)}\cap D(w,R)} \int_{D(z,R)} |k_w(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} dA(\zeta) dA(z)\\
&\leq \frac{C_1 M^p 2\pi}{pR^2} \|k_w\|_p^p \int_{G^{\epsilon}_{(g,\psi)}\cap D(w,R)} dA(z)\\
&= \frac{C_1 M^p 2\pi}{pR^2} A\big(G^{\epsilon}_{(g,\psi)} \cap D(w, R)\big).
\end{aligned}$$

This together with (3.12) imply

$$\frac{A\big(G^{\epsilon}_{(g,\psi)}\cap D(w,R)\big)}{R^2}\ge \frac{\alpha|a|^2}{cC_1M^p}-\frac{\epsilon^p}{M^pC_1}-\frac{p}{2\pi C_1}\int_{|z-w|>R}|k_w(z)|^pe^{-\frac{p}{2}|z|^2}dA(z).$$

Since the integral above is convergent, we can choose R big enough and ϵ so small (if need be) such that

$$\frac{A\big(G^{\epsilon}_{(g,\psi)}\cap D(w,R)\big)}{R^2}\ge\beta>0,$$

where

$$\beta:=\frac{\alpha|a|^2}{cC_1M^p}-\frac{\epsilon^p}{M^pC_1}-\frac{p}{2\pi C_1}\int_{|z-w|>R}|k_w(z)|^pe^{-\frac{p}{2}|z|^2}dA(z)$$

and completes the proof for finite p. Let now $p=\infty$ and $\alpha\le\|V_{(g,\psi)}k_w\|_\infty$ for all w in $\mathbb{C}$. Since $V_{(g,\psi)}$ is bounded on both $\mathcal{F}_1$ and $\mathcal{F}_\infty$, by the inclusion property on Fock paces we have $\alpha\le\|V_{(g,\psi)}k_w\|_\infty\le\|V_{(g,\psi)}k_w\|_1$ and then we argue as above setting $p=1$ to arrive at the claim.

(ii) The proof of this part follows from a simple variant of the proof of part (i). $\square$

Lemma 3.5 *Let $g,\psi\in\mathcal{H}(\mathbb{C})$ be nonconstant, and $1\le p\le\infty$.*

(i) *Let $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. If there exist positive numbers ϵ, r, and σ such that (3.9) holds, then $M_{(g,\psi)}$ is essentially bounded away from zero on $\mathbb{C}$.*

(ii) *Let $J_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. If there exist positive numbers ϵ, r, and σ such that (3.10) holds, then $\widetilde{M}_{(g,\psi)}$ is essentially bounded away from zero on $\mathbb{C}$.*

Proof (i) The sufficiency of the condition follows easily. We prove the necessity and suppose that $V_{(g,\psi)}$ is bounded below. Let ϵ be as in Proposition 3.4 and $E=\{z\in\mathbb{C}:M_{(g,\psi)}(z)<\frac{\epsilon}{2}\}$. Then we aim to show that $A(E)=0$. Suppose on the contrary that $A(E)>0$. Then we can find a disc D of radius δ_1 and center w contained in $\{z\in\mathbb{C}:M_{(g,\psi)}(z)<\delta_1\}$ for some $0<\delta_1<\frac{\epsilon}{2}$ such that $0<A(D)=\delta_1^2$. Now using the constant β in Proposition 3.4 and setting $\delta_1=(|a|\epsilon)/2$,

$$\begin{aligned}\beta A(D(w,\delta_1))&\le A(G^{\epsilon}_{(g,\psi)}\cap D(w,\delta_1))=\int_{G^{\epsilon}_{(g,\psi)}\cap D(w,\delta_1)}dA(z)\\&=|a|^{-2}\int_{\Omega^{\epsilon}_{(g,\psi)}\cap\psi(D(w,\delta_1))}dA(z)=|a|^{-2}\int_{\Omega^{\epsilon}_{(g,\psi)}\cap D\left(w,\delta_1|a|^{-1}\right)}dA(z)\\&=|a|^{-2}A(\Omega^{\epsilon}_{(g,\psi)}\cap D\big(w,\delta_1|a|^{-1})\big)=0\end{aligned}$$

since $\Omega^{\epsilon}_{(g,\psi)}\cap D\big(w,\frac{\delta_1}{|a|}\big)$ has no element in it. Hence $A(E)=0$.

