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Original Articles

Convergence of a fitted finite volume method for pricing two dimensional assets with stochastic volatilities

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Abstract

In this article, we provide the rigorous mathematical convergence proof both in space and time of the two dimensional Black Scholes equation with stochastic volatility. The spatial approximation of this three dimensional problem is performed using the finite volume method coupled with a fitted technique to tackle the degeneracy in the Black Scholes operator, while the temporal discretization is performed using implicit Euler method. We provide a mathematical rigorous convergence proof in space and time of the full discretized scheme. Numerical results are presented to validate our theoretical results.

Keywords: Stochastic volatility; Black–Scholes equation; Fitted finite volume method; Finite element method; Stability and convergence

1. Introduction

In financial markets many different products are traded. Focusing on the foreign exchange market, we are basically faced with exchange rates of different currency pairs and derivative products which depend on the underlying rates. The most common examples are swaps, forwards and different kind of options [10]. At time of maturity T, the value of a call option for instance is by contract equal to the value of an underlying minus a predefined so called strike value if the underlying exceeds the strike and otherwise zero, so the price of the option is based on the value of the portfolio at the beginning of the contract. For an accurate assessment of the price it is essential to have a realistic stochastic model. Black and Scholes [4] in 1973 proposed a simple model based on geometric Brownian motion and derived options prices under no arbitrage principle. Their work was a breakthrough and the results are still widely used. Since then, several improvements and extensions were suggested in order to obtain a more realistic model and hence more accurate option prices [7,11]. We concentrate on stochastic volatility models [11] where as the name suggests the volatility is not constant like in the Black Scholes model but a stochastic process itself and examine the numerical approximation as in many cases the exact solution is not available. The first numerical approach to Black Scholes equations is the lattice technique [5,12]. This approach is equivalent to an explicit time-stepping scheme, and therefore is very unstable. Other numerical schemes based on classical finite

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difference methods applied to heat equations with constant coefficients have also been developed [2,3]. Since the Black Scholes equation is degenerated, classical finite difference methods may fail to give accurate approximations when the asset is small. To overcome this drawback, some authors [19,20] suggest to solve the differential equation in a truncated space interval excluding the singularity points. In [20], S. Wang presents a novel discretization method for the Black–Scholes based on a finite volume formulation coupled with a fitted local approximation to handle the degeneracy of the operator. In Huang et al. [8] the authors proposed a fitted finite volume method for the one dimensional Black–Scholes equation in Hull–White's model of stochastic volatility. In [9], the authors studied the spatial convergence of the one dimensional Black Scholes in Hull–White's model of stochastic volatility. Note that to the best of our knowledge only full convergence proof of the non stationary case has been presented in [19] for one dimensional options as the proof in [9] is only for stationary (independent of time) two dimensional Black Scholes model. Recently three dimensional fitted finite volume method has been extended in [18] for Hamilton–Jacobi–Bellman equation. The local approximation is determined by a set of two-point boundary values problems defined on the element edges.

Here, we applied the fitted finite volume method [18] to three dimensional problem, which is indeed the two dimensional Black–Scholes Hull–White's stochastic volatility model. Furthermore, we provide the mathematical convergence proof of the fully discretization where the time discretization is performed using the implicit Euler method. Furthermore, we prove that the discrete Black Scholes operator obtained after the fitted finite volume space discretization is an M-matrix. To analyze the method, we formulate the scheme as a finite element method in which each of the basis functions of the trial space is determined by a set of two-point boundary value problems defined on element edges. Using this formulation, we establish the stability of the method with respect to a discrete energy norm, and show that the error of the numerical solution in the energy norm is bounded by $\mathcal{O}(h+\Delta\tau)$, where h denotes the mesh parameter and $\Delta\tau$ the time discretization stepsize.

The rest of this article is organized as follows. In Section 2, preliminaries and mathematical setting are provided and the proof of theoretical results on existence and uniqueness of the continuous problem. In Section 3, we develop the finite element framework for the fitted finite volume method. The coercivity proofs of the corresponding discrete bilinear form are also provided to ensure the existence and uniqueness of the discrete solution. In Section 4, fully discretization scheme is provided along with the proof of our main theorem based on convergence of the numerical solution toward the continuous solution. We present some numerical results to demonstrate the convergence of the numerical scheme. Finally, our findings are summarized in Section 6

2. Preliminaries and mathematical setting

Two assets European option pricing with a stochastic volatility developed in [11] is given by

$$\begin{aligned}&\frac{\partial f}{\partial t}+\frac{1}{2}x^2 z\frac{\partial^2 f}{\partial x^2}+\frac{1}{2}y^2 z\frac{\partial^2 f}{\partial y^2}+\frac{1}{2}z^2\xi^2\frac{\partial^2 f}{\partial z^2}+z\,x\,y\rho_{1,2}\frac{\partial^2 f}{\partial x\partial y}\\&+z^{3/2}x\,\xi\rho_{1,3}\frac{\partial^2 f}{\partial x\partial z}+z^{3/2}y\,\xi\rho_{2,3}\frac{\partial^2 f}{\partial y\partial z}+r\,x\frac{\partial f}{\partial x}+r\,y\frac{\partial f}{\partial y}+\mu\,z\frac{\partial f}{\partial z}-r\,f=0\end{aligned}\tag{1}$$

on $\Omega=[0,X]\times[0,Y]\times[\zeta,Z]$,where $\sqrt{z}=\sigma$ and T the expiry date.

We will use the following final condition and Dirichlet boundary conditions

$$\begin{aligned}&f(T,x,y,z)=f_T(x,y,z),\ \ (x,y,z)\in\Omega,\\&f(t,x,y,z)=f_D(t,x,y,z),\ \ (t,x,y,z)\in[0,T]\times\partial\Omega.\end{aligned}$$

Note that x, y denotes the price of the underlying stock, ξ and μ are constants from the stochastic process governing the variance z, $\rho_{1,2}$ is the instantaneous correlation between x_t[1] and y_t, $\rho_{1,3}$ is the instantaneous correlation between x_t and z_t, $\rho_{2,3}$ is the instantaneous correlation between y_t and z_t, X, ζ, Y, Z and T are positive constants. Note that the volatility of the assets $\sigma=\sigma(t)$ and the interest rate $r=r(t)$ depend of t. The function $f_T(x,y,z)$ is the payoff and $f_D(t)$ the Dirichlet boundary condition value. The independent variable z is strictly positive ($z>0$). However the case that $z=0$ is trivial because it means that the volatility of the stock is zero in the market. The stock then becomes deterministic, which is impossible unless the stock is a risk-less asset. In this

[1] This is indeed the stochastic price of the asset x at time t.

case the price of the option is deterministic. Therefore it is reasonable to assume that $z > \zeta$ for a given (small) positive constant ζ. The solution domain of the above problem contains six boundary conditions surfaces defined by $x = 0,\ x = X,\ y = 0,\ y = Y,\ z = \zeta, z = Z$. For the boundary condition at $z = \zeta$ and $z = Z$, we need to solve the PDE in (1) by taking $\xi = \mu = 0$ for two particular values $\sigma = \sqrt{\zeta}$ and $\sigma = \sqrt{Z}$ with the boundary and final conditions defined above. By using the changing variable $f := v\, e^{\alpha t}$ where α is a positive arbitrary constant, and setting $t = T - \tau$ Eq. (1) becomes

$$\begin{aligned}&\frac{\partial v}{\partial \tau} - \frac{1}{2}x^2 z\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}y^2 z\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}z^2\xi^2\frac{\partial^2 v}{\partial z^2} - z\, x\, y\rho_{1,2}\frac{\partial^2 v}{\partial x\partial y} \\ &- z^{3/2}x\,\xi\rho_{1,3}\frac{\partial^2 v}{\partial x\partial z} - z^{3/2}y\,\xi\rho_{2,3}\frac{\partial^2 v}{\partial y\partial z} - r\,x\frac{\partial v}{\partial x} - r\,y\frac{\partial v}{\partial y} - \mu\, z\frac{\partial v}{\partial z} + (r+\alpha)\,v = 0.\end{aligned} \tag{2}$$

By including the boundary condition, (2) can be written as the following inhomogeneous divergence form

$$\frac{\partial v(\tau,x,y,z)}{\partial \tau} - \nabla\cdot(k(v(\tau,x,y,z))) + c\, v(\tau,x,y,z) = g, \tag{3}$$

where g is a known function coming from the contribution of the boundary condition, $k(v(\tau,x,y,z)) = A\,\nabla v(\tau,x,y,z) + b\, v(\tau,x,y,z)$ is the flux, $b = (x\, b_1, y\, b_2, z\, b_3)^T$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix},$$

with

$$\begin{aligned}&a_{11} = \frac{1}{2}\sigma^2 x^2,\ a_{22} = \frac{1}{2}\sigma^2 y^2,\ a_{33} = \frac{1}{2}\xi^2 z^2,\\ &a_{12} = a_{21} = \frac{1}{2}z\,x\,y\rho_{1,2},\ a_{13} = a_{31} = \frac{1}{2}z^{\frac{3}{2}}x\,\xi\rho_{1,3},\ a_{23} = a_{32} = \frac{1}{2}z^{\frac{3}{2}}y\,\xi\rho_{2,3}.\end{aligned} \tag{4}$$

$$\begin{cases} b_1 = r - \frac{1}{2}z\,\rho_{1,2} - \frac{3}{4}z^{\frac{1}{2}}\,\xi\rho_{1,3} - z,\\ b_2 = r - \frac{1}{2}z\,\rho_{1,2} - \frac{3}{4}z^{\frac{1}{2}}\,\xi\rho_{2,3} - z,\\ b_3 = \mu\ - \xi^2 - \frac{1}{2}z^{\frac{1}{2}}\,\xi\rho_{1,3} - \frac{1}{2}z^{\frac{1}{2}}\,\xi\rho_{2,3}\\ c = 3\,r + \alpha + \mu - \xi^2 - z\,\rho_{1,2} - \frac{3}{2}z^{\frac{1}{2}}\,\xi\rho_{1,3} - \frac{3}{2}z^{\frac{1}{2}}\,\xi\rho_{2,3} - 2\,z.\end{cases} \tag{5}$$

Let $L^p(\Omega)$ be the space of all p-integral functions on Ω for $p > 1$ with usual modification for $p = \infty$. When p = 2, we denote the inner product on $L^2(\Omega)$ by $(v,u) = \int_\Omega u\,v\,d\Omega$ and the norm $\|v\|_0^2 := \int_\Omega v^2\,d\Omega$. Let $l = 1,2,\ldots$, $W_p^l(\Omega)$ denote the usual Sobolev space such that for $p = 2$ we simply denote $W_2^l(\Omega)$ by $H^l(\Omega)$, $|\cdot|_l$ and $\|\cdot\|_l$ respectively the semi norm and norm on $H^l(\Omega)$. We also use $\|\cdot\|_{0,\infty,\Omega} := \|\cdot\|_{0,\infty}$ to denote the L^∞ - norm on Ω and $\|v\|_{1,\infty,\Omega} := \|v\|_{1,\infty}$ to denote the sup-norm of ∇v on the (open) set Ω. We shall use the standard notation for function spaces $C^m(\Omega)$ and $C^m(\overline{\Omega})$ of which a function and its derivatives up to order m are continuous on Ω(Resp. $\overline{\Omega}$). For any Hilbert space $H(\Omega)$ of classes of functions defined on Ω, we denote

$$L^2(0,T;H(\Omega)) = \left\{v(\tau,\cdot):\ v(\tau,\cdot)\in H(\Omega)\,\text{a.e in}\,[0,T];\ \|v(\tau,\cdot)\|_H\ \in\ L^2([0,T])\right\}. \tag{6}$$

Clearly, $L^2(0,T;L^2(\Omega)) = L^2([0,T]\times\Omega)$. To handle the degeneracy in the Black–Scholes equation we introduce a weighted inner product on $L^2(\Omega)$ by $(u,v)_{\hat{w}} = \int_\Omega(z\,x^2\,u_1\,v_1 + z\,y^2\,u_2\,v_2 + z^2\,u_3\,v_3)d\Omega$ for any $u = (u_1,u_2,u_3)^T$ and $v = (v_1,v_2,v_3)^T\ \in\ \left(L^2(\Omega)\right)^3$. The corresponding weighted L^2-norm is given by

$$\|v\|_{0,\hat{w}} = \left(\int_\Omega z\,x^2\,v_1^2 + z\,y^2\,v_2^2 + z^2\,v_3^2\,d\Omega\right)^{1/2}. \tag{7}$$

The space of all weighted square-integrable functions defined as

$$L^2_{\hat{w}}(\Omega) := \left\{v\in\ \left(L^2(\Omega)\right)^3;\ \|v\|_{0,\hat{w}} < \infty\right\}, \tag{8}$$

is a Hilbert space according to [15]. Using $L^2(\Omega)$ and $L^2_{\hat{w}}(\Omega)$, we define the weighted Sobolev space $H^1_{\hat{w}}(\Omega)$, by

$$H^1_{\hat{w}}(\Omega) := \left\{v:\ v\ \in\ L^2(\Omega), \nabla v\ \in\ L^2_{\hat{w}}(\Omega)\right\},$$

where the derivative in ∇v is understood in weak sense. Furthermore, using the inner product in $L^2(\Omega)$ and $L^2_{\hat{w}}(\Omega)$, we define a weighted inner product on $H^1_{\hat{w}}(\Omega)$ by

$$(u, v)_{H^1_{\hat{w}}(\Omega)} := (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2_{\hat{w}}(\Omega)}$$

which corresponds to the norm

$$\|v\|^2_{1,\hat{w}} = \|v\|^2_0 + \|\nabla v\|^2_{0,\hat{w}}.$$

Following again [15], one can prove that $\left(H^1_{\hat{w}}(\Omega), (\cdot,\cdot)_{H^1_{\hat{w}}(\Omega)}\right)$ is a Hilbert space.

Let $\partial\Omega_D = \{(x, y, z) \in \partial\Omega : x \neq 0,\ y \neq 0\}$ denote the boundary segments of Ω with $x = X,\ y = Y,\ z = \zeta$ and $z = Z$. Let

$$H^1_{0,\hat{w}}(\Omega) := \left\{v : v \in H^1_{\hat{w}}(\Omega) \text{ and } v|_{\partial\Omega_D} = 0\right\},$$

The weak formulation of our problem (2) is given by

Problem 2.1. Find $v(\tau) \in H^1_{0,\hat{w}}(\Omega)$ such that for all $u \in H^1_{0,\hat{w}}(\Omega)$,

$$\left(\frac{\partial v(\tau)}{\partial \tau}, u\right) + B(v(\tau), u; \tau) = (g, u), \tag{9}$$

where $B(\cdot,\cdot;\tau)$ is a bilinear form defined by

$$B(v(\tau), u; \tau) = (A\,\nabla v + b\,v,\ \nabla u) + (c\,v, u)\,. \tag{10}$$

To prove the existence and uniqueness of Problem 2.1, we need the following assumption and lemma.

Assumption 2.1. We assume that

$$\rho_{i,j} \geq 0 \quad\text{and}\quad \sum_{i\neq j} \rho_{i,j} \in [0, 1). \tag{11}$$

Lemma 2.1. *Let $v, u \in H^1_{0,\hat{w}}(\Omega)$. Then*

1- There exists positive constant M, independent of v and u such that,

$$|B(v, u; \tau)| \leq M\|v\|_{H^1_{0,\hat{w}}(\Omega)}\|u\|_{H^1_{0,\hat{w}}(\Omega)}. \tag{12}$$

2- There exists positive constant γ such that

$$|B(v, v; \tau)| \geq \gamma\|v\|^2_{H^1_{0,\hat{w}}(\Omega)}, \tag{13}$$

uniformly with respect to $\tau \in [0, T]$.