(ii) This follows from a similar argument as in the proof of part (i). $\square$

Lemma 3.6 *Let $g,\psi\in\mathcal{H}(\mathbb{C})$ be nonconstant and $1\le p\le\infty$.*

(i) *Let $V_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. If $M_{(g,\psi)}$ is essentially bounded away from zero on $\mathbb{C}$, then $V_{(g,\psi)}$ has closed range and its range is given by*

$$\mathcal{R}(V_{(g,\psi)}) = \mathcal{F}_p^0. \tag{3.13}$$

(ii) *Let $J_{(g,\psi)}$ be bounded on $\mathcal{F}_p$. If $\widetilde{M}_{(g,\psi)}$ is essentially bounded away from zero on $\mathbb{C}$, then it has closed range and its range is given by*

$$\mathcal{R}(J_{(g,\psi)}) = \mathcal{F}_p^0. \tag{3.14}$$

Proof (i) Let $\psi(z) = az + b$, $0 < |a| \leq 1$ and $\gamma > 0$ be an essential lower bound for $M_{(g,\psi)}$. Applying (3.2), for each f in $\mathcal{F}_p$ and $p < \infty$,

$$\begin{aligned}\|V_{(g,\psi)} f\|_p^p &\simeq |a|^{-2} \int_{\mathbb{C}} \frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p} |f(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z)\\ &= |a|^{-2} \int_{\mathbb{C}} M_{(g,\psi)}^p(\psi^{-1}(z)) |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &\geq \gamma^p |a|^{-2} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \gtrsim \gamma^p |a|^{-2} \|f\|_p^p.\end{aligned}$$

Therefore, the operator is bounded below and hence has closed range.

To prove (3.13), for each f in $\mathcal{F}_p^0$, consider the function

$$h_f(z) = \begin{cases} \frac{f'(\psi^{-1}(z))}{g'(\psi^{-1}(z))}, & g'(\psi^{-1}(z)) \neq 0\\ \lim_{w\to z} \frac{f'(\psi^{-1}(w))}{g'(\psi^{-1}(w))}, & g'(\psi^{-1}(z)) = 0. \end{cases}$$

Clearly, $V_{(g,\psi)} h_f = f$. We claim that $h_f \in \mathcal{F}_p^0$. Since g' is entire, it vanishes at most in a set of measure zero. Then we estimate the norm of h_f as

$$\begin{aligned}\|h_f\|_p^p &= \frac{p}{2\pi} \int_{\mathbb{C}} \frac{|f'(\psi^{-1}(z))|^p}{|g'(\psi^{-1}(z))|^p} e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \frac{p}{2\pi} \int_{\mathbb{C}} M_{(g,\psi)}^{-p}(\psi^{-1}(z)) \frac{|f'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p} e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z)\\ &\leq \frac{p\gamma^{-p}}{2\pi} \int_{\mathbb{C}} \frac{|f'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p} e^{-\frac{p}{2}|\psi^{-1}(z)|^2} dA(z) \lesssim \|f\|_p^p < \infty.\end{aligned}$$

For $p = \infty$, we replace the integral above by the supremum and argue similarly.

(ii) If $\alpha_2 > 0$ is an essential lower bound for $\widetilde{M}$, then arguing as above,

$$\begin{aligned}\|J_{(g,\psi)}f\|_p^p &\simeq \frac{1}{|a|^2}\int_{\mathbb{C}} \frac{|g(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}|f'(z)|^p e^{-\frac{p}{2}|\psi^{-1}(z)|^2}dA(z)\\ &= \frac{1}{|a|^2}\int_{\mathbb{C}} \widetilde{M}^p(\psi^{-1}(z))\frac{|f'(z)|^p}{(1+|z|)^p}e^{-\frac{p}{2}|z|^2}dA(z)\\ &\geq \frac{\alpha_2^p}{|a|^2}\int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p}e^{-\frac{p}{2}|z|^2}dA(z) \simeq \alpha_2^p|a|^{-2}\|f\|_p^p.\end{aligned}$$

and $p < \infty$. For $p = \infty$, we argue with the supremum in stead of integral again. To prove (3.14), for each f in $\mathcal{F}_p^0$, consider the function

$$t_f(z) = \begin{cases} \frac{f(\psi^{-1}(z))}{g(\psi^{-1}(z))}, & g(\psi^{-1}(z)) \neq 0\\ \lim_{w\to z}\frac{f(\psi^{-1}(w))}{g(\psi^{-1}(w))}, & g(\psi^{-1}(z)) = 0,\end{cases}$$

and argue as in the proof of (3.13). □

3.3 Proof of Theorem 2.5

In this section we prove the results related to reverse Fock–Carleson measures. By Proposition 3.3, $V_{(g,\psi)}$ has closed range on $\mathcal{F}_p$ if and only if there exists $\epsilon > 0$ such that $G^{\epsilon}_{(g,\psi)}$ is a $(p, 0)$ sampling set for $\mathcal{F}_p$. Consequently, there exists a positive δ such that

$$\int_{G^{\epsilon}_{(g,\psi)}} |f(z)|^p e^{-\frac{p|z|^2}{2}}dA(z) = \int_{\mathbb{C}} |f(z)|^p \chi_{G^{\epsilon}_{(g,\psi)}}(z)e^{-\frac{p|z|^2}{2}}dA(z) \geq \delta\|f\|_p^p.$$

for all f in $\mathcal{F}_p$. This shows $\mu^{\epsilon}_{(g,\psi)}$ is a $(p, 0)$ reverse Fock–Carleson measure, where

$$d\mu^{\epsilon}_{(g,\psi)}(z) = \chi_{G^{\epsilon}_{(g,\psi)}}(z)e^{-\frac{p}{2}|z|^2}dA(z),$$

and this proves the equivalency of (a) and (b).
Next, we show the equivalency of (a) and (c). Since $\psi(z) = az + b$ with $a \neq 0$ is bijective on the complex plane,

$$\begin{aligned}\|V_{(g,\psi)}f\|_p^p &\simeq \int_{\mathbb{C}} |g'(z)(1+|z|)^{-1}|^p|f(\psi(z))|^p e^{-\frac{p|z|^2}{2}}dA(z)\\ &= |a|^{-2}\int_{\mathbb{C}} |f(z)|^p\left(\frac{|g'(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}e^{-\frac{p|\psi^{-1}(z)|^2}{2}}\right)dA(z)\\ &= \int_{\mathbb{C}} |f(z)|^p d\mu_{(g,\psi,p)}(z) \geq C\|f\|_p^p\end{aligned}$$

for some $C > 0$, where

$$d\mu_{(g,\psi,p)}(z) = |g'(\psi^{-1}(z))|^p(|a|^2(1+|\psi^{-1}(z)|)^{-p}e^{-\frac{p|\psi^{-1}(z)|^2}{2}}dA(z).$$

Hence, the assertion follows.

ii) The equivalency of the statements in this part can be established by following similar arguments as above. By Proposition 3.3, the operator $J_{(g,\psi)}$ has closed range on $\mathcal{F}_p$ if and only if there exists $\epsilon > 0$ such that $\Gamma^{\epsilon}_{(g,\psi)}$ is a $(p, 1)$ sampling set for $\mathcal{F}^0_p$. Thus, there exists a positive β such that

$$\int_{\Gamma^{\epsilon}_{(g,\psi)}} \frac{|f'(z)|^p}{(1+|z|)^p}e^{-\frac{p|z|^2}{2}}dA(z) = \int_{\mathbb{C}} |f'(z)|^p \chi_{\Gamma^{\epsilon}_{(g,\psi)}}(z)\frac{e^{-\frac{p|z|^2}{2}}}{(1+|z|)^p}dA(z) \geq \beta\|f\|^p_p.$$

for all f in $\mathcal{F}_p$. Therefore, $\theta^{\epsilon}_{(g,\psi)}$ is a $(p, 1)$ reverse Fock–Carleson measure, where

$$d\theta^{\epsilon}_{(g,\psi)}(z) = \chi_{\Gamma^{\epsilon}_{(g,\psi)}}(z)(1+|z|)^{-p}e^{-\frac{p}{2}|z|^2}dA(z).$$

Thus, the statements in (a) and (b) are equivalent.

Similarly, applying (3.2) again

$$\begin{aligned}\|J_{(g,\psi)}f\|^p_p &\simeq \int_{\mathbb{C}} |g(z)(1+|z|)^{-1}|^p|f'(\psi(z))|^p e^{-\frac{p|z|^2}{2}}dA(z)\\ &= |a|^{-2}\int_{\mathbb{C}} |f'(z)|^p\left(\frac{|g(\psi^{-1}(z))|^p}{(1+|\psi^{-1}(z)|)^p}e^{-\frac{p|\psi^{-1}(z)|^2}{2}}\right)dA(z)\\ &= \int_{\mathbb{C}} |f(z)|^p d\theta_{(g,\psi,p)}(z) \geq \alpha\|f\|^p_p\end{aligned}$$

for some $\alpha > 0$, where

$$d\theta_{(g,\psi,p)}(z) = |a|^{-2}(1+|\psi^{-1}(z)|)^{-p}|g(\psi^{-1}(z))|^p e^{-\frac{p|\psi^{-1}(z)|^2}{2}}dA(z).$$

This gives the equivalency of the statements in (a) and (c).