Proof.

1. For any $v, u \in H^1_{0,\hat{w}}(\Omega)$ we have

$$|B(v, u; \tau)| \leq |\,(A\,\nabla v, \nabla u)\,| + |\,(b\,v, \nabla u)\,| + |\,(c\,v, u)\,| \tag{14}$$

For the term $|\,(A\,\nabla v, \nabla u)\,|$ in (14) we split it into two parts as follows:

$$\begin{aligned} |\,(A\,\nabla v, \nabla u)\,| \leq &\left|\int_\Omega \frac{1}{2} z\,x^2\,v_x u_x + \frac{1}{2} y^2\,z\,v_y u_y + \frac{1}{2}\xi^2\,z^2\,z\,v_z u_z d\Omega\right| \\ &+ \left|\int_\Omega \frac{1}{2} z\,x\,y\,\rho_{1,2}(v_x\,u_y + u_x\,v_y) + \frac{1}{2}\,x\,z^{\frac{3}{2}}\,\xi\,\rho_{1,3}(v_x\,u_z + u_x\,v_z)\right. \\ &\left.+\frac{1}{2}\,y\,z^{\frac{3}{2}}\,\xi\,\rho_{2,3}(v_y\,u_z + u_y\,v_z) d\Omega\right|. \end{aligned} \tag{15}$$

For the first term on the right-hand side of (15), by Cauchy–Schwarz inequality we obtain

$$\left|\int_\Omega \frac{1}{2} z\, x^2\, v_x u_x + \frac{1}{2} y^2\, z\, v_y u_y + \frac{1}{2}\xi^2\, z^2\, v_z u_z d\Omega\right| \tag{16}$$
$$\leq \frac{1}{2}\left[\int_\Omega z\, x^2\, v_x^2 + z\, y^2\, v_y^2 + \xi^2\, z^2\, v_z^2\, d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega z\, x^2\, u_x^2 + z\, y^2\, u_y^2 + \xi^2\, z^2\, u_z^2\, d\Omega\right]^{\frac{1}{2}}$$
$$\leq M\, \|v\|_{H^1_{0,\hat{w}}(\Omega)} \cdot \|u\|_{H^1_{0,\hat{w}}(\Omega)}.$$

For the second term,

$$\left|\int_\Omega \frac{1}{2} z\, x\, y\, \rho_{1,2}(v_x\, u_y + u_x\, v_y) + \frac{1}{2}\, x\, z^{\frac{3}{2}}\, \xi\, \rho_{1,3}(v_x\, u_z + u_x\, v_z)\right.$$
$$\left. + \frac{1}{2}\, y\, z^{\frac{3}{2}}\, \xi\, \rho_{2,3}(v_y\, u_z + u_y\, v_z) d\Omega\right| \leq \left|\int_\Omega \frac{1}{2} z\, x\, y\, \rho_{1,2}(v_x\, u_y + u_x\, v_y) d\Omega\right|$$
$$+ \left|\int_\Omega \frac{1}{2}\, x\, z^{\frac{3}{2}}\, \xi\, \rho_{1,3}(v_x\, u_z + u_x\, v_z) d\Omega\right| + \left|\int_\Omega \frac{1}{2}\, y\, z^{\frac{3}{2}}\, \xi\, \rho_{2,3}(v_y\, u_z + u_y\, v_z) d\Omega\right|,$$

so

$$\left|\int_\Omega \frac{1}{2} z\, x\, y\, \rho_{1,2}(v_x\, u_y + u_x\, v_y) d\Omega\right|$$
$$\leq \left|\int_\Omega \frac{1}{2} z\, x\, y\, \rho_{1,2}\, v_x\, u_y\, d\Omega\right| + \left|\int_\Omega \frac{1}{2} z\, x\, y\, \rho_{1,2}\, u_x\, v_y d\Omega\right|$$
$$= \frac{\rho_{1,2}}{2}\left|\left(x\, z^{1/2}\, v_x,\, y\, z^{1/2}\, u_y\right)\right| + \frac{\rho_{1,2}}{2}\left|\left(y\, z^{1/2}\, v_y,\, x\, z^{1/2}\, u_x\right)\right|$$
$$\leq \frac{\rho_{1,2}}{2}\left[\int_\Omega z\, x^2\, v_x^2\, d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega z\, y^2\, u_y^2\, d\Omega\right]^{\frac{1}{2}}$$
$$+ \frac{\rho_{1,2}}{2}\left[\int_\Omega z\, y^2\, v_y^2\, d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega z\, x^2\, u_x^2\, d\Omega\right]^{\frac{1}{2}}$$

Using the same technique yields

$$\left|\int_\Omega \frac{1}{2}\Big(z\, x\, y\, \rho_{1,2}(v_x\, u_y + u_x\, v_y) + x\, z^{\frac{3}{2}}\, \xi\, \rho_{1,3}(v_x\, u_z + u_x\, v_z)\right.$$
$$\left. + y\, z^{\frac{3}{2}}\, \xi\, \rho_{2,3}(v_y\, u_z + u_y\, v_z)\Big) d\Omega\right|.$$

$$\leq M\left[\int_\Omega (z\, x^2\, v_x^2 + z\, y^2\, v_y^2 + \xi^2\, z^2\, v_z^2)\, d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega (z\, x^2\, u_x^2 + z\, y^2\, u_y^2 + \xi^2\, z^2\, u_z^2)\, d\Omega\right]^{\frac{1}{2}} \tag{17}$$
$$+ M\left[\int_\Omega (z\, x^2\, v_x^2 + z\, y^2\, v_y^2 + \xi^2\, z^2\, v_z^2)\, d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega (z\, x^2\, u_x^2 + z\, y^2\, u_y^2 + \xi^2\, z^2\, u_z^2)\, d\Omega\right]^{\frac{1}{2}}$$
$$\leq M\, \|v\|_{H^1_{0,\hat{w}}(\Omega)} \cdot \|u\|_{H^1_{0,\hat{w}}(\Omega)}.$$

Thus combining (16) and (17). We obtain

$$|\,(A\, \nabla v, \nabla u)\,| \leq M\, \|v\|_{H^1_{0,\hat{w}}(\Omega)} \cdot \|u\|_{H^1_{0,\hat{w}}(\Omega)}. \tag{18}$$

For $|(b\, v, \nabla u)|$, using integration by parts, we have

$$\left|\int_\Omega bv \cdot \nabla u\, d\Omega\right| = \left|\underbrace{\int_{\partial\Omega} b\, v\, u \cdot n\, ds}_{0} - \int_\Omega u\, \nabla \cdot (b\, v)\, d\Omega\right|$$
$$\leq M\, \|v\|_0 \|u\|_0 + \left|\int_\Omega u\, b \cdot \nabla v\, d\Omega\right|.$$

Furthermore, since $b = (x\, b_1, y\, b_2, z\, b_3)^T$, by Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_\Omega u\, b\cdot\nabla v\, d\Omega\right| &\le \left|\int_\Omega u\, x\, v_x \left[r - \frac{1}{2} z\,\rho_{1,2} - \frac{3}{4} z^{\frac{1}{2}}\,\xi\rho_{1,3} - z\right] d\Omega\right| \\ &+ \left|\int_\Omega u\, y\, v_y \left[r - \frac{1}{2} z\,\rho_{1,2} - \frac{3}{4} z^{\frac{1}{2}}\,\xi\rho_{2,3} - z\right] d\Omega\right| \\ &+ \left|\int_\Omega u\, z\, v_z \left[\mu - \xi^2 - \frac{1}{2} z^{\frac{1}{2}}\,\xi\rho_{1,3} - \frac{1}{2} z^{\frac{1}{2}}\,\xi\rho_{2,3}\right] d\Omega\right| \\ &\le M\left[|(x\, v_x, u)| + \left|(y\, v_y, u)\right| + |(z\, v_z, u)|\right] \\ &\le M\left[\int_\Omega x^2 v_x^2 + y^2 v_y^2 + z^2 v_z^2\, d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega u^2 + u^2 + u^2\, d\Omega\right]^{\frac{1}{2}} \\ &\le M\,\|v\|_{H^1_{0,\hat{w}}(\Omega)} \cdot \|u\|_{H^1_{0,\hat{w}}(\Omega)}.\end{aligned}$$

Therefore we have

$$\left|\int_\Omega u\, b\cdot\nabla v\, d\Omega\right| \le M\,\|v\|_{H^1_{0,\hat{w}}(\Omega)} \cdot \|u\|_{H^1_{0,\hat{w}}(\Omega)} \tag{19}$$

$$|(c\, v, u)| \le M\,\|v\|_0 \|u\|_0. \tag{20}$$

2. For any $v \in H^1_{0,\hat{w}}(\Omega)$

$$\begin{aligned}\int_\Omega b\, v\cdot\nabla v\, d\Omega &= \frac{1}{2}\int_{\partial\Omega} v^2\, b\cdot\mathbf{n}\, ds - \frac{1}{2}\int_\Omega v^2\,\nabla\cdot b\, d\Omega \\ &= -\frac{1}{2}\int_\Omega v^2\,\cdot\nabla\cdot b\, d\Omega\end{aligned} \tag{21}$$

since $b\cdot\mathbf{n} = 0$ on $\partial\Omega\backslash\partial\Omega_D$ and $v = 0$ on $\partial\Omega_D$. Using (21), we have

$$\begin{aligned}B(v, v; \tau) &= (A\,\nabla v, \nabla v) + (b\, v, \nabla v) + (c\, v, v) \\ &= (A\,\nabla v, \nabla v) + \left(\left(c - \frac{1}{2}\nabla\cdot b\right) v, v\right),\end{aligned}$$

$$\begin{aligned}&\left(\left(c - \frac{1}{2}\nabla\cdot b\right) v, v\right) \\ &= \left(\left[2\, r + \alpha + \frac{\mu}{2} - \frac{\xi^2}{2} - \frac{1}{2} z\,\rho_{1,2} - \frac{3}{4}\sigma\,\xi\rho_{1,3} - \frac{3}{4} z^{1/2}\,\xi\rho_{2,3} - z\right] v, v\right) \\ &\ge K\,\|v\|^2_{L^2(\Omega)},\end{aligned}$$

with $K = K(\alpha) > 0$, since α is arbitrary. Assumption 2.1, we also have

$$\begin{aligned}&\frac{1}{2}\left[\int_\Omega z\,(x^2 v_x^2 + y^2 v_y^2 + 2\, x\, y\,\rho_{1,2}\, v_x v_y) + \xi^2 z^2 v_z^2 + 2\, x\, z^{\frac{3}{2}}\,\xi\,\rho_{1,3}\, v_x\, v_z \right.\\ &\left. + 2\, y\, z^{\frac{3}{2}}\,\xi\,\rho_{2,3}\, v_y\, v_z\right] d\Omega \\ &= \frac{1}{2}\int_\Omega (1 - \rho_{1,2} - \rho_{1,3}) z\, x^2 v_x^2 + (1 - \rho_{1,2} - \rho_{2,3})\, z\, y^2 v_y^2 + (1 - \rho_{1,3} - \rho_{2,3})\xi^2 z^2 v_z^2\, d\Omega \\ &+ \frac{1}{2}\int_\Omega \rho_{1,3}(z^{1/2}\, x\, v_x + \xi\, z\, v_z)^2 + \rho_{2,3}(z^{1/2}\, y\, v_y + \xi\, z\, v_z)^2 + \rho_{1,2}(z^{1/2}\, x\, v_x + z^{1/2}\, y\, v_y)^2\, d\Omega \\ &\ge C\,\|\nabla v\|^2_{L^2_{\hat{w}}(\Omega)}.\end{aligned}$$

We finally have

$$\begin{aligned}|B(v, v, \tau)| &\ge C\,\|\nabla v\|^2_{L^2_{\hat{w}}(\Omega)} + K\|v\|^2_{L^2(\Omega)} \\ &\ge \gamma\|v\|^2_{H^1_{0,\hat{w}}(\Omega)}. \quad \blacksquare\end{aligned}$$

Theorem 2.1. *Under Assumption 2.1, the Problem 2.1 has a unique solution.*

Proof. The proof uses the previous Lemma and can be found in [6, Theorem 1.33 page 40]. ■

3. The finite element formulation of the fitted scheme and its consistency

3.1. Representation of the projection of the exact solution on the fitted finite volume grid

We are now discussing the finite element formulation for the discretization scheme. We follow the works developed in [9,16,17] with the difference that we are working here in the three dimensional Black Scholes model. As usual, we truncate the problem in the finite interval $I_x = [0, x_{\max}]$, $I_y = [0, y_{\max}]$ and $I_z = [\zeta, z_{\max}]$. Let the intervals $I_x = [0, x_{\max}]$, $I_y = [0, y_{\max}]$ and $I_z = [\zeta, z_{\max}]$ be divided into N_1, N_2 and N_3 sub-intervals:

$$I_{x_i} := (x_i, x_{i+1}),\ \ I_{y_j} := (y_j, y_{j+1}),\ \ I_{z_k} := (z_k, z_{k+1}),$$

$i = 0 \cdots N_1 - 1$, $j = 0 \cdots N_2 - 1$, $k = 0 \cdots N_3 - 1$ with $0 = x_0 < x_1 < \cdots\cdots < x_{N_1} = x_{\max}$, $0 = y_0 < y_1 < \cdots\cdots < y_{N_2} = y_{\max}$ and $\zeta = z_0 < z_1 < \cdots\cdots < z_{N_3} = z_{\max}$. We also let

$$\begin{aligned} x_{i+1/2} &= \frac{x_i + x_{i+1}}{2},\ \ x_{i-1/2} = \frac{x_i + x_{i-1}}{2},\ \ y_{j+1/2} = \frac{y_j + y_{j+1}}{2},\\ y_{j-1/2} &= \frac{y_j + y_{j-1}}{2},\ \ z_{k+1/2} = \frac{z_k + z_{k+1}}{2},\ \ z_{k-1/2} = \frac{z_k + z_{k-1}}{2}, \end{aligned} \tag{22}$$

for each $i = 1 \cdots N_1 - 1$ $\ j = 1 \cdots N_2 - 1$ and each $k = 1 \cdots N_3 - 1$. These mid-points form a second partition of $I_x \times I_y \times I_z$ if we define $x_{-1/2} = x_0$, $x_{N_1+1/2} = x_{\max}$, $y_{-1/2} = y_0$, $y_{N_2+1/2} = y_{\max}$ and $z_{-1/2} = z_0$, $z_{N_3+1/2} = z_{\max}$. For each $i = 0, 1, \ldots, N_1$, $\ j = 0, 1, \ldots, N_2$ and $k = 0, 1, \ldots, N_3$, we define

$$h_{x_i} = x_{i+1/2} - x_{i-1/2},\ \ h_{y_j} = y_{j+1/2} - y_{j-1/2} \tag{23}$$

$$h_{z_k} = z_{k+1/2} - z_{k-1/2},\ \ h = \max_{i,j,k}\{h_{x_i}, h_{y_j}, h_{z_k}\}, \tag{24}$$

$$\mathcal{R}_{i,j,k} = \left[x_{i-1/2}, x_{i+1/2}\right] \times \left[y_{j-1/2}, y_{j+1/2}\right] \times \left[z_{k-1/2}, z_{k+1/2}\right]. \tag{25}$$

Integrating both size of (3) over $\mathcal{R}_{i,j,k}$, $i = 1, \ldots, N_1 - 1$, $\ j = 1, \ldots, N_2 - 1$ and $k = 1, \ldots, N_3 - 1$ we have $(N_1 - 1) \times (N_2 - 1) \times (N_3 - 1)$ balance equations

$$\int_{\mathcal{R}_{i,j,k}} \frac{\partial v}{\partial \tau}\, dx\, dy\, dz + \left[-\int_{\mathcal{R}_{i,j,k}} \nabla \cdot (k(v))\, dx\, dy\, dz + \int_{\mathcal{R}_{i,j,k}} c\, v\, dx\, dy\, dz \right] = \int_{\mathcal{R}_{i,j,k}} g\, dx\, dy\, dz. \tag{26}$$

Multiplying the $\{i, j, k\}$th Eq. (26) with an arbitrary real number, say $u_{i,j,k}$, and adding the results, Ostrogradski Theorem, integrating by parts and using the definition of flux $k(v)$ where $\mathbf{n}$ denote the unit vector outward-normal to $\partial\mathcal{R}_{i,j,k}$, we have

$$\begin{aligned} &\sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} \sum_{k=1}^{N_3-1} \int_{\mathcal{R}_{i,j,k}} \frac{\partial v}{\partial \tau} u_{i,j,k}\, dx\, dy\, dz - \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} \sum_{k=1}^{N_3-1} \left(\int_{\partial\mathcal{R}_{i,j,k}} k(v) \cdot \mathbf{n}\, ds \right) u_{i,j,k} \\ &+ \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} \sum_{k=1}^{N_3-1} \int_{\mathcal{R}_{i,j,k}} c\, v\, u_{i,j,k}\, dx\, dy\, dz = \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} \sum_{k=1}^{N_3-1} \int_{\mathcal{R}_{i,j,k}} g\, u_{i,j,k}\, dx\, dy\, dz. \end{aligned} \tag{27}$$

For any arbitrary function $u \in C^0(\overline{\Omega})$, we define the mass lumping operator P from $C^0(\overline{\Omega})$ to $L^\infty(\Omega)$ by

$$P(u)\big|_{\mathcal{R}_{i,j,k}} := u(x_i, y_j, z_k),\ \ i = 0, 1 \cdots N_1 - 1\ \ j = 0, 1 \cdots N_2 - 1,\ \ k = 0, 1 \cdots N_3 - 1.$$

If in addition, the function u satisfies homogeneous Dirichlet boundary conditions, we have $P(u)|_{\partial\Omega_D} = 0$. Therefore, using the mass lumping P, (27) can be written as follows:

$$\left(\frac{\partial v}{\partial \tau}, P(u) \right) + \hat{B}_h(v, P(u); \tau) = (g, P(u)), \quad \text{where} \tag{28}$$

$$\hat{B}_h(v, P(u); \tau) =$$

$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(\bar{a_1}\,x\,\frac{\partial v}{\partial x}+b_1\,v\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{(x_{i-1/2},y_j,z_k)}^{(x_{i+1/2},y_j,z_k)} P(u(x_i,y_j,z_k))$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[y\left(\bar{a_2}\,y\,\frac{\partial v}{\partial y}+b_2\,v\right)\cdot h_{x_i}\cdot h_{z_k}\right]_{(x_i,y_{j-1/2},z_k)}^{(x_i,y_{j+1/2},z_k)} P(u(x_i,y_j,z_k))$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[z\left(\bar{a_3}\,z\,\frac{\partial v}{\partial z}+b_3\,v\right)\cdot h_{y_j}\cdot h_{x_i}\right]_{(xi,y_j,z_{k-1/2})}^{(x_i,y_j,z_{k+1/2})} P(u(x_i,y_j,z_k))$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(d_1\,\frac{\partial v}{\partial y}+d_2\,\frac{\partial v}{\partial z}\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{(x_{i-1/2},y_j,z_k)}^{(x_{i+1/2},y_j,z_k)} P(u(x_i,y_j,z_k))$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[y\left(d_3\,\frac{\partial v}{\partial x}+d_4\,\frac{\partial v}{\partial z}\right)\cdot h_{x_i}\cdot h_{z_k}\right]_{(x_i,y_{j-1/2},z_k)}^{(x_i,y_{j+1/2},z_k)} P(u(x_i,y_j,z_k))$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[z\left(d_5\,\frac{\partial v}{\partial x}+d_6\,\frac{\partial v}{\partial y}\right)\cdot h_{y_j}\cdot h_{x_i}\right]_{(x_i,y_j,z_{k-1/2})}^{(x_i,y_j,z_{k+1/2})} P(u(x_i,y_j,z_k))+(c(\tau)\,v,P(u)),$$

$$(P(u),P(v))=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3}u_{i,j,k}\,v_{i,j,k}\,l_{i,j,k},\;\; u,\;v\;\in C^0(\overline{\Omega}), \tag{29}$$

$l_{i,j,k} = \left(x_{i+1/2}-x_{i-1/2}\right)\times\left(y_{j+1/2}-y_{j-1/2}\right)\times\left(z_{k+1/2}-z_{k-1/2}\right)$ is the volume of $\mathcal{R}_{i,j,k}$, $d_1 = \frac{1}{2}z\,\rho_{1,2}\,y$, $d_2=\frac{1}{2}\xi\,\rho_{1,3}\,z^{3/2}$, $d_3=\frac{1}{2}z\,\rho_{1,2}\,x$, $d_4=\frac{1}{2}\xi\,z^{3/2}\,\rho_{2,3}$, $d_5=\frac{1}{2}\sigma\,\rho_{1,2}\,x\,\xi\rho_{1,3}$, $d_6=\frac{1}{2}\xi\,\sigma\,y\,\rho_{2,3}$, $\bar{a_1}=\frac{1}{2}z$, $\bar{a_2}=\frac{1}{2}z$ and $\bar{a_3}=\frac{1}{2}\xi^2$. Note that

$$\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(d_1\,\frac{\partial v}{\partial y}\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{(x_{i-1/2},y_j,z_k)}^{(x_{i+1/2},y_j,z_k)} P(u(x_i,y_j,z_k))$$
$$=\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(d_1\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2},y_j,z_k} P(u(x_i,y_j,z_k))\,l_{i,j,k}.$$

For the sake of simplicity, we will denote

$$\begin{aligned}&\left(d_n\frac{\partial v}{\partial y},P(u)\right):=\left(P\left(d_n\frac{\partial v}{\partial y}\right),P(u)\right)=\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(d_n\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2},y_j,z_k} P(u(x_i,y_j,z_k))\,l_{i,j,k},\\&\left(d_n\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2}}:=\left(d_n\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2},y_j,z_k},\quad n\in\{1,2,3,4,5,6\}.\end{aligned} \tag{30}$$

and therefore the rewritten value of $\hat{B}_h$ will be

$$\hat{B}_h(v,P(u);\tau)=$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(\bar{a_1}\,x\,\frac{\partial v}{\partial x}+b_1\,v\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{x_{i-1/2}}^{x_{i+1/2}} P(u(x_i,y_j,z_k))$$
$$- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[y\left(\bar{a_2}\,y\frac{\partial v}{\partial y}+b_2\,v\right)\cdot h_{x_i}\cdot h_{z_k}\right]_{y_{j-1/2}}^{y_{j+1/2}} P(u(x_i,y_j,z_k))$$

$$
\begin{aligned}
&- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[z\left(\bar{a_3}\, z\,\frac{\partial v}{\partial z}+b_3\, v\right)\cdot h_{y_j}\cdot h_{x_i}\right]_{z_{k-1/2}}^{z_{k+1/2}} P(u(x_i,y_j,z_k))\\
&-\left(d_1\frac{\partial v}{\partial y},P(u)\right)-\left(d_2\frac{\partial v}{\partial z},P(u)\right)-\left(d_3\frac{\partial v}{\partial x},P(u)\right)-\left(d_4\frac{\partial v}{\partial z},P(u)\right)\\
&-\left(d_5\frac{\partial v}{\partial x},P(u)\right)-\left(d_6\frac{\partial v}{\partial y},P(u)\right)+(c(\tau)\,v,P(u)).
\end{aligned} \tag{31}
$$

Applying the composite (w.r.t. the partition $\{\mathcal{R}_{i,j,k},\ i=1,\dots,N_1-1;\quad j=1,\dots,N_2-1;\ k=1,\dots,N_3-1\}$), the mid-point quadrature rule to all terms except the second one we obtain the above for all $u\in C^0(\overline{\Omega})$

$$
\begin{aligned}
&\left(\frac{\partial v}{\partial \tau},u\right)_h-\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(\bar{a_1}\, x\,\frac{\partial v}{\partial x}+b_1\, v\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{x_{i-1/2}}^{x_{i+1/2}} P(u(x_i,y_j,z_k))\\
&- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[y\left(\bar{a_2}\, y\,\frac{\partial v}{\partial y}+b_2\, v\right)\cdot h_{x_i}\cdot h_{z_k}\right]_{y_{j-1/2}}^{y_{j+1/2}} P(u(x_i,y_j,z_k))\\
&- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[z\left(\bar{a_3}\, z\,\frac{\partial v}{\partial z}+b_3\, v\right)\cdot h_{y_j}\cdot h_{x_i}\right]_{z_{k-1/2}}^{z_{k+1/2}} P(u(x_i,y_j,z_k))\\
&-\left(d_1\frac{\partial v}{\partial y},P(u)\right)-\left(d_2\frac{\partial v}{\partial z},P(u)\right)-\left(d_3\frac{\partial v}{\partial x},P(u)\right)-\left(d_4\frac{\partial v}{\partial z},P(u)\right)-\left(d_5\frac{\partial v}{\partial x},P(u)\right)-\\
&\left(d_6\frac{\partial v}{\partial y},P(u)\right)+(c(\tau)\,v,u)_h\approx(g,u)_h,\quad \text{where}
\end{aligned} \tag{32}
$$

$(u,v)_h := (P(u),P(v))$. The expression (32) is a representation of the projection of the exact solution Pu on the fitted finite volume grid and will play a key role in the sequel of this paper.

3.2. *Fitted finite volume scheme*

We will denote by $v_{i,j,k}$ the nodal approximation of $v(\tau,x_i,y_j,z_k) := v_{i,j,k}$. Applying the fitted finite volume approximation to (32) as [18], we then arrive to the system

$$
\begin{aligned}
&\frac{dv_{i,j,k}}{d\tau}+e^{i,j}_{i-1,j,k}\,v_{i-1,j,k}+e^{i,j,k}_{i,j,k}\,v_{i,j,k}+e^{i,j,k}_{i+1,j,k}\,v_{i+1,j,k}+e^{i,j,k}_{i,j-1,k}\,v_{i,j-1,k}\\
&+e^{i,j,k}_{i,j+1,k}v_{i,j+1}+e^{i,j,k}_{i,j,k-1}\,v_{i,j-1,k}+e^{i,j,k}_{i,j,k+1}v_{i,j,k+1}=g_{i,j,k},
\end{aligned} \tag{33}
$$

for $i=1,\dots,N_1-1\quad j=1,\dots,N_2-1$ and $k=1,\dots,N_3-1$. This can be rewritten as the Ordinary Differential Equation (ODE)

$$
\frac{d\mathbf{v}(\tau)}{d\tau}+E(\tau)\,\mathbf{v}(\tau)=\mathbf{g}(\tau), \tag{34}
$$

where $E(\tau)$ is an $N\times N$ matrix with $N=(N_1-1)\times(N_2-1)\times(N_3-1)$, $\mathbf{v}=(v_{i,j,k})$, $\mathbf{g}=(g_{i,j,k})$, $i=1,\dots,N_1-1$, $j=1,\dots,N_2-1$ and $k=1,\dots,N_3-1$. By setting $n_1=N_1-1,\ n_2=N_2-1;\ n_3=N_3-1,\quad I:=I(i,j,k)=i+(j-1)n_1+(k-1)n_1n_2$ and $J:=J(i',j',k')=i'+(j'-1)n_1+(k'-1)n_1n_2$, we have $E(\tau)(I,J)=\left(e^{i,j,k}_{i',j',k'}\right)$, $i',i=1,\dots,N_1-1,\quad j',j=1,\dots,N_2-1$ and $k',k=1,\dots,N_3-1$ where the coefficients are defined by

$$
e^{i,j,k}_{i+1,j,k}=-\frac{d_{5i,j,k}}{h_{x_i}}-\frac{d_{3i,j,k}}{h_{x_i}}-x_{i+1/2}\frac{b_{1i+1/2,j,k}\,x_{i+1}^{\beta_{i,j,k}}}{h_{x_i}\left(x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}}\right)},
$$

$$
e^{i,j,k}_{i-1,j,k}=-x_{i-1/2}\frac{b_{1i-1/2,j,k}\,x_{i-1}^{\beta_{i-1,j,k}}}{h_{x_i}\left(x_i^{\beta_{i-1,j,k}}-x_{i-1}^{\beta_{i-1,j,k}}\right)},
$$

$$
\begin{aligned}
e_{i,j+1,k}^{i,j,k} &= -\frac{d_{1i,j,k}}{h_{y_j}} - \frac{d_{6i,j,k}}{h_{y_j}} - y_{j+1/2}\frac{b_{2i,j+1/2,k}\, y_{j+1}^{\beta_{1i,j,k}}}{h_{y_j}\left(y_{j+1}^{\beta_{1i,j,k}} - y_j^{\beta_{1i,j,k}}\right)},\\
e_{i,j-1,k}^{i,j,k} &= -y_{j-1/2}\frac{b_{2i,j-1/2,k}\, y_{j-1}^{\beta_{1i,j-1,k}}}{h_{y_j}\left(y_j^{\beta_{1i,j-1,k}} - y_{j-1}^{\beta_{1i,j-1,k}}\right)},\\
e_{i,j,k+1}^{i,j,k} &= -\frac{d_{2i,j,k}}{h_{z_k}} - \frac{d_{4i,j,k}}{h_{z_k}} - z_{k+1/2}\frac{b_{3i,j,k+1/2}\, z_{k+1}^{\beta_{2i,j,k}}}{h_{z_k}\left(z_{k+1}^{\beta_{2i,j,k}} - z_k^{\beta_{2i,j,k}}\right)},
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
e_{i,j,k-1}^{i,j,k} &= -z_{k-1/2}\frac{b_{3i,j,k-1/2}\, z_{k-1}^{\beta_{2i,j,k-1}}}{h_{z_k}\left(z_k^{\beta_{2i,j,k-1}} - z_{k-1}^{\beta_{2i,j,k-1}}\right)},\\
e_{i,j,k}^{i,j,k} &= c_{i,j,k} + \frac{d_{1i,j,k}}{h_{y_j}} + \frac{d_{2i,j,k}}{h_{z_k}} + \frac{d_{3i,j,k}}{h_{x_i}} + \frac{d_{4i,j,k}}{h_{z_k}} + \frac{d_{5i,j,k}}{h_{x_i}} + \frac{d_{5i,j,k}}{h_{y_j}}\\
&+ x_{i-1/2}\frac{b_{1i-1/2,j,k}\, x_i^{\beta_{i-1,j,k}}}{h_{x_i}\left(x_i^{\beta_{i-1,j,k}} - x_{i-1}^{\beta_{i-1,j,k}}\right)} + y_{j-1/2}\frac{b_{2i,j-1/2,k}\, y_j^{\beta_{1i,j-1,k}}}{h_{y_j}\left(y_j^{\beta_{1i,j-1,k}} - y_{j-1}^{\beta_{1i,j-1,k}}\right)}\\
&+ z_{k-1/2}\frac{b_{3i,j,k-1/2}\, z_k^{\beta_{2i,j,k-1}}}{h_{z_k}\left(z_k^{\beta_{2i,j,k-1}} - z_{k-1}^{\beta_{2i,j,k-1}}\right)}, + x_{i+1/2}\frac{b_{1i+1/2,j,k}\, x_i^{\beta_{i,j,k}}}{h_{x_i}\left(x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}\right)}\\
&+ y_{j+1/2}\frac{b_{1i,j+1/2,k}\, y_j^{\beta_{1i,j,k}}}{h_{y_j}\left(y_{j+1}^{\beta_{1i,j,k}} - y_j^{\beta_{1i,j,k}}\right)}\\
&+ z_{k+1/2}\frac{b_{3i,j,k+1/2}\, z_k^{\beta_{2i,j,k}}}{h_{z_k}\left(z_{k+1}^{\beta_{2i,j,k}} - z_k^{\beta_{2i,j,k}}\right)},
\end{aligned}
\tag{36}
$$