3.4 Proof of Theorem 2.6

(i) By Lemma 1.2, for $|a| = 1$ we have

$$g'(z) = (cz+d)K_{-\overline{a}b}(z) \tag{3.15}$$

for some $c, d \in \mathbb{C}$ and g' vanishes at most at a point. To prove the sufficiency, consider f in $\mathcal{F}_p$ and using (3.2) for $p < \infty$ and $c \neq 0$,

$$\begin{aligned}\|V_{(g,\psi)} f\|_p^p &\simeq \int_{\mathbb{C}} \left|K_{-\overline{a}b}(z)\right|^p |f(az+b)|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \int_{\mathbb{C}} \left|K_{-\overline{a}b}(z) e^{\frac{1}{2}|az+b|^2-\frac{1}{2}|z|^2}\right|^p |f(az+b)|^p e^{-\frac{p}{2}|az+b|^2} dA(z)\\ &= \int_{\mathbb{C}} |f(az+b)|^p e^{-\frac{p}{2}|az+b|^2} dA(z) \simeq \|f\|_p^p.\end{aligned}$$

Therefore, the operator is bounded below and hence the assertion.
For the case $p = \infty$, we argue with supremum in stead of the integral to arrive at the same conclusion.

For the necessity, first suppose for the sake of contradiction that $c = 0$ in (3.15). Using the sequence of kernel function K_n, (3.2) and boundedness from below,

$$\begin{aligned}\|V_{(g,\psi)} K_n\|_p^p &\simeq \int_{\mathbb{C}} \frac{|d|^p}{(1+|z|)^p} \left|K_{-\overline{a}b}(z)\right|^p |K_n(az+b)|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \frac{|de^{nb}|^p}{|(n-b)\overline{a}|^p} \int_{\mathbb{C}} \frac{|K'_{\overline{a}(n-b)}(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2} dA(z)\\ &\simeq |de^{\frac{|b|^2}{2}}|^p |n-b|^{-p} \|K_n\|_p^p \gtrsim \|K_n\|_p^p\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we arrive at a contradiction and hence $c \neq 0$.
Next, we consider the case when $0 < |a| < 1$. By Theorem 2.3, $M_{(g,\psi)}$ is essentially bounded away from zero. Since g' is entire, its zero set does not affect the essential boundedness of $M_{(g,\psi)}$. Thus, we can assume that g' is non-vanishing. Then by Lemma 1.3, the function g' has the form in (1.6), $g'(z) = e^{a_0+a_1z+a_2z^2}$, for some constants a_0, a_1, a_2 in $\mathbb{C}$ such that $|a_2| \leq \frac{1-|a|^2}{2}$.
Case 1: If $|a_2| < \frac{1-|a|^2}{2}$, then the operator is compact, and it is known that a compact operator can have closed range if and only if its range is finite dimensional. On the other hand, $V_{(g,\psi)}$ is injective on $\mathcal{F}_p$ which is infinite dimensional. Thus, the operator cannot have closed range in this case.
For the next cases, we may first write

$$\begin{aligned}M_{(g,\psi)}(z) &= |g'(z)|(1+|z|)^{-1} e^{\frac{1}{2}(|az+b|^2-|z|^2)}\\ &= C(1+|z|)^{-1} e^{\Re((a_1+a\overline{b})z)+\Re(a_2z^2)+\frac{|a|^2-1}{2}|z|^2}\end{aligned}$$

for all $z \in \mathbb{C}$, where $C = e^{\Re(a_0)+\frac{|b|^2}{2}}$.
Case 2: If $|a_2| = \frac{1-|a|^2}{2}$ and $a_1 + a\overline{b} = 0$, then

$$M_{(g,\psi)}(z) = C(1+|z|)^{-1} e^{\Re(a_2z^2)+\frac{|a|^2-1}{2}|z|^2}.$$

We write $a_2 = |a_2|e^{-2i\theta_2}$, where $0 \leq \theta_2 < \pi$ and replace z by $e^{i\theta_2}w$ above to get

$$M_{(g,\psi)}(e^{i\theta_2}w) = C(1+|w|)^{-1}e^{\frac{1-|a|^2}{2}(\Re(w^2)-|w|^2)}$$

for all w in $\mathbb{C}$. This clearly shows that $M_{(g,\psi)}$ is not essentially bounded away from zero in this case.
Case 3: If $a_1 + a\overline{b} \neq 0$ and

$$a_2 = -\frac{(1-|a|^2)(a_1+a\overline{b})^2}{2|a_1+a\overline{b}|^2},$$

then using the above polar notation for a_2, we write

$$a_2 = -\frac{(1-|a|^2)(a_1+a\overline{b})^2}{2|a_1+a\overline{b}|^2} = |a_2|e^{-2i\theta_2}$$

and observe that $(a_1+a\overline{b})e^{-i\theta_2} = \pm i|a_1+a\overline{b}|$ is a purely imaginary number. Setting $(a_1+a\overline{b})e^{-i\theta_2} = iy$ for some $y \in \mathbb{R}$ and $w = u+iv$, we have

$$\begin{aligned} M_{(g,\psi)}(e^{i\theta_2}w) &= C(1+|w|)^{-1}e^{-yv+(|a|^2-1)(\Re(w^2)-|w|^2)} \\ &= C(1+|w|)^{-1}e^{-yv+(|a|^2-1)v^2}. \end{aligned}$$

Thus, $M_{(g,\psi)}$ cannot be essentially bounded away from zero in this case either.

ii) The proof of this part follows in a similar way as above using Lemma 1.2 and Lemma 1.3.

3.5 Proof of Corollary 2.8

First note that because of the extra term in the relation $C_\psi f = J_{(1,\psi)}f + f(\psi(0))$, some of the conditions in Corollary 2.8 do not directly follow from Theorem 2.4. Thus, we may first verify the assertion (i) implies (ii). Let $\psi(z) = az+b,\ |a| < 1$ and suppose C_ψ has closed range and hence bounded below. If σ is such a bound, then using the pointwise estimate,

$$|f(z)| \leq e^{\frac{|z|^2}{2}}\|f\|_p, \tag{3.16}$$

for each f in $\mathcal{F}_p$ we have

$$\begin{aligned} \|J_{(1,\psi)}f\|_p = \|C_\psi f - f(\psi(0))\|_p &\geq \Big|\|C_\psi f\|_p - |f(\psi(0))|\Big| \\ &\geq \Big|\sigma\|f\|_p - e^{\frac{|\psi(0)|^2}{2}}\|f\|_p\Big| = \Big|\sigma - e^{\frac{|b|^2}{2}}\Big|\|f\|_p, \end{aligned}$$

where σ can be chosen such that $\sigma \neq e^{\frac{|b|^2}{2}}$. It follows that $J_{(1,\psi)}$ is bounded from below. Consequently, by Theorem 2.4

$$\widetilde{M}_{(1,\psi)}(z) = (1 + |\psi(z)|)(1 + |z|)^{-1} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \simeq e^{\frac{1}{2}((|a|^2-1)|z|^2 + \Re(az\overline{b}) + |b|^2)}$$

is essentially bounded away from zero, and this happens only if $|a| = 1$, and hence $b = 0$ by boundedness of the operator.

Next, we show (ii) implies (iii). Setting $|a| = 1$ and $b = 0$, we have $\widetilde{M}_{(1,\psi)}(z) = 1$ for all z in $\mathbb{C}$. It follows that

$$\Lambda^{\epsilon_1}_{(1,\psi)} = \{z \in \mathbb{C} : \widetilde{M}_{(1,\psi)}(z) \geq \epsilon_1\} = \mathbb{C}$$

for any $\epsilon_1 \leq 1$ and hence $\Gamma^{\epsilon_1}_{(1,\psi)} = \mathbb{C}$. Using the relation, $J_{(1,\psi)} f = C_\psi f - f(\psi(0)) = C_\psi f - f(0)$, Theorem 2.4, and Theorem 2.6, we deduce that $\mathbb{C}$ is a $(p, 1)$ sampling set for each f in $\mathcal{F}_p$.

The assertion (iii) implies (iv) follow by the same argument as above by considering the operator $J_{(1,\psi)}$ and using its relation with C_ψ. The case (iv) implies (v) follows easily since (iv) implies that $a \neq 0$ and hence for each f in $\mathcal{F}_p$, the function $h_f = f \circ \psi^{-1} \in \mathcal{F}_p$, and $C_\psi(h_f) = f(\psi^{-1}(\psi)) = f$.