for $i = 2, \ldots, N_1 - 1,\ j = 2, \ldots, N_2 - 1$ and $k = 2, \ldots, N_3 - 1$ and

$$
\begin{aligned}
e_{0,j,k}^{1,j,k} &= -\frac{1}{2\,x_2}\, x_1(\bar{a}_{1k} - b_{11/2,j,k})\, v_{0,j,k},\\
e_{1,j,k}^{1,j,k} &= \frac{1}{2\,x_2}\, x_1(\bar{a}_{1k} + b_{11/2,j,k}) + \frac{1}{3}\, c_{1,j,k} + \frac{d_{51,j,k}}{h_{x_1}} + \frac{d_{31,j,k}}{h_{x_1}} + x_{1+1/2}\frac{b_{11+1/2,j,k}\, x_1^{\beta_{1,j,k}}}{h_{x_1}\left(x_2^{\beta_{1,j,k}} - x_1^{\beta_{1,j,k}}\right)},\\
e_{2,j,k}^{1,j,k} &= -\frac{d_{51,j,k}}{h_{x_1}} - \frac{d_{31,j,k}}{h_{x_1}} - x_{1+1/2}\frac{b_{11+1/2,j,k}\, x_2^{\beta_{1,j,k}}}{h_{x_1}\left(x_2^{\beta_{1,j,k}} - x_1^{\beta_{1,j,k}}\right)},\\
e_{i,0,k}^{i,1,k} &= -\frac{1}{2\,y_2}\, y_1(\bar{a}_{2k} - b_{2i,1/2,k})\, v_{i,0,k},\\
e_{i,1,k}^{i,1,k} &= \frac{1}{2\,y_2}\, y_1(\bar{a}_{2k} + b_{2i,1/2,k}) + \frac{1}{3}\, c_{i,1,k} + \frac{d_{1i,1,k}}{h_{y_1}} + \frac{d_{6i,1,k}}{h_{y_1}} + y_{1+1/2}\frac{b_{2i,1+1/2,k}\, y_1^{\beta_{1i,1,k}}}{h_{y_1}\left(y_2^{\beta_{1i,1,k}} - y_1^{\beta_{1i,1,k}}\right)},\\
e_{i,2,k}^{i,1,k} &= -\frac{d_{1i,1,k}}{h_{y_1}} - \frac{d_{6i,1,k}}{h_{y_1}} - y_{1+1/2}\frac{b_{2i,1+1/2,k}\, y_2^{\beta_{1i,1,k}}}{h_{y_1}\left(y_2^{\beta_{1i,1,k}} - y_1^{\beta_{1i,1,k}}\right)},\\
e_{i,j,0}^{i,j,1} &= -z_{1/2}\frac{b_{3i,j,1/2}\, z_0^{\beta_{2i,j,0}}}{h_{z_1}\left(z_1^{\beta_{2i,j,0}} - z_0^{\beta_{2i,j,0}}\right)},
\end{aligned}
$$

$$e^{i,j,1}_{i,j,1} = \frac{1}{3}c_{i,j,1} + \frac{d_{2i,j,1}}{h_{z_1}} + \frac{d_{4i,j,1}}{h_{z_1}} + z_{1+1/2}\frac{b_{3i,j,1+1/2}\, z_1^{\beta_{2i,j,1}}}{h_{z_1}\left(z_2^{\beta_{2i,j,1}} - z_1^{\beta_{2i,j,1}}\right)} + z_{1/2}\frac{b_{3i,j,1/2}\, z_1^{\beta_{2i,j,0}}}{h_{z_1}\left(z_1^{\beta_{2i,j,0}} - z_0^{\beta_{2i,j,0}}\right)},$$

$$e^{i,j,1}_{i,j,2} = -\frac{d_{2i,j,1}}{h_{z_1}} - \frac{d_{4i,j,1}}{h_{z_1}} - z_{1+1/2}\frac{b_{3i,j,1+1/2}\, z_2^{\beta_{2i,j,1}}}{h_{z_1}\left(z_2^{\beta_{2i,j,1}} - z_1^{\beta_{2i,j,1}}\right)}.$$

Let

$$D_{i,j,k,m,n,l} = \left\{(x, y, z) \mid h_y\, h_z \left|x - x_{i-\frac{1}{2}+m}\right| + h_x\, h_z \left|y - y_{j-\frac{1}{2}+n}\right| + h_x\, h_y \left|z - z_{k-\frac{1}{2}+l}\right| \le h_x\, h_y\, h_z\right\}.$$

Following the work in [9, Theorem 3.1], we define the trial space using the following function

$$\phi_{i,j,k}(x, y, z) = \begin{cases} \left(\frac{x_i}{x}\right)^{\beta_{i-1,j,k}} \left[1 - \left(\frac{x_i}{x_{i-1}}\right)^{\beta_{i-1,j,k}}\right]^{-1} \left[1 - \left(\frac{x}{x_{i-1}}\right)^{\beta_{i-1,j,k}}\right], & (x, y, z) \in D_{i,j,k,0,\frac{1}{2},\frac{1}{2}}, \\ \left(\frac{x_i}{x}\right)^{\beta_{i,j,k}} \left[1 - \left(\frac{x_i}{x_{i+1}}\right)^{\beta_{i,j,k}}\right]^{-1} \left[1 - \left(\frac{x}{x_{i+1}}\right)^{\beta_{i,j,k}}\right], & (x, y, z) \in D_{i,j,k,1,\frac{1}{2},\frac{1}{2}}, \\ \left(\frac{y_j}{y}\right)^{\beta_{1i,j-1,k}} \left[1 - \left(\frac{y_j}{y_{j-1}}\right)^{\beta_{1i,j-1,k}}\right]^{-1} \left[1 - \left(\frac{y}{y_{j-1}}\right)^{\beta_{1i,j-1,k}}\right], & (x, y, z) \in D_{i,j,k,\frac{1}{2},0\frac{1}{2}}, \\ \left(\frac{y_j}{y}\right)^{\beta_{1i,j,k}} \left[1 - \left(\frac{y_j}{y_{j+1}}\right)^{\beta_{1i,j,k}}\right]^{-1} \left[1 - \left(\frac{y}{y_{j+1}}\right)^{\beta_{1i,j,k}}\right], & (x, y, z) \in D_{i,j,k,\frac{1}{2},1,\frac{1}{2}}, \\ \left(\frac{z_k}{z}\right)^{\beta_{2i,j,k-1}} \left[1 - \left(\frac{z_k}{z_{k-1}}\right)^{\beta_{2i,j,k-1}}\right]^{-1} \left[1 - \left(\frac{z}{z_{k-1}}\right)^{\beta_{2i,j,k-1}}\right], & (x, y, z) \in D_{i,j,k,\frac{1}{2},\frac{1}{2},0}, \\ \left(\frac{z_k}{z}\right)^{\beta_{2i,j,k}} \left[1 - \left(\frac{z_k}{z_{k+1}}\right)^{\beta_{2i,j,k}}\right]^{-1} \left[1 - \left(\frac{z}{z_{k+1}}\right)^{\beta_{2i,j,k}}\right], & (x, y, z) \in D_{i,j,k,\frac{1}{2},\frac{1}{2},1}, \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

for $i = 2, \cdots, N_1 - 1$, $j = 2, \cdots, N_2 - 1$ and $k = 1, \cdots, N_3 - 1$.

The finite element trial space is chosen to be

$$S_h = \text{span}\left\{\phi_{i,j,k}(x, y, z),\ i = 1, \cdots, N_1 - 1,\ \ j = 1, \cdots, N_2 - 1,\ \text{and}\ k = 1, \cdots, N_3 - 1\right\},$$

and for any $u_h \in S_h$, we have the following the representation $u_h = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} u_{h\,i,j,k}\,\phi_{i,j,k}$, where $u_{h\,i,j,k} := u_h(x_i, y_j, z_k)$.

Let Q_1, Q_2 and Q_3 denote three mass lumping operators from $C^0(\overline{\Omega})$ to $L^\infty(\Omega)$ such that for any $v \in C^1(\Omega)$

$$Q_1\left(\frac{\partial v}{\partial y}\right)\bigg|_{\mathcal{R}_{i,j,k}} := \frac{v(x_i, y_{j+1}, z_k) - v(x_i, y_j, z_k)}{h_{y_j}}, \tag{38}$$

$$Q_2\left(\frac{\partial v}{\partial x}\right)\bigg|_{\mathcal{R}_{i,j,k}} := \frac{v(x_{i+1}, y_j, z_k) - v(x_i, y_j, z_k)}{h_{x_i}}, \tag{39}$$

$$Q_3\left(\frac{\partial v}{\partial z}\right)\bigg|_{\mathcal{R}_{i,j,k}} := \frac{v(x_i, y_j, z_{k+1}) - v(x_i, y_j, z_k)}{h_{z_k}} \tag{40}$$

$$Q_1\left(d_1\frac{\partial v}{\partial y}\right)\bigg|_{\mathcal{R}_{i,j,k}} := d_{1i,j,k}\left(\frac{v(x_i, y_{j+1}, z_k) - v(x_i, y_j, z_k)}{h_{y_j}}\right), \tag{41}$$

$i = 0, 1, \ldots, N_1,\ j = 0, 1, \ldots, N_2,\ k = 0, 1, \ldots, N_3$, the same for other terms. Furthermore, using the flux approximations in Section 3, this motivates the following semi discretization of (28) in the space S_h

$$\left(\frac{\partial v_h(\tau)}{\partial \tau}, u_h\right)_h + B_h(v_h, P(u_h); \tau) = (g, u_h)_h\,, \quad \forall\, u_h \in S_h \text{ where} \tag{42}$$

$$\begin{aligned} B_h(v_h, P(u_h); \tau) :=& -\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(\bar{a}_1\, x\,\frac{\partial v_h}{\partial x} + b_1\, v_h\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{x_{i-1/2}}^{x_{i+1/2}} P(u_h(x_i, y_j, z_k)) \\ &-\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[y\left(\bar{a}_2\, y\,\frac{\partial v_h}{\partial y} + b_2\, v_h\right)\cdot h_{x_i}\cdot h_{z_k}\right]_{y_{j-1/2}}^{y_{j+1/2}} P(u_h(x_i, y_j, z_k)) \\ &-\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[z\left(\bar{a}_3\, z\,\frac{\partial v_h}{\partial z} + b_3\, v_h\right)\cdot h_{y_j}\cdot h_{x_i}\right]_{z_{k-1/2}}^{z_{k+1/2}} P(u_h(x_i, y_j, z_k)) \\ &-\left(Q_1\left(d_1\frac{\partial v_h}{\partial y}\right), P(u_h)\right) - \left(Q_3\left(d_2\frac{\partial v_h}{\partial z}\right), P(u_h)\right) \\ &-\left(Q_2\left(d_3\frac{\partial v_h}{\partial x}\right), P(u_h)\right) - \left(Q_3\left(d_4\frac{\partial v_h}{\partial z}\right), P(u_h)\right) \\ &-\left(Q_2\left(d_5\frac{\partial v_h}{\partial x}\right), P(u_h)\right) - \left(Q_1\left(d_6\frac{\partial v_h}{\partial y}\right), P(u_h)\right) + (c(\tau)\, v_h, u_h)_h\,. \end{aligned}$$

We introduce the natural interpolation operator $I_h : C^0(\overline{\Omega}) \to S_h$ by

$$I_h\, u = \sum_{i,j,k} u_{i,j,k}\phi_{i,j,k}, \quad \text{where} \quad u(x_i, y_j, z_k) := u_{i,j,k}, \tag{43}$$

$$i = 1, \ldots, N_1 - 1\ \ j = 1, \ldots, N_2 - 1, \quad k = 1, \ldots, N_3 - 1. \tag{44}$$

Before further discussion, we will first define some norms and semi-norms on S_h. Note that (28) is the representation of the exact solution on $\mathcal{R}_{i,j,k}$ and will play a key role in our error analysis.

Let us define the discrete functionals $\|\cdot\|_{1,h_x}$, $\|\cdot\|_{1,h_y}$, $\|\cdot\|_{1,h_z}$ and $\|\cdot\|_{0,h}$ on S_h

$$\|u_h\|^2_{1,h_x} = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i+1/2}\, b_{1_{i+1/2,j,k}} h_{y_j} h_{z_k} \frac{x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}}}{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}\left(u_{h\,i+1,j,k} - u_{h\,i,j,k}\right)^2, \tag{45}$$

$$\|u_h\|^2_{1,h_y} = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} y_{j+1/2}\, b_{2_{i,j+1/2,k}} h_{x_i} h_{z_k} \frac{y_{j+1}^{\beta_{1i,j,k}} + y_j^{\beta_{1i,j,k}}}{y_{j+1}^{\beta_{1i,j,k}} - y_j^{\beta_{1i,j,k}}}\left(u_{h\,i,j+1,k} - u_{h\,i,j,k}\right)^2, \tag{46}$$

$$\|u_h\|^2_{1,h_z} = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} z_{k+1/2}\, b_{3_{i,j,k+1/2}} h_{y_j} h_{x_i} \frac{z_{k+1}^{\beta_{2i,j,k}} + z_k^{\beta_{2i,j,k}}}{z_{k+1}^{\beta_{2i,j,k}} - z_k^{\beta_{2i,j,k}}}\left(u_{h\,i,j,k+1} - u_{h\,i,j,k}\right)^2, \tag{47}$$

$$\|u_h\|^2_{0,h} = \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} u_{h\,i,j,k}^2\, l_{i,j,k}, \tag{48}$$

for any $u_h \in S_h$, where $l_{i,j,k} = \left(x_{i+1/2} - x_{i-1/2}\right) \times \left(y_{j+1/2} - y_{j-1/2}\right) \times \left(z_{k+1/2} - z_{k-1/2}\right)$. It is easy to show that $\|.\|_{1,h_x}$, $\|.\|_{1,h_y}$ and $\|.\|_{1,h_z}$ are the semi-norms on S_h. We define the weighted discrete $H^1_{\hat{w}}$-norm:

$$\|u_h\|^2_h = \|u_h\|^2_{1,h_x} + \|u_h\|^2_{1,h_y} + \|u_h\|^2_{1,h_z} + \|u_h\|^2_{0,h}. \tag{49}$$

The following result, extends [9, Theorem 4.1] and shows the coercivity of the bilinear form $B_h(\cdot, \cdot; \tau)$ with respect to this norm,

Theorem 3.1. *If h is sufficiently small then, for all $u_h \in S_h$, we have*

$$B_h(u_h, P(u_h); \tau) \geq C\|u_h\|^2_h, \tag{50}$$

where C denotes a positive constant independent of h and u_h.

Proof. The proof is done exactly as for [9, Theorem 4.1]. ∎

Remark 3.1. Note that using the coercivity property in (50) with the fact that the linear mapping $v \to (g, u_h)_h$ is continuous in S_h, the existence and uniqueness of the discrete solution u_h in (42) is ensured (see Theorem 1.33, [6]).