3.6 Proofs of Theorem 2.9, Corollary 2.10, and Corollary 2.11

(i) Suppose $M_{(g,\psi)} \in L^q(\mathbb{C}, dA)$. We need to show $V_{(g,\psi)}$ is order bounded. For f in $\mathcal{F}_p$, applying the pointwise estimate in (3.16)

$$|V_{(g,\psi)} f(z)| = \left| \int_0^z g'(w) f(\psi(w)) dw \right| \leq \|f\|_p \left| \int_0^z g'(w) e^{\frac{|\psi(w)|^2}{2}} dw \right|. \tag{3.17}$$

Setting

$$h(z) = \left| \int_0^z g'(w) e^{\frac{|\psi(w)|^2}{2}} dw \right|,$$

and applying (3.2) for $q < \infty$,

$$\begin{aligned} \int_{\mathbb{C}} h(z)^q e^{-\frac{q}{2}|z|^2} dA(z) &\simeq \int_{\mathbb{C}} |g'(z)(1 + |z|)^{-1}|^q e^{\frac{q}{2}(|\psi(z)|^2 - |z|^2)} dA(z) \\ &= \int_{\mathbb{C}} M^q_{(g,\psi)}(z) dA(z) < \infty. \end{aligned} \tag{3.18}$$

Similarly for $q = \infty$, we have

$$\sup_{z \in \mathbb{C}} h(z) e^{-\frac{1}{2}|z|^2} \simeq \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + |z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} = \sup_{z \in \mathbb{C}} M_{(g,\psi)}(z) < \infty. \tag{3.19}$$

By (3.18) and (3.19), it follows h belongs to $L^q\big(\mathbb{C}, e^{-\frac{q}{2}|z|^2}dA(z)\big)$. Furthermore, (3.17) implies

$$|V_{(g,\psi)}f(z)| \leq h(z)$$

for all $z \in \mathbb{C}$ and $f \in \mathcal{F}_p$ such that $\|f\|_p \leq 1$.

Conversely, suppose $V_{(g,\psi)}$ is order bounded. Then there exists a positive function $h \in L^q\big(\mathbb{C}, e^{-\frac{q}{2}|z|^2}dA(z)\big)$ such that

$$|V_{(g,\psi)}f(z)| \leq h(z)$$

for all most all z in $\mathbb{C}$. In particular for the normalized kernel $k_w = K_w/\|K_w\|_2$,

$$|V_{(g,\psi)}k_w(z)| = \left|\int_0^z g'(\zeta)k_w(\psi(\zeta))d\zeta\right| = \left|\int_0^z g'(\zeta)e^{\frac{|\psi(\zeta)|^2}{2}}f_w(\zeta)d\zeta\right| \leq h(z)$$

for all $w \in \mathbb{C}$, where

$$f_w(\zeta) = K_w(\psi(\zeta))e^{-\frac{|\psi(\zeta)|^2+|w|^2}{2}}.$$

Observe that f_w is bounded as a sequence of w, and the maximum happens when $w = \psi(\zeta)$. It follows that

$$\sup_{w\in\mathbb{C}}|V_{(g,\psi)}k_w(z)| = \sup_{w\in\mathbb{C}}\left|\int_0^z g'(\zeta)e^{\frac{|\psi(\zeta)|^2}{2}}f_w(\zeta)d\zeta\right| = \left|\int_0^z g'(\zeta)e^{\frac{|\psi(\zeta)|^2}{2}}d\zeta\right| \leq h(z).$$

Now, if $q < \infty$, integration using (3.2) gives

$$\begin{aligned}\int_{\mathbb{C}}\left|\int_0^z g'(\zeta)e^{\frac{|\psi(\zeta)|^2}{2}}d\zeta\right|^q e^{-\frac{q|z|^2}{2}}dA(z) &\simeq \int_{\mathbb{C}}\frac{|g'(z)|^q}{(1+|z|)^q}e^{\frac{q|\psi(z)|^2}{2}}e^{-\frac{q|z|^2}{2}}dA(z)\\ &= \int_{\mathbb{C}}M^q_{(g,\psi)}(z)dA(z) \leq \int_{\mathbb{C}}h(z)^q e^{-\frac{q|z|^2}{2}}dA(z) < \infty.\end{aligned}$$

To prove (ii), in stead of the estimate in (3.16), we use

$$|f'(z)| \leq e^2(1+|z|)e^{\frac{|z|^2}{2}}\|f\|_p,$$

which follows from Cauchy integral formula, and (3.16), and argue as in part (i).

For Corollary 2.10, we set $\psi(z) = z$ and observe

$$\int_{\mathbb{C}}M^q_{(g,\psi)}(z)dA(z) = \int_{\mathbb{C}}|g'(z)|^q(1+|z|)^{-q}dA(z) < \infty$$

if and only if g' is a constant and $q > 2$.