The coercivity result in Theorem 3.1 will be important in our convergence analysis in the next section.

3.3. Consistency of the fitted finite volume method

Let $\varrho_1(u)$, $\varrho_2(u)$ and $\varrho_3(u)$ be defined, respectively by

$$\varrho_1(u) := \bar{a}_1\, x \frac{\partial u}{\partial x} + b_1\, u, \quad \varrho_2(u) := \bar{a}_2\, y \frac{\partial u}{\partial y} + b_2\, u, \quad \varrho_3(u) := \bar{a}_3\, z \frac{\partial u}{\partial z} + b_3\, u.$$

We denote by $\varrho_{l_{i,j,k}}(u)$ the approximation value of $\varrho_l(u)$ at $\left(x_{i+1/2}, y_j, z_k\right)$ for $l = 1$, $\left(x_i, y_{j+1/2}, z_k\right)$ for $l = 2$ and $\left(x_i, y_j, z_{k+1/2}\right)$ for $l = 3$ using the fitted scheme in Section 3.2 for $l \in \{1, 2, 3\}$

$$\varrho_{1_{i,j,k}}(u) = \begin{cases} \dfrac{1}{2}\left[(\bar{a}_{1k} + b_{1_{x_{1/2},j,k}})u_{1,j,k} - (\bar{a}_{1k} - b_{1_{x_{1/2},j,k}})u_{0,j,k}\right], \ i = 0, \\ \dfrac{b_{1_{i+1/2,j,k}}\left(x_{i+1}^{\beta_{i,j,k}}\, u_{i+1,j,k} - x_i^{\beta_{i,j,k}}\, u_{i,j,k}\right)}{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}, \ i = 1, \ldots, N_1 - 1, \\ j = 1, \ldots, N_2 - 1,\ k = 1, \ldots, N_3 - 1 \end{cases} \tag{51}$$

$$\varrho_{2_{i,j,k}}(u) = \begin{cases} \dfrac{1}{2}\left[(\bar{a}_{2k} + b_{2_{i,y_{1/2},k}})u_{i,1,k} - (\bar{a}_{2k} - b_{2_{i,y_{1/2},k}})u_{i,0,k}\right], \ j = 0, \\ \dfrac{b_{2_{i,j+1/2,k}}\left(y_{j+1}^{\beta_{1_{i,j,k}}}\, u_{i,j+1,k} - y_j^{\beta_{1_{i,j,k}}}\, u_{i,j,k}\right)}{y_{j+1}^{\beta_{1_{i,j,k}}} - y_j^{\beta_{1_{i,j,k}}}}, \ j = 1, \ldots, N_2 - 1, \\ i = 1, \ldots, N_1 - 1,\ k = 1, \ldots, N_3 - 1 \end{cases}$$
$$\varrho_{3_{i,j,k}}(u) = \begin{cases} \dfrac{b_{3_{i,j,k+1/2}}\left(z_{k+1}^{\beta_{2_{i,j,k}}}\, u_{i,j,k+1} - z_k^{\beta_{2_{i,j,k}}}\, u_{i,j,k}\right)}{z_{k+1}^{\beta_{2_{i,j,k}}} - z_k^{\beta_{2_{i,j,k}}}}, \ i = 1, \ldots, N_1 - 1, \\ j = 1, \ldots, N_2 - 1,\ k = 0,\ 1, \ldots, N_3 - 1 \end{cases} \tag{52}$$

As in [8], we define the global approximation $\varrho_{1_h}(u)$, $\varrho_{2_h}(u)$, $\varrho_{3_h}(u)$ where in long notation, $\varrho_{1_h}(u)(x, y, z) = \varrho_{1_h}(u(\tau, x, y, z))$, $(x, y, z) \in I_{x_i} \times I_y \times I_z$, by

$$\begin{aligned} \varrho_{1_h}(u)|_{I_{x_i} \times I_y \times I_z} &:= \varrho_{1_{i,j,k}}(u), \\ \varrho_{2_h}(u)|_{I_x \times I_{y_j} \times I_z} &:= \varrho_{2_{i,j,k}}(u) \\ \varrho_{3_h}(u)|_{I_x \times I_y \times I_{z_k}} &:= \varrho_{3_{i,j,k}}(u), \end{aligned}$$

for $i = 0, 1, \ldots, N_1 - 1$, $j = 0, 1, \ldots, N_2 - 1$ and $k = 0, 1, \ldots, N_3 - 1$. We will need the following assumption of the local quasi-uniformity of the spatial mesh [14]:

Assumption 3.1. There exist constants $c_1 > 0, c_2 > 0$ and $c_3 > 0$ such that $c_1^{-1}\, h_{x_{i+1}} \le h_{x_i} \le c_1\, h_{x_{i+1}}$, $c_2^{-1}\, h_{y_{j+1}} \le h_{y_j} \le c_2\, h_{y_{j+1}}$, $c_3^{-1}\, h_{z_{k+1}} \le h_{z_k} \le c_3\, h_{z_{k+1}}$ $i = 0, \ldots, N_1 - 1$ $j = 0, \ldots, N_2 - 1$, $k = 0, \ldots, N_3 - 1$.

For the sake of simplicity, let us rewrite the discrete value ϱ_{1_h}, ϱ_{2_h}, ϱ_{3_h} as follows

$$\varrho_{1_h}(u_h)|_{I_{x_i} \times I_y \times I_z} = \begin{cases} \bar{a}_{1k}\left(1 + \beta_{0,j,k}\right) x_{1/2} \dfrac{u_{h1,j,k} - u_{h0,j,k}}{h_{x_0}} + b_{1_{1/2,j,k}} u_{h0,j,k}, \ i = 0,\ j = 1, \ldots, N_2 - 1,\ k = 1, \ldots, N_3 - 1 \\ \bar{a}_{1k}\left(1 + \gamma_{1_{i,j,k}}\right) x_{i+1} \dfrac{u_{hi+1,j,k} - u_{hi,j,k}}{h_{x_i}} + b_{1_{i+1/2,j,k}} u_{hi,j,k}, \ i = 1, \ldots, N_1 - 1,\ j = 1, \ldots, N_2 - 1, \\ k = 1 \cdots, N_3 - 1 \end{cases}$$

$$\varrho_{2h}(u_h)|_{I_x\times I_{y_j}\times I_z} = \begin{cases} \bar{a}_{2k}\left(1+\beta_{1i,0,k}\right)y_{1/2}\dfrac{u_{hi,1,k}-u_{hi,0,k}}{h_{y_0}} + b_{2i,1/2,k}u_{hi,0,k} \;\; i=1,\ldots,N_1-1,\; j=0,\; k=1,\ldots,N_3-1 \\ \bar{a}_{2k}\left(1+\gamma_{2i,j,k}\right)y_{j+1}\dfrac{u_{hi,j+1,k}-u_{hi,j,k}}{h_{y_j}} + b_{2i,j+1/2,k}u_{hi,j,k}, \;\; i=1,\ldots,N_1-1,\; j=1,\ldots,N_2-1, \\ k=1,\ldots,N_3-1 \end{cases}$$

$$\varrho_{3h}(u_h)|_{I_x\times I_y\times I_{z_k}} = \begin{cases} \bar{a}_3\left(1+\gamma_{3i,j,k}\right)z_{k+1}\dfrac{u_{hi,j,k+1}-u_{hi,j,k}}{h_{z_k}} + b_{3i,j,k+1/2}u_{hi,j,k} \;\; i=1,\ldots,N_1-1,\; j=1,\ldots,N_2-1, \\ k=0,1,\ldots,N_3-1 \end{cases}$$

where

$$\gamma_{1i,j,k} = \frac{\beta_{i,j,k}\,h_{x_i}\,x_{i+1}^{\beta_{i,j,k}-1}}{(x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}})}-1,\;\; \gamma_{2i,j,k} = \frac{\beta_{1i,j,k}\,h_{y_j}\,y_{j+1}^{\beta_{1i,j,k}-1}}{(y_{j+1}^{\beta_{1i,j,k}}-y_j^{\beta_{1i,j,k}})}-1,\;\; \gamma_{3i,j,k} = \frac{\beta_{2i,j,k}\,h_{z_k}\,z_{k+1}^{\beta_{2i,j,k}-1}}{(z_{k+1}^{\beta_{2i,j,k}}-z_k^{\beta_{2i,j,k}})}-1. \tag{53}$$

Proof. We start with the case $i=0$. By (51),

$$\begin{aligned}\varrho_{1h}(u_h)|_{I_{x_0}\times I_y\times I_z} &= \frac{1}{2}\left[(\bar{a}_{1k}+b_{1x_{1/2},j,k})u_{h1,j,k}-(\bar{a}_{1k}-b_{1x_{1/2},j,k})u_{h0,j,k}\right]\\ &= \frac{1}{2}\bar{a}_{1k}\left[u_{h1,j,k}-u_{h0,j,k}\right]+\frac{1}{2}b_{1x_{1/2},j,k}\left[u_{h1,j,k}+u_{h0,j,k}\right]\\ &= \frac{1}{2}\bar{a}_{1k}h_{x_0}\frac{\left[u_{h1,j,k}-u_{h0,j,k}\right]}{h_{x_0}}+\frac{1}{2}b_{1x_{1/2},j,k}\left[u_{h1,j,k}+u_{h0,j,k}\right].\end{aligned}$$

Since $u_{h1,j,k}=u_{h0,j,k}+h_{x_0}\dfrac{u_{h1,j,k}-u_{h0,j,k}}{h_{x_0}}$, $x_{1/2}=\dfrac{h_{x_0}}{2}$ we obtain

$$\varrho_{1h}(u_h)|_{I_{x_0}\times I_y\times I_z} = (\bar{a}_{1k}+b_{1x_{1/2},j,k})x_{1/2}\frac{\left[u_{h1,j,k}-u_{h0,j,k}\right]}{h_{x_0}}+b_{1x_{1/2},j,k}u_{h0,j,k},$$

using the fact that $(\bar{a}_{1k}+b_{1x_{1/2},j,k})=(\bar{a}_{1k}+\bar{a}_{1k}\beta_{0,j,k})=\bar{a}_{1k}(1+\beta_{0,j,k})$, we have the result.

For the case $i=1,\ldots,N_1-1$

$$\begin{aligned}\varrho_{1h}(u_h)|_{I_{x_i}\times I_y\times I_z} &= \frac{b_{1i+1/2,j,k}\left(x_{i+1}^{\beta_{i,j,k}}\,u_{hi+1,j,k}-x_i^{\beta_{i,j,k}}\,u_{hi,j,k}\right)}{x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}}}\\ &= \frac{b_{1i+1/2,j,k}}{x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}}}\left[x_{i+1}^{\beta_{i,j,k}-1}x_{i+1}\left(h_{x_i}\frac{u_{hi+1,j,k}-u_{hi,j,k}}{h_{x_i}}+u_{hi,j,k}\right)-x_i^{\beta_{i,j,k}}\right]\\ &= \left[\frac{b_{1i+1/2,j,k}h_{x_i}x_{i+1}^{\beta_{i,j,k}-1}}{x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}}}x_{i+1}\left(\frac{u_{hi+1,j,k}-u_{hi,j,k}}{h_{x_i}}\right)+b_{1i+1/2,j,k}u_{hi,j,k}\right]\\ &= \left[\bar{a}_{1k}\left(1+\gamma_{1i,j,k}\right)x_{i+1}\left(\frac{u_{hi+1,j,k}-u_{hi,j,k}}{h_{x_i}}\right)+b_{1i+1/2,j,k}u_{hi,j,k}\right],\end{aligned}$$

since $b_{1i+1/2,j,k}=\bar{a}_{1k}\beta_{i,j,k}$. ■

Lemma 3.1. *Under Assumption 3.1, there exist constants $C_{\gamma_1}>0$, $C_{\gamma_2}>0$ and $C_{\gamma_3}>0$ depending only respectively on* $\max_{\substack{i=1,\ldots,N_1-1\\ j=1,\ldots,N_2-1\\ k=1,\ldots,N_3-1}}\beta_{i,j,k}$, $\max_{\substack{i=1,\ldots,N_1-1\\ j=1,\ldots,N_2-1\\ k=1,\ldots,N_3-1}}\beta_{1i,j,k}$, $\max_{\substack{i=1,\ldots,N_1-1\\ j=1,\ldots,N_2-1\\ k=0,1,\ldots,N_3-1}}\beta_{2i,j,k}$ *and c_1, c_2, c_3 such that the following estimates hold*

$$\begin{aligned}&\left|\gamma_{1i,j,k}x_{i+1}\right|\le C_{\gamma_1}h_{x_i},\;\; i=1,\ldots,N_1-1,\; j=1,\ldots,N_2-1,\; k=1,\ldots,N_3-1,\\ &\left|\gamma_{2i,j,k}y_{j+1}\right|\le C_{\gamma_2}h_{y_j},\;\; i=1,\ldots,N_1-1,\; j=1,\ldots,N_2-1,\; k=1,\ldots,N_3-1,\end{aligned}$$

$$\left|\gamma_{3_{i,j,k}} z_{k+1}\right| \leq C_{\gamma_3} h_{z_k},\ \ i=1,\dots,N_1-1,\ j=1,\dots,N_2-1,\ k=0,1,\dots,N_3-1.$$

Proof. We follow the same procedure as in [1, Lemma 2] and use (53). Let us just prove the first inequality as the others follow the same lines. Note that

$$\frac{b_{1_{i+1/2,j,k}} h_{x_i} x_{i+1}^{\beta_{i,j,k}-1}}{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}} = \bar{a}_{1k}\left(1+\gamma_{1_{i,j,k}}\right)$$

with $\gamma_{1_{i,j,k}}$ being defined in (53). Using the Taylor expansion

$$\psi^{\beta_{i,j,k}} = x_i^{\beta_{i,j,k}} + \beta_{i,j,k}\xi^{\beta_{i,j,k}-1}(\psi - x_i)\ \text{ where } x_i < \xi < \psi.$$

For $\psi = x_{i+1}$, we have

$$\gamma_{1_{i,j,k}} = \left(\frac{x_{i+1}}{\xi}\right)^{\beta_{i,j,k}-1} - 1 = \frac{x_{i+1}^{\beta_{i,j,k}-1} - \xi^{\beta_{i,j,k}-1}}{x_i^{\beta_{i,j,k}-1}}.$$

Similarly, by the Taylor expansion

$$\psi^{\beta_{i,j,k}-1} = \xi^{\beta_{i,j,k}-1} + (\beta_{i,j,k}-1)\eta^{\beta_{i,j,k}-2}(\psi - \xi),\ \ \text{where } \xi < \eta < \psi.$$

We also have for $\psi = x_{i+1}$

$$\gamma_{1_{i,j,k}} = (\beta_{i,j,k}-1)\frac{x_{i+1}-\xi}{\xi}\left(\frac{\eta}{\xi}\right)^{\beta_{i,j,k}-2}$$

and, thus

$$|\gamma_{1_{i,j,k}} x_{i+1}| = |\beta_{i,j,k}-1|\frac{x_{i+1}}{\xi}\left(\frac{\eta}{\xi}\right)^{\beta_{i,j,k}-2} h_{x_i},\ \ \text{where } x_i < \xi < \eta < x_{i+1}.$$

To estimate the factor $(\eta/\xi)^{\beta_{i,j,k}-2}$, we have to distinguish between the cases $\beta_{i,j,k} < 2$ and $\beta_{i,j,k} > 2$. In the first case, we use that $\xi < \eta$ and obtain

$$\left(\frac{\eta}{\xi}\right)^{\beta_{i,j,k}-2} = \left(\frac{\xi}{\eta}\right)^{2-\beta_{i,j,k}} < \left(\frac{\eta}{\eta}\right)^{2-\beta_{i,j,k}} = 1.$$

In the second case, we use that $\eta < x_{i+1}$ and obtain

$$\left(\frac{\eta}{\xi}\right)^{\beta_{i,j,k}-2} \leq \left(\frac{x_{i+1}}{x_i}\right)^{\beta_{i,j,k}-2}.$$

Since $x_{i-1} \geq 0$ for all $i = 1,\dots,N_1-1$, we have the following estimate

$$\frac{x_{i+1}}{x_i} = \frac{x_i + h_{x_i}}{x_i} = 1 + \frac{h_{x_i}}{x_i} = 1 + \frac{h_{x_i}}{h_{x_{i-1}} + x_{i-1}} \leq 1 + \frac{h_{x_i}}{h_{x_{i-1}}} \leq 1 + c_1$$

by Assumption 3.1. Therefore

$$\left(\frac{\eta}{\xi}\right)^{\beta_{i,j,k}-2} \leq (1+c_1)^{\beta_{i,j,k}-2}.$$

Similarly,

$$\frac{x_{i+1}}{\xi} \leq \frac{x_{i+1}}{x_i} \leq (1+c_1).$$

So we finally obtain the estimate. ■

The consistency results of the fitted finite volume method are given in the following Lemma.