Similarly for the operator J_g, we have

$$\int_{\mathbb{C}} \widetilde{M}^q_{(g,\psi)}(z)dA(z) = \int_{\mathbb{C}} |g(z)|^q dA(z) < \infty$$

only if g is identically zero.

Proof of Corollary 2.11

Suppose C_ψ is order bounded with a bound function h. Then using its relation with the integral operator and (3.16)

$$\begin{aligned} |J_{(1,\psi)} f(z)| &= |C_\psi f(z) - f(\psi(0))| \leq |C_\psi f(z)| + |f(\psi(0))| \\ &\leq h(z) + e^{\frac{|\psi(0)|^2}{2}} \|f\|_p \leq h(z) + e^{\frac{|\psi(0)|^2}{2}} =: h_1(z) \end{aligned}$$

almost for every $z \in \mathbb{C}$ and $f \in \mathcal{F}_p$ such that $\|f\|_p \leq 1$. Note that $h_1 \in L^q\left(\mathbb{C}, e^{-\frac{q}{2}|z|^2} dA(z)\right)$ since the function h belongs to it. It follows that $J_{(1,\psi)}$ is order bounded. Then by Theorem 2.9, the function

$$\widetilde{M}_{(1,\psi)}(z) = (1 + |\psi(z)|)(1 + |z|)^{-1} e^{\frac{|\psi(z)|^2}{2} - \frac{|z|^2}{2}} \tag{3.20}$$

belongs to L^q which further implies $\widetilde{M}_{(1,\psi)}$ is bounded. By Lemma 1.2, it follows that $\psi(z) = az + b$, $|a| \leq 1$. Now if $|a| = 1$, then a simplification shows $\widetilde{M}_{(1,\psi)}$ is not L^q integrable. Thus, the necessity of the condition $|a| < 1$ is proved. Conversely, suppose $\psi(z) = az + b$ and $|a| < 1$. We need to show that the resulting composition operator is order bounded. The assumption implies

$$\widetilde{M}_{(1,\psi)}(z) = (1 + |\psi(z)|)(1 + |z|)^{-1} e^{\frac{|\psi(z)|^2}{2} - \frac{|z|^2}{2}}$$

belongs to L^q. Then, by Theorem 2.9, the integral operator $J_{(1,\psi)}$ is order bounded with a bound function h_2. On the other hand,

$$\begin{aligned} |C_\psi f(z)| &= |J_{(1,\psi)} f(z) + f(\psi(0))| \leq |J_{(1,\psi)} f(z)| + |f(\psi(0))| \\ &\leq h_2(z) + e^{\frac{|b|^2}{2}} \|f\|_p \leq h_2(z) + e^{\frac{|b|^2}{2}} =: h_3(z) \end{aligned}$$

almost for every z in $\mathbb{C}$ and f in $\mathcal{F}_p$ such that $\|f\|_p \leq 1$. It is easy to see that h_3 is an order bound for C_ψ and hence the claim.

4 Some Discussions on the Main Results

In this section we discuss further some of the main results presented in Sect. 2. In [13, 14], it was showed that the operators in (1.2) are bounded from above if and only if certain Berezin-type integral transforms are bounded. Part (iii) of the condition in Theorem 2.3 asserts that such transforms play important role in determining when the operators have closed ranges and bounded below structures. Note that, this is also called the reproducing kernel thesis property; the property of having closed range or bounded below can be determined only from the operator's action on the kernel functions.

Condition (iv) of the theorem ensures every sampling set $G^{\epsilon}_{(g,\psi)}$ for the space is dense in the sense that for all z in $\mathbb{C}$, there exists a disc $D(z,r)$ which contains a positive measure subset of $G^{\epsilon}_{(g,\psi)}$. This condition is in addition independent of the Fock exponents p. Thus, if $V_{(g,\psi)}$ has a closed range on some Fock space $\mathcal{F}_p$, then it has closed range on all the other Fock spaces.

By Theorem 1.1, the boundedness property of $V_{(g,\psi)}$ is described by the boundedness of the function $M_{(g,\psi)}$ on the complex plane. Similarly, condition (v) of Theorem 2.3 asserts that the essential boundedness from below of $M_{(g,\psi)}$ completely characterizes the closed range and bounded below structures of the operator again. By the relation in (2.3), we also deduce that $V_{(g,\psi)}$ cannot be surjective on $\mathcal{F}_p$ for any choice of g and ψ in $\mathcal{H}(\mathbb{C})$.