Lemma 3.2. *Let $v \in H^q(\Omega) \cap H^1_{0,\hat{w}}(\Omega)$ with $q > 5/2$, under Assumption 3.1 there exist positive constants C_1, C_2, C_3 depending on c_1, c_2, c_3, C_{γ_1} C_{γ_2}, C_{γ_3}[2] and independent of h such that*

$$\begin{aligned}
&\left|\left(\varrho_1(v) - \varrho_{1h}(I_h v)\right)_{x_{i+1/2},y_j,z_k}\right| \leq C_1 \int_{x_i}^{x_{i+1}} \left[\left|\frac{\partial \varrho_1}{\partial x}\right| + \left|\frac{\partial v(\tau,\cdot,y,z)}{\partial x}\right| + |v(\tau,\cdot,y,z)|\right] dx,\\
&\qquad i = 0,\dots,N_1-1,\ j = 1,\dots,N_2-1,\ k = 1,\dots,N_3-1\\
&\left|\left(\varrho_2(v) - \varrho_{2h}(I_h v)\right)_{x_i,y_{j+1/2},z_k}\right| \leq C_2 \int_{y_j}^{y_{j+1}} \left[\left|\frac{\partial \varrho_2}{\partial y}\right| + \left|\frac{\partial v(\tau,x,\cdot,z)}{\partial y}\right| + |v(\tau,x,\cdot,z)|\right] dy,\\
&\qquad i = 1,\dots,N_1-1,\ j = 0,1,\dots,N_2-1\ \ k = 1,\dots,N_3-1\\
&\left|\left(\varrho_3(v) - \varrho_{3h}(I_h v)\right)_{x_i,y_j,z_{k+1/2}}\right| \leq C_3 \int_{z_k}^{z_{k+1}} \left[\left|\frac{\partial \varrho_3}{\partial z}\right| + \left|\frac{\partial v(\tau,x,y,\cdot)}{\partial z}\right| + |v(\tau,x,y,\cdot)|\right] dz,\\
&\qquad i = 1,\dots,N_1-1,\ j = 1,\dots,N_2-1,\ k = 0,1\cdots,N_3-1.
\end{aligned} \tag{54}$$

Proof. As we are working in three dimension, we have $H^q(\Omega) \hookrightarrow C^1(\Omega)$ with $q > 5/2$ according to the Sobolev's embedding Theorem. The proof of our Theorem follows the same lines as [1, Lemma 3]. Note that in that proof they only need $v \in H^2(\Omega)$ as their proof is done in one dimension. ■

4. Full discretization and error estimate

Let $0 =: \tau_0 < \tau_1 < \cdots < \tau_M := T$ be a subdivision of the time interval $[0, T]$ with the step sizes $\Delta\tau_m = \tau_{m+1} - \tau_m > 0$, $m \in \{0,\dots,M-1\}$ and $\Delta\tau = \max_{0\leq m\leq M-1} \Delta\tau_m$. The fully discrete method with the parameter $\theta \in [0,1]$ for (2) reads as follows: Find a sequence $v_h^1, v_h^2, \dots, v_h^m \in S_h$ such that for $m \in \{0,\dots,M-1\}$

$$\begin{cases}
\left(\dfrac{v_h^{m+1} - v_h^m}{\Delta\tau_m}, u_h\right)_h + B_h\left(\theta\, v_h^{m+1} + (1-\theta) v_h^m, P(u_h); \tau_{m+\theta}\right)\\
= \left(\theta\, g^{m+1} + (1-\theta)\, g^m, u_h\right)_h \quad \forall u_h \in S_h,\\
v_h^0 = v_{0h},
\end{cases} \tag{55}$$

where $\tau_{m+\theta} = \theta\,\tau^{m+1} + (1-\theta)\,\tau^m = \tau_m + \theta\Delta\tau_m$, $g^m = g(\tau_m)$ and $v_{0h} \in S_h$ is an approximation to v_0. By representing the elements v_h^m is terms of the basis $\{\phi_{i,j,k}\}_{i,j,k=1}^{N_1-1,N_2-1,N_3-1}$ of S_h and choosing $u_h = \phi_{p,q,r}$, for $p = 1,\dots,N_1-1$, $q = 1,\dots,N_2-1$ and $r = 1,\dots,N_3-1$, we have the following corresponding form

$$\left(\frac{M_h\, v_h^{m+1} - M_h\, v_h^m}{\Delta\tau_m}\right) + \theta\, B_h^{m+\theta} v_h^{m+1} + (1-\theta)\, B_h^{m+\theta} v_h^m = \theta\, b_h^{m+1} + (1-\theta)\, b_h^m, \ \text{where} \tag{56}$$

$$\begin{aligned}
M_h &:= \left((\phi_{i,j,k}, \phi_{p,q,r})_h\right)_{\substack{p,q,r=1\\ i,j,k=1}}^{N_1-1,N_2-1,N_3-1},\\
B_h^m &:= \left(B_h(\phi_{i,j,k}, \phi_{p,q,r}; \tau_m)\right)_{\substack{p,q,r=1\\ i,j,k=1}}^{N_1-1,N_2-1,N_3-1},\\
b_h^m &:= \left((g(\tau_m), \phi_{i,j,k})_h\right)_{i,j,k=1}^{N_1-1,N_2-1,N_3-1}.
\end{aligned}$$

The initial condition v_h^0 is obtained from the representing of v_{0h} in the basis of S_h and the matrix M_h is diagonal.

Remark 4.1. $M_h + \Delta\tau_m\,\theta\, B_h^{m+\theta}$ is an M-matrix since $B_h^{m+\theta}$ is an M matrix and $\Delta\tau_m > 0$. Therefore, the above problem (56) is uniquely solvable and our method preserves the positivity.

Remark 4.2. In the sequel we will focus on the estimate the error for $\theta = 1$. The proof for $\theta \in [\frac{1}{2}, 1)$ follows the same lines.

[2] Constants in Assumption 3.1 and Lemma 3.1 respectively.

Our error analysis for $\theta = 1$ is given in the following theorem under the assumption that the solution v of (3) is sufficiently smooth.

Theorem 4.1. *Let $v_h^0 = v_{0h} = I_h v(0)$ and assume (11) in Assumption 2.1 holds. Let v_h^m be the numerical solution of the fully discretized scheme (55) with $\theta = 1$. Then if the solution v of (9) is such that $v \in C(0,T;H^q(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap H^2(0,T;L^2(\Omega))$ with $q > 5/2$ and $\varrho_1(v), \varrho_2(v), \varrho_3(v) \in C(0,T;H^1(\Omega))$, then there exists a positive constant C such that the following estimate holds*

$$\|v(\tau_m) - v_h^m\|_{0,h} \leq C\,(h + \Delta\tau). \tag{57}$$

Proof. We follow the same procedure as in [1, Theorem 7] for the one dimensional penalized Black–Scholes equation. Remember that $I_h\, v$ is the S_h interpolant of v defined in (43), let us notice that

$$\|v(\tau_m) - v_h^m\|_{0,h} \leq \|v(\tau_m) - I_h\, v(\tau_m)\|_{0,h} + \|I_h\, v(\tau_m) - v_h^m\|_{0,h} \tag{58}$$

For the first term on the right-hand side of (58), since $v(\tau) \in H^2(\Omega)$ then there exists a positive constant C_3 depending on v (see [13, Theorem 3.29 page 138]) such that

$$\|v(\tau) - I_h\, v(\tau)\|_{0,h} \leq C_3\, h^2 |v(\tau)|_2, \tag{59}$$

where $|\cdot|_2$ is the semi-norm on $H^2(\Omega)$. Furthermore, for $v \in C(0,T;H^2(\Omega))$ there exists a positive constant C_{31} depending on $v, T, x_{\max}, y_{\max}$ and $z_{\max}$ such that

$$\|v(\tau_m) - I_h v(\tau_m)\|_{0,h} \leq C_{31}\, h. \tag{60}$$

To complete the proof, we need to estimate the second term on the right-hand side of (58), more precisely $\Theta^m := I_h\, v(\tau_m) - v_h^m$ in the discrete L^2-norm (45). By adding and subtracting appropriate terms and by using Eq. (55) (with the test function $u_h \in S_h$), we easily derive the following equation respect to Θ^{m+1} :

$$\left(\frac{\Theta^{m+1} - \Theta^m}{\Delta\tau_m}, u_h\right)_h + B_h\left(\Theta^{m+1}, P(u_h); \tau_{m+1}\right) \tag{61}$$

$$\begin{aligned}
&= \left[\left(\frac{I_h\, v(\tau_{m+1}) - I_h\, v(\tau_m)}{\Delta\tau_m}, u_h\right)_h - \left(\frac{\partial v(\tau_{m+1})}{\partial\tau}, P(u_h)\right)\right] \\
&+ \left[B_h\left(I_h\, v(\tau_{m+1}), P(u_h); \tau_{m+1}\right) - \hat{B}_h\left(v(\tau_{m+1}), P(u_h); \tau_{m+1}\right)\right] \\
&+ \left[\left(g^{m+1}, P(u_h)\right) - \left(g^{m+1}, u_h\right)_h\right] := Y_1^m + Y_2^m + Y_3^m,
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
Y_1^m &= \left[\left(\frac{I_h\, v(\tau_{m+1}) - I_h\, v(\tau_m)}{\Delta\tau_m}, u_h\right)_h - \left(\frac{\partial v(\tau_{m+1})}{\partial\tau}, P(u_h)\right)\right] \\
Y_2^m &= \left[B_h\left(I_h\, v(\tau_{m+1}), P(u_h); \tau_{m+1}\right) - \hat{B}_h\left(v(\tau_{m+1}), P(u_h); \tau_{m+1}\right)\right] \\
Y_3^m &= \left[\left(g^{m+1}, P(u_h)\right) - \left(g^{m+1}, u_h\right)_h\right].
\end{aligned}$$

The estimation of Y_3^m is done exactly as in [1, Y_4^m] and we have

$$|Y_3^m| \leq C_4\, h\, \|u_h\|_{0,h}. \tag{63}$$

The estimation of Y_1^m is done exactly as in [1, (54)]. Since $P(I_h\, v) = P(v)$, we have $Y_1^m = (w^m, P(u_h))$ with

$$w^m := \frac{P(v(\tau_{m+1})) - P(v(\tau_m))}{\Delta\tau_m} - \frac{\partial v(\tau_{m+1})}{\partial\tau} \tag{64}$$

By Cauchy–Schwarz inequality

$$|Y_1^m| \leq \|w^m\|_0 \|u_h\|_{0,h}, \tag{65}$$

where

$$\|w^m\|_0 \leq \Gamma_1^m(\Delta\tau_m, h) := \frac{1}{\Delta\tau_m}\int_{\tau_m}^{\tau_{m+1}} \left\|(P - I)\left(\frac{\partial v(s)}{\partial s}\right)\right\|_0 ds + \int_{\tau_m}^{\tau_{m+1}} \left\|\frac{\partial^2 v(s)}{\partial s^2}\right\|_0 ds. \tag{66}$$

In the following subsection we will estimate the term $|Y_2^m|$ as it contains new terms not appearing in [1]. We follow [9, Theorem 4.3]. **Estimation of** $|Y_2^m|$ we have

$$Y_2^m = B_h\left(I_h\, v(\tau_{m+1}), P(u_h); \tau_{m+1}\right) - \hat{B}_h\left(v(\tau_{m+1}), P(u_h); \tau_{m+1}\right)$$

Let us set

$$\delta_{21}(I_h\, v, u_h, \tau) := \hat{B}_h\left(v, P(u_h); \tau_{m+1}\right) - B_h\left(I_h\, v, P(u_h); \tau_{m+1}\right).$$

Indeed from the definitions of $\hat{B}_h$ and B_h, we have

$$\begin{aligned}
|\delta_{21}(I_h\, v, u_h, \tau)| = \Bigg| &- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(\varrho_1(v) - \varrho_{1h}(I_h\, v)\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{x_{i-1/2}}^{x_{i+1/2}} P(u_h(x_i, y_j, z_k)) - \\
&\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[y\left(\varrho_2(v) - \varrho_{2h}(I_h\, v)\right)\cdot h_{x_i}\cdot h_{z_k}\right]_{y_{j-1/2}}^{y_{j+1/2}} P(u_h(x_i, y_j, z_k)) \\
&- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[z\left(\varrho_3(v) - \varrho_{3h}(I_h\, v)\right)\cdot h_{y_j}\cdot h_{x_i}\right]_{z_{k-1/2}}^{z_{k+1/2}} P(u_h(x_i, y_j, z_k)) \\
&- \left(d_1\frac{\partial v}{\partial y} - Q_1\left(d_1\frac{\partial I_h\, v}{\partial y}\right), P(u_h)\right) - \left(d_2\frac{\partial v}{\partial z} - Q_3\left(d_2\frac{\partial I_h\, v}{\partial z}\right), P(u_h)\right) \\
&- \left(d_3\frac{\partial v}{\partial x} - Q_2\left(d_3\frac{\partial I_h\, v}{\partial x}\right), P(u_h)\right) - \left(d_4\frac{\partial v}{\partial z} - Q_3\left(d_4\frac{\partial I_h\, v}{\partial z}\right), P(u_h)\right) \\
&- \left(d_5\frac{\partial v}{\partial x} - Q_2\left(d_5\frac{\partial I_h\, v}{\partial x}\right), P(u_h)\right) - \left(d_6\frac{\partial v}{\partial y} - Q_1\left(d_6\frac{\partial I_h\, v}{\partial y}\right), P(u_h)\right) \\
&+ \left(c(\tau)\, v - P(c(\tau)\, I_h\, v), P(u_h)\right)\Bigg|.
\end{aligned} \tag{67}$$

For the first term, by changing variable $i = i+1$, since $u_{h0,j,k}$ and $u_{hN_1,j,k}$ are equal to zero, we have

$$\begin{aligned}
&- \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left[x\left(\varrho_{1h}(I_h\, v)\right)\cdot h_{y_j}\cdot h_{z_k}\right]_{x_{i-1/2}}^{x_{i+1/2}} u_{hi,j,k} \\
&= - \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i+1/2}\left(\frac{b_{1i+1/2,j,k}\left(x_{i+1}^{\beta_{i,j,k}}\, v_{i+1,j,k} - x_i^{\beta_{i,j,k}}\, v_{i,j,k}\right)}{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}\right)\cdot h_{y_j}\cdot h_{z_k}\, u_{hi,j,k} \\
&\quad + \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i-1/2}\left(\frac{b_{1i-1/2,j,k}\left(x_i^{\beta_{i-1,j,k}}\, v_{i,j,k} - x_{i-1}^{\beta_{i-1,j,k}}\, v_{i-1,j,k}\right)}{x_i^{\beta_{i-1,j,k}} - x_{i-1}^{\beta_{i-1,j,k}}}\right)\cdot h_{y_j}\cdot h_{z_k}\, u_{hi,j,k} \\
&= \sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{1/2}\left(\frac{1}{2}(\bar{a}_{1k} + b_{1x_{1/2},j,k})v_{1,j,k}\right)\cdot h_{y_j}\cdot h_{z_k} u_{h1,j,k} + \\
&\quad \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i+1/2}\,\frac{b_{1i+1/2,j,k}\left(x_{i+1}^{\beta_{i,j,k}}\, v_{i+1,j,k} - x_i^{\beta_{i,j,k}}\, v_{i,j,k}\right)}{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}\cdot h_{y_j}\cdot h_{z_k}\left(u_{hi+1,j,k} - u_{hi,j,k}\right) \\
&= \sum_{i=0}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i+1/2}\left(\varrho_{1h}(I_h\, v)\right)_{x_{i+1/2}}\left(u_{hi+1,j,k} - u_{hi,j,k}\right)\cdot h_{y_j}\cdot h_{z_k}.
\end{aligned}$$

Then, re-arranging the three first sum in (67) and using boundary conditions $u_h = 0$ on $\partial\Omega_D$, we have

$$
\begin{aligned}
|\delta_{21}(I_h\, v, u_h, \tau)| \leq\ & \underbrace{|(c(\tau)\, v - P(c(\tau)\, I_h\, v), P(u_h))|}_{R_1} \\
& + \underbrace{\left|-\left(d_1(s)\,\frac{\partial v}{\partial y} - Q_1\left(d_1(s)\frac{\partial I_h\, v}{\partial y}\right), P(u_h)\right) - \left(d_2(s)\,\frac{\partial v}{\partial z} - Q_3\left(d_2(s)\frac{\partial I_h\, v}{\partial z}\right), P(u_h)\right)\right|}_{R_2} \\
& + \underbrace{\left|-\left(d_3(s)\,\frac{\partial v}{\partial x} - Q_2\left(d_3(s)\frac{\partial I_h\, v}{\partial x}\right), P(u_h)\right) - \left(d_4(s)\,\frac{\partial v}{\partial z} - Q_3\left(d_4(s)\frac{\partial I_h\, v}{\partial z}\right), P(u_h)\right)\right|}_{R_3} \\
& + \underbrace{\left|-\left(d_5(s)\,\frac{\partial v}{\partial x} - Q_2\left(d_5(s)\frac{\partial I_h\, v}{\partial x}\right), P(u_h)\right) - \left(d_6(s)\,\frac{\partial v}{\partial y} - Q_1\left(d_6(s)\frac{\partial I_h\, v}{\partial y}\right), P(u_h)\right)\right|}_{R_4} \\
& + \left|\sum_{i=0}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i+1/2}\left(\varrho_1(v) - \varrho_{1h}(I_h\, v)\right)_{x_{i+1/2}} (u_{h\,i+1,j,k} - u_{h\,i,j,k})\cdot h_{y_j}\cdot h_{z_k}\right| \\
& + \left|\sum_{i=1}^{N_1-1}\sum_{j=0}^{N_2-1}\sum_{k=1}^{N_3-1} y_{j+1/2}\left(\varrho_2(v) - \varrho_{2h}(I_h\, v)\right)_{y_{j+1/2}} (u_{h\,i,j+1,k} - u_{h\,i,j,k})\cdot h_{x_i}\cdot h_{z_k}\right| \\
& + \left|\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=0}^{N_3-1} z_{k+1/2}\left(\varrho_3(v) - \varrho_{3h}(I_h\, v)\right)_{z_{k+1/2}} (u_{h\,i,j,k+1} - u_{h\,i,j,k})\cdot h_{y_j}\cdot h_{x_i}\right|.
\end{aligned}
$$

We have that $|\delta_{21}(I_h\, v, u_h, \tau)| \leq R_1 + R_2 + R_3 + R_4 + R_5$.

As in [1, δ_{212}], we have

$$R_1 \leq C_{2_{1_1}}\, h\, (|c(\tau)|_1 + |v|_1)\, \|u_h\|_h.$$

By setting $\delta_{1i,j,k} = \dfrac{v_{i,j+1,k} - v_{i,j,k}}{h_{y_j}}$, $\delta_{2i,j,k} = \dfrac{v_{i,j,k+1} - v_{i,j,k}}{h_{z_k}}$, we also have

$$
\begin{aligned}
R_2 \leq\ & \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\left(d_1\,\frac{\partial v}{\partial y} - Q_1\left(d_1\frac{\partial I_h\, v}{\partial y}\right)\right)_{x_{i+1/2},y_j,z_k}\right|\right) u_{h\,i,j,k}\, l_{i,j,k} \\
& + \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\left(d_2\,\frac{\partial v}{\partial z} - Q_3\left(d_2\frac{\partial I_h\, v}{\partial z}\right)\right)_{x_{i+1/2},y_j,z_k}\right|\right) u_{h\,i,j,k}\, l_{i,j,k}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\left(d_1\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2},y_j,z_k} - d_{1i,j,k}\,\delta_{1i,j,k}\right|\right) u_{h\,i,j,k}\, l_{i,j,k} \\
& + \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\left(d_2\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2},y_j,z_k} - d_{1i,j,k}\,\delta_{2i,j,k}\right|\right) u_{h\,i,j,k}\, l_{i,j,k} \\
\leq\ & \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\left(d_1\,\frac{\partial v}{\partial y}\right)_{x_{i+1/2},y_j,z_k} - d_{1i,j,k}\left(\frac{v_{i,j+1,k} - v_{i,j,k}}{h_{y_j}}\right)\right|\right) u_{h\,i,j,k}\, l_{i,j,k} \\
& + \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\left(d_2\,\frac{\partial v}{\partial z}\right)_{x_{i+1/2},y_j,z_k} - d_{2i,j,k}\left(\frac{v_{i,j,k+1} - v_{i,j,k}}{h_{z_k}}\right)\right|\right) u_{h\,i,j,k}\, l_{i,j,k}
\end{aligned}
$$

$$\leq \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\frac{v_{i,j+1,k}-v_{i,j,k}}{h_{y_j}}\left(d_{1i+1/2,j,k}-d_{1i,j,k}\right)\right|\right)u_{hi,j,k}\,l_{i,j,k}$$
$$+\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} d_{1i+1/2,j,k}\left(\left|\frac{\partial v}{\partial y}\Big|_{x_{i+1/2},y_j,z_k}-\left(\frac{v_{i,j+1,k}-v_{i,j,k}}{h_{y_j}}\right)\right|\right)u_{hi,j,k}\,l_{i,j,k}$$
$$+\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\frac{v_{i,j,k+1}-v_{i,j,k}}{h_{z_k}}\left(d_{2i+1/2,j,k}-d_{2i,j,k}\right)\right|\right)u_{hi,j,k}\,l_{i,j,k}$$
$$+\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} d_{2i+1/2,j,k}\left(\left|\frac{\partial v}{\partial z}\Big|_{x_{i+1/2},y_j,z_k}-\left(\frac{v_{i,j,k+1}-v_{i,j,k}}{h_{z_k}}\right)\right|\right)u_{hi,j,k}\,l_{i,j,k}.$$

Taylor expansion with integral remainder gives us

$$|v-v_{i,j,k}|\leq|\mathcal{R}_0(v)|,\quad \text{with} \tag{68}$$
$$|\mathcal{R}_0(v)|=\int_{x_i}^{x}\int_{y_j}^{y}\int_{z_k}^{z}|v^1(s)|ds,$$

setting $x=x_i,\ y=y_{j+1}$ and $z=z_k$

$$|\mathcal{R}_0(v)|\leq h_{y_j}|v|_1\leq h\,|v|_1,\quad \text{where } |v|_1\ is \text{ the semi-norm } of\ \ H^1(\Omega).$$

Therefore

$$\frac{|v_{i,j+1,k}-v_{i,j,k}|}{h_{y_j}}\leq|v|_1\,. \tag{69}$$

For $v\in H^q(\Omega)\subset C^1(\Omega)$[3], Taylor expansion with integral remainder yields

$$v_{i,\psi,k}=v_{i,j,k}+(\psi-y_j)\frac{\partial v}{\partial y}\Big|_{x_{i+1/2},j,k}+\int_{\psi}^{y_j}\frac{\partial^2 v}{\partial y^2}\times(\psi-y)dy$$

for $\psi=y_{j+1}$

$$v_{i,j+1,k}-v_{i,j,k}=h_{y_j}\,\frac{\partial v}{\partial y}\Big|_{x_{i+1/2},j,k}+\mathcal{R}_1(v)$$

with

$$\mathcal{R}_1(v)=\int_{y_j}^{y_{j+1}}(y_j+h_{y_j}-u)\,v^{(2)}(u)du$$

Let us set $u=y_j+h_{y_j}s$, when $u\to y_j$, $s\to 0$ and when $u\to y_j+h_{y_j}$, $s\to 1$. Therefore

$$\mathcal{R}_1(v)=h_{y_j}^2\int_0^1(1-s)\,v^{(2)}(y_j+h_{y_j}s)ds,$$

we finally have

$$\left|\frac{\partial v}{\partial y}\Big|_{x_{i+1/2},j,k}-\left(\frac{v_{i,j+1,k}-v_{i,j,k}}{h_{y_j}}\right)\right|\leq h\,|v|_2\int_0^1(1-s)ds\leq\frac{h}{2}\,|v|_2\leq h\,|v|_2\,. \tag{70}$$

Therefore since in S_h, the discrete L^2 norm and the norm $\|\cdot\|_{0,h}$ are equal,

$$\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(\left|\frac{v_{i,j+1,k}-v_{i,j,k}}{h_{y_j}}\left(d_{1i+1/2,j,k}-d_{1i,j,k}\right)\right|\right)u_{hi,j,k}\,l_{i,j,k}$$
$$\leq C\,|v|_1\ h\ |d_1(\tau)|_1\,\|u_h\|_{0,h}.$$

[3] As we have already seen we need $q>5/2$

and

$$\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} d_{1\,i+1/2,j,k}\left(\left|\frac{\partial v}{\partial y}\right|_{x_{i+1/2},y_j,z_k} - \left(\frac{v_{i,j+1,k}-v_{i,j,k}}{h_{y_j}}\right)\right|\right) u_{h\,i,j,k}\, l_{i,j,k}$$
$$\leq h\ |v|_2\ \|d_1(\tau)\|_{0,h}\ \|u_h\|_{0,h} \leq h\ |v|_2\ \|d_1(\tau)\|_0\ \|u_h\|_{0,h}.$$

According to (70)

$$\left\|(I-Q_1)\left(\frac{\partial v}{\partial y}\right)\right\|_0 \leq C\,h\ |v|_2\,,$$

and according to (68)

$$\|(I-P)v\|_0 \leq C\,h\ |v|_1\,, \tag{71}$$

then we have

$$R_2 \leq C\,|v|_1\ h\ |d_1(\tau)|_1\ \|u_h\|_{0,h} + h\ |v|_2\ \|d_1(\tau)\|_0\ \|u_h\|_{0,h}$$
$$+\ C\,|v|_1\ h\ |d_2(\tau)|_1\ \|u_h\|_{0,h} + h\ |v|_2\ \|d_2(\tau)\|_0\ \|u_h\|_{0,h}$$
$$R_2 \leq C_{21_2}\,h\|u_h\|_h,$$

where $|\cdot|_2$ is the semi-norm on $H^2(\Omega)$. Using the same procedure applying for R_2, we get

$$R_3 \leq C_{21_3}\,h\|u_h\|_h,$$
$$R_4 \leq C_{21_4}\,h\|u_h\|_h.$$

Let us estimate the last term R_5:

$$R_5 = \underbrace{\left|\sum_{i=0}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} x_{i+1/2}\left(\varrho_1(v)-\varrho_{1h}(I_h\,v)\right)_{x_{i+1/2}}(u_{h\,i+1,j,k}-u_{h\,i,j,k})\cdot h_{y_j}\cdot h_{z_k}\right|}_{a}$$
$$+\underbrace{\left|\sum_{i=1}^{N_1-1}\sum_{j=0}^{N_2-1}\sum_{k=1}^{N_3-1} y_{j+1/2}\left(\varrho_2(v)-\varrho_{2h}(I_h\,v)\right)_{y_{j+1/2}}(u_{h\,i,j+1,k}-u_{h\,i,j,k})\cdot h_{x_i}\cdot h_{z_k}\right|}_{e}$$
$$+\underbrace{\left|\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=0}^{N_3-1} z_{k+1/2}\left(\varrho_3(v)-\varrho_{3h}(I_h\,v)\right)_{z_{k+1/2}}(u_{h\,i,j,k+1}-u_{h\,i,j,k})\cdot h_{y_j}\cdot h_{x_i}\right|}_{f}$$
$$:= a+e+f.$$

We will focus our attention to estimate the a term of R_5 and the terms e and f will follow the same procedure

$$a \leq \sum_{i=0}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} |\left(\varrho_1(v)-\varrho_{1h}(I_h\,v)\right)_{x_{i+1/2}}|\,x_{i+1/2}\,|(u_{h\,i+1,j,k}-u_{h\,i,j,k})|h_{y_j}\cdot h_{z_k}$$
$$\leq \sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} |\left(\varrho_1(v)-\varrho_{1h}(I_h\,v)\right)_{x_{1/2}}|\,x_{1/2}\,|u_{h\,i+1,j,k}|h_{y_j}\cdot h_{z_k}$$
$$+\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} |\left(\varrho_1(v)-\varrho_{1h}(I_h\,v)\right)_{x_{i+1/2}}|\,x_{i+1/2}\,|(u_{h\,i+1,j,k}-u_{h\,i,j,k})|h_{y_j}\,h_{z_k}.$$

Using the fact that $x_{1/2} = h_{x_0} = \frac{x_1}{2}$, and according to [1, (59)]

$$x_{i+1/2} = x_{i+1/2}^{1/2}\left(\frac{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}{b_{1\,i+1/2,j,k}(x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}})}\right)^{1/2} \times \left(\frac{x_{i+1/2}^{1/2} b_{1\,i+1/2,j,k}^{1/2}(x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}})^{1/2}}{(x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}})^{1/2}}\right).$$

Then

$$\begin{aligned}
a &\le h_{x_0}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left|\left(\varrho_1(v) - \varrho_{1h}(I_h\, v)\right)_{x_{1/2}}\right| |u_{h\,1,j,k}| h_{y_j}\cdot h_{z_k} \\
&+ \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left|\left(\varrho_1(v) - \varrho_{1h}(I_h\, v)\right)_{x_{i+1/2}}\right| x_{i+1/2}^{1/2}\left(\frac{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}{b_{1\,i+1/2,j,k}(x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}})}\right)^{1/2} \times \\
&\left(\frac{x_{i+1/2}^{1/2} b_{1\,i+1/2,j,k}^{1/2}(x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}})^{1/2}}{(x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}})^{1/2}}\right) |(u_{h\,i+1,j,k} - u_{h\,i,j,k})| h_{y_j}\cdot h_{z_k} \\
&\le a_{11} + a_{12},
\end{aligned} \tag{72}$$

with

$$a_{11} = h_{x_0}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left|\left(\varrho_1(v) - \varrho_{1h}(I_h\, v)\right)_{x_{1/2}}\right| |u_{h\,1,j,k}| h_{y_j}\cdot h_{z_k}.$$