We note in passing that by Theorems 1.1 and 2.3, the discussions made above applies to the operator $J_{(g,\psi)}$ as well. It is interesting to note that while the range of the nontrivial integral operator $J_{(g,\psi)}$ contain only functions in $\mathcal{F}_p$ which vanish at the origin, the extra term in the relation $C_\psi f = J_{(1,\psi)}f + f(\psi(0))$ can make the composition operator surjective as stated in Corollary 2.8

Theorem 2.6 is rather interesting in the sense that the conditions are simpler to apply than those listed in Theorems 2.3, 2.4 and 2.5. It is known that a compact operator on an infinite dimensional space cannot have closed range. By Theorem 1.1, compactness of the operators implies $\psi(z) = az+b$ such that $|a| < 1$. But the converse of this fails. For example set $\psi_1(z) = z/2$ and $g_1(z) = e^{\frac{3}{8}z^2}$ and apply Lemma 1.3 to observe that the operator $J_{(g_1,\psi_1)}$ is not compact. The same counterexample holds for $V_{(g,\psi)}$ by simply replacing g_1 by g_1'. In this regard, the interesting question was whether there exists closed range noncompact integral operators whenever $|a| < 1$. Theorem 2.6 answers the question negatively; ensuring that closed range happens only when a belongs to the unit circle.

Carleson measures have proved to be useful tools in the study of several operators. For Fock spaces, such measures were characterized in [9]. Now setting a pullback measure

$$\mu_{(g',\psi,p)}(E) := \int_{\psi^{-1}(E)} \big(|g'(z)|(1+|z|)^{-1}\big)^p e^{-\frac{p}{2}|z|^2} dA(z) \tag{4.1}$$

for every Borel subset E of $\mathbb{C}$and applying (3.2), we note that for each f in $\mathcal{F}_p$

$$\begin{aligned} \|V_{(g,\psi)}f\|_p^p &\simeq \int_{\mathbb{C}} \big(|g'(z)|(1+|z|)^{-1}\big)^p |f(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z) \\ &\simeq \int_{\mathbb{C}} |f(z)|^p d\mu_{(g',\psi,p)}(z), \end{aligned}$$

where

$$d\mu_{(g',\psi,p)}(z) = \frac{|g'(\psi^{-1}(z))|^p e^{-\frac{p}{2}|(\psi^{-1}(z))|^2}}{(1+|\psi^{-1}(z))|)^p} dA(\psi^{-1}(z))$$

and $\psi^{-1}(z) = (z-b)/a$. It follows that $V_{(g,\psi)}$ is bounded on $\mathcal{F}_p$ if and only if $\mu_{(g,\psi,p)}$ is a $(p,0)$ Fock-Carleson measure. By replacing g' by g in (4.1), one can also deduce that $J_{(g,\psi)}$ is bounded on $\mathcal{F}_p$ if and only if $\mu_{(g,\psi,p)}$ is a $(p,1)$ Fock-Carleson measure. Similarly, Theorem 2.5 describes the closed range and bounded below properties of both $V_{(g,\psi)}$ and $J_{(g,\psi)}$ in terms of the notion of reverse Fock-Carleson measures.

We now turn to Theorem 2.9. Like the other main results, the order boundedness of the integral operators are characterized in terms of the functions in (1.3). While the closed range conditions require $\psi(z) = az + b$ with $|a| = 1$, on the contrary order boundedness happens only when $|a| < 1$. Clearly, the conditions in the theorem imply the corresponding conditions in Theorem 1.1. Indeed, if both p and q are finite, then order boundedness implies the stronger compactness conditions as well. On the other hand, if at least one of the exponents p or q is infinite, then the conditions in the two theorems coincide. Furthermore, unlike the conditions in Theorem 1.1, the order boundedness condition is independent of whether $p \le q$ or $p > q$. Corollary 2.10, which is a special case of Theorem 2.9, ensures that the operator $V_g : \mathcal{F}_p \to \mathcal{F}_p$ is order bounded if and only if g is a polynomial of at most degree one, and $p > 2$. By [5, Thoeorem 2], this condition coincides with the Schatten $\mathcal{S}_p$ class membership characterization of the operator when it acts on the Hilbert space $\mathcal{F}_2$. In this context, the requirement $p > 2$ implies the operator fails to be Hilbert–Schmidt. Therefore, not every compact V_g is order bounded. Another consequence of Theorem 1.1 is Corollary 2.11 which together with [4, Proposition 3] implies $C_\psi : \mathcal{F}_p \to \mathcal{F}_q$ is order bounded if and only if it is compact.

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