From Lemma 3.2 and using Cauchy–Schwarz,

$$\begin{aligned}
a_{11} &\le C_1 h_{x_0}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left\{\int_{x_0}^{x_1}\left[\left|\frac{\partial \varrho_1}{\partial x}\right| + \left|\frac{\partial v}{\partial x}\right| + |v(\tau,\cdot,y,z)|\right] dx\right\} h_{y_j}\, h_{z_k} |u_{h\,1,j,k}| \\
&\le \left(C_1 h_{x_0}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left\{\int_{x_0}^{x_1}\left[\left|\frac{\partial \varrho_1}{\partial x}\right| + \left|\frac{\partial v}{\partial x}\right| + |v(\tau,\cdot,y,z)|\right]^2 dx\right\}^{1/2} h_{y_j}\, h_{z_k}\sqrt{u_{h\,1,j,k}^2 h_{x_0}}\right) \\
&\le C_1 h_{x_0}\left\{\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} h_{y_j} h_{z_k}\int_{x_0}^{x_1}\left[\left|\frac{\partial \varrho_1}{\partial x}\right| + \left|\frac{\partial v}{\partial x}\right| + |v|\right]^2 dx\right\}^{1/2}\sqrt{\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} u_{h\,1,j,k}^2 h_{x_0} h_{y_j} h_{z_k}}.
\end{aligned}$$

Therefore

$$a_{11} \le C_{11} h_{x_0}\left\{\int_{x_0}^{x_1}\left[\left|\frac{\partial \varrho_1}{\partial x}\right| + \left|\frac{\partial v}{\partial x}\right| + |v|\right]^2 dx\right\}^{1/2}\sqrt{\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1} u_{h\,1,j,k}^2 h_{x_0} h_{y_j} h_{z_k}}. \tag{73}$$

For the a_{12} term,

$$\begin{aligned}
a_{12} &= \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left|\left(\varrho_1(v) - \varrho_{1h}(I_h\, v)\right)_{x_{i+1/2}}\right| x_{i+1/2}^{1/2}\left(\frac{x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}}}{b_{1\,i+1/2,j,k}(x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}})}\right)^{1/2} \times \\
&\left(\frac{x_{i+1/2}^{1/2} b_{1\,i+1/2,j,k}^{1/2}(x_{i+1}^{\beta_{i,j,k}} + x_i^{\beta_{i,j,k}})^{1/2}}{(x_{i+1}^{\beta_{i,j,k}} - x_i^{\beta_{i,j,k}})^{1/2}}\right) |u_{h\,i+1,j,k} - u_{h\,i,j,k}| h_{y_j}\cdot h_{z_k}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{x_{i+1}}{x_{i+1/2}} = 1 + \frac{h_{x_i}}{2\,x_{i+1/2}}; \quad \frac{x_i}{x_{i+1/2}} = 1 - \frac{h_{x_i}}{2\,x_{i+1/2}}, \\
&\frac{h_{x_i}}{2\,x_{i+1/2}} = \frac{h_{x_i}}{(x_i + x_{i+1})} = \frac{h_{x_i}}{(x_i + x_{i+1} - x_i + x_i)} = \frac{h_{x_i}}{(2\,x_i + h_{x_i})} < 1, \ i \ge 1.
\end{aligned}$$

Using Taylor's expansion we have

$$x_{i+1/2}\frac{x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}}}{x_{i+1}^{\beta_{i,j,k}}+x_i^{\beta_{i,j,k}}}=x_{i+1/2}\left(\frac{\left(1+\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)^{\beta_{i,j,k}}-\left(1-\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)^{\beta_{i,j,k}}}{\left(1+\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)^{\beta_{i,j,k}}+\left(1-\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)^{\beta_{i,j,k}}}\right)$$

$$=x_{i+1/2}\left(\frac{\left(1+\beta_{i,j,k}\,\mathcal{O}\left(\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)\right)-\left(1-\beta_{i,j,k}\,\mathcal{O}\left(\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)\right)}{\left(1+\beta_{i,j,k}\,\mathcal{O}\left(\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)\right)+\left(1-\beta_{i,j,k}\,\mathcal{O}\left(\dfrac{h_{x_i}}{2\,x_{i+1/2}}\right)\right)}\right)\le C_{28}\,\beta_{i,j,k}\,h_{x_i},$$

because $\beta_{i,j,k}=\dfrac{b_{1\,i+1/2,j,k}}{\bar{a}_{1k}}$,

$$x_{i+1/2}\frac{x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}}}{b_{1\,i+1/2,j,k}(x_{i+1}^{\beta_{i,j,k}}+x_i^{\beta_{i,j,k}})}\le C_{29}\,h_{x_i}. \tag{74}$$

From Lemma 3.2 and Eq. (74) and using the Cauchy–Schwarz inequality,

$$a_{12}\le C_1\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left\{\int_{x_i}^{x_{i+1}}\left[\left|\frac{\partial\varrho_1}{\partial x}\right|+\left|\frac{\partial v}{\partial x}\right|+|v(\tau,\cdot,y,z)|\right]dx\right\}h_{x_i}^{1/2}h_{y_j}\cdot h_{z_k}\times$$
$$\left(\frac{x_{i+1/2}^{1/2}b_{1\,i+1/2,j,k}^{1/2}(x_{i+1}^{\beta_{i,j,k}}+x_i^{\beta_{i,j,k}})^{1/2}}{(x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}})^{1/2}}\right)|u_{h\,i+1,j,k}-u_{h\,i,j,k}|$$

$$\le C_1\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left\{\int_{x_i}^{x_{i+1}}\left[\left|\frac{\partial\varrho_1}{\partial x}\right|+\left|\frac{\partial v}{\partial x}\right|+|v(\tau,\cdot,y,z)|\right]^2dx\right\}^{1/2}h_{x_i}h_{y_j}\cdot h_{z_k}\times$$
$$\left(\frac{x_{i+1/2}^{1/2}b_{1\,i+1/2,j,k}^{1/2}(x_{i+1}^{\beta_{i,j,k}}+x_i^{\beta_{i,j,k}})^{1/2}}{(x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}})^{1/2}}\right)|u_{h\,i+1,j,k}-u_{h\,i,j,k}|$$
$$\le C_1\,h\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left\{h_{y_j}\cdot h_{z_k}\int_{x_i}^{x_{i+1}}\left[\left|\frac{\partial\varrho_1}{\partial x}\right|+\left|\frac{\partial v}{\partial x}\right|+|v(\tau,\cdot,y,z)|\right]^2dx\right\}^{1/2}\times$$
$$\left(h_{y_j}\cdot h_{z_k}\frac{x_{i+1/2}b_{1\,i+1/2,j,k}(x_{i+1}^{\beta_{i,j,k}}+x_i^{\beta_{i,j,k}})}{(x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}})}\right)^{1/2}|u_{h\,i+1,j,k}-u_{h\,i,j,k}|$$
$$\le C_1\,h\left\{\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}h_{y_j}\cdot h_{z_k}\int_{x_i}^{x_{i+1}}\left[\left|\frac{\partial\varrho_1}{\partial x}\right|+\left|\frac{\partial v}{\partial x}\right|+|v(\tau,\cdot,y,z)|\right]^2dx\right\}^{1/2}\times$$
$$\left\{\sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1}\sum_{k=1}^{N_3-1}\left(h_{y_j}\cdot h_{z_k}\frac{x_{i+1/2}b_{1\,i+1/2,j,k}(x_{i+1}^{\beta_{i,j,k}}+x_i^{\beta_{i,j,k}})}{(x_{i+1}^{\beta_{i,j,k}}-x_i^{\beta_{i,j,k}})}\right)(u_{h\,i+1,j,k}-u_{h\,i,j,k})^2\right\}^{1/2}$$
$$\le C_{72}\,h\left\{\int_{x_1}^{x_{N_1}}\left[\left|\frac{\partial\varrho_1}{\partial x}\right|+\left|\frac{\partial v}{\partial x}\right|+|v(\tau,\cdot,y,z)|\right]^2dx\right\}^{1/2}\times\|u_h\|_{1,h_x}.$$

Then, we have

$$a \leq \left(C_{71} h_{x_0} \left\{ \int_{x_0}^{x_1} \left[\left| \frac{\partial \varrho_1}{\partial x} \right| + \left| \frac{\partial v}{\partial x} \right| + |v(\tau, \cdot, y, z)| \right]^2 dx \right\}^{1/2} \sqrt{\sum_{j=1}^{N_2-1} \sum_{K=1}^{N_3-1} u_{h_{1,j,k}}^2 h_{x_0} h_{y_j} h_{z_k}} \right)$$

$$+ C_{72}\, h \left\{ \int_{x_1}^{x_{N_1}} \left[\left| \frac{\partial \varrho_1}{\partial x} \right| + \left| \frac{\partial v}{\partial x} \right| + |v(\tau, \cdot, y, z)| \right]^2 dx \right\}^{1/2} \times \|u_h\|_{1,h_x} .$$

Coming back to R_5

$$\begin{aligned}
R_5 &= a + e + f \\
&\leq \left(C_{71} h_{x_0} \left\{ \int_{x_0}^{x_1} \left[\left| \frac{\partial \varrho_1}{\partial x} \right| + \left| \frac{\partial v}{\partial x} \right| + |v(\tau, \cdot, y, z)| \right]^2 dx \right\}^{1/2} \sqrt{\sum_{j=1}^{N_2-1} \sum_{K=1}^{N_3-1} u_{h_{1,j,k}}^2 h_{x_0} h_{y_j} h_{z_k}} \right) \\
&+ C_{72}\, h \left\{ \int_{x_1}^{x_{N_1}} \left[\left| \frac{\partial \varrho_1}{\partial x} \right| + \left| \frac{\partial v}{\partial x} \right| + |v(\tau, \cdot, y, z)| \right]^2 dx \right\}^{1/2} \times \|u_h\|_{1,h_x} \\
&+ \left(C_{271} h_{y_0} \left\{ \int_{y_0}^{y_1} \left[\left| \frac{\partial \varrho_2}{\partial y} \right| + \left| \frac{\partial v}{\partial y} \right| + |v(\tau, x, \cdot, z)| \right]^2 dy \right\}^{1/2} \sqrt{\sum_{k=1}^{N_3-1} \sum_{i=1}^{N_1-1} u_{h_{i,1,k}}^2 h_{y_0} h_{x_i} h_{z_k}} \right) \\
&+ C_{272}\, h \left\{ \int_{y_1}^{y_{N_2}} \left[\left| \frac{\partial \varrho_2}{\partial y} \right| + \left| \frac{\partial v}{\partial y} \right| + |v(\tau, x, \cdot, z)| \right]^2 dy \right\}^{1/2} \times \|u_h\|_{1,h_y} \\
&+ \left(C_{371} h_{z_0} \left\{ \int_{z_0}^{z_1} \left[\left| \frac{\partial \varrho_3}{\partial z} \right| + \left| \frac{\partial v}{\partial z} \right| + |v(\tau, x, y, \cdot)| \right]^2 dz \right\}^{1/2} \sqrt{\sum_{j=1}^{N_2-1} \sum_{i=1}^{N_1-1} u_{h_{i,j,1}}^2 h_{z_0} h_{y_j}\, h_{x_i}} \right) \\
&+ C_{372}\, h \left\{ \int_{z_1}^{z_{N_3}} \left[\left| \frac{\partial \varrho_3}{\partial z} \right| + \left| \frac{\partial v}{\partial z} \right| + |v(\tau, x, y, \cdot)| \right]^2 dz \right\}^{1/2} \times \|u_h\|_{1,h_z} \\
&\leq C_4\, h \left\{ \int_{y_0}^{y_{N_2}} \left[\left| \frac{\partial \varrho_2}{\partial y} \right| + \left| \frac{\partial v}{\partial y} \right| + |v(\tau, x, \cdot, z)| \right]^2 dy \right\}^{1/2} \times \\
&\left(\sqrt{\sum_{k=1}^{N_3-1} \sum_{j=1}^{N_2-1} u_{h_{i,1,k}}^2 h_{y_0} h_{x_i}\, h_{z_k}} + \|u_h\|_{1,h_y} \right) + \left(\sqrt{\sum_{j=1}^{N_2-1} \sum_{k=1}^{N_3-1} u_{h_{1,j,k}}^2 h_{x_0} h_{y_j}\, h_{z_k}} + \|u_h\|_{1,h_x} \right) \\
&\times C_5\, h \left\{ \int_{x_0}^{x_{N_1}} \left[\left| \frac{\partial \varrho_1}{\partial x} \right| + \left| \frac{\partial v}{\partial x} \right| + |v(\tau, \cdot, y, z)| \right]^2 dx \right\}^{1/2} \\
&+ \left(\sqrt{\sum_{j=1}^{N_2-1} \sum_{i=1}^{N_1-1} u_{h_{i,j,1}}^2 h_{z_0} h_{y_j}\, h_{x_i}} + \|u_h\|_{1,h_z} \right) \times \\
&C_6\, h \left\{ \int_{z_0}^{z_{N_3}} \left[\left| \frac{\partial \varrho_3}{\partial z} \right| + \left| \frac{\partial v}{\partial z} \right| + |v(\tau, x, y, \cdot)| \right]^2 dz \right\}^{1/2} \\
R_5 &\leq C_7\, h \left(|\varrho_1|_2 + |\varrho_2|_2 + |\varrho_3|_2 + \|v\|_1 \right) \|u_h\|_h ,
\end{aligned}$$

where $\|v\|_1$ correspond to the norm on $H^1(\Omega)$. Therefore

$$|\delta_{21}(v, u_h, \tau)| \leq C_{2_1} h\, \|u_h\|_h \tag{75}$$

Using (75) to estimate $|{Y_2}^m|$, we have

$$|{Y_2}^m| \leq C_{21}\, h\, \|u_h\|_h. \tag{76}$$

Coming back to the equation

$$\left(\frac{\Theta^{m+1}-\Theta^m}{\Delta\tau_m}, u_h\right)_h + B_h\left(\Theta^{m+1}, P(u_h); \tau_{m+1}\right) = Y_1^m + Y_2^m + Y_3^m,$$

combining the estimates (63), (65) and (76), we get from (61)

$$\left(\frac{\Theta^{m+1}-\Theta^m}{\Delta\tau_m}, u_h\right)_h + B_h\left(\Theta^{m+1}, P(u_h); \tau_{m+1}\right) \leq \Gamma^m(\Delta\tau_m, h)\|u_h\|_h, \tag{77}$$

where

$$\Gamma^m(\Delta\tau_m, h) := \Gamma_1^m(\Delta\tau_m, h) + h\,(C_4 + C_{21}),$$

and according to (64)

$$\Gamma_1^m(\Delta\tau_m, h) := \frac{1}{\Delta\tau_m}\int_{\tau_m}^{\tau_{m+1}}\left\|(P-I)\left(\frac{\partial v(s)}{\partial s}\right)\right\|_0 ds + \int_{\tau_m}^{\tau_{m+1}}\left\|\frac{\partial^2 v(s)}{\partial s^2}\right\|_0 ds.$$

We now choose in (77) the particular test function $u_h = \Theta^{m+1}$. As in [1], using the coercivity property, we get

$$\left(\frac{\Theta^{m+1}-\Theta^m}{\Delta\tau_m}, u_h\right)_h + B_h\left(u_h, P(u_h); \tau_{m+1}\right) \geq \frac{1}{2\,\Delta\tau_m}\left[\|\Theta^{m+1}\|_{0,h}^2 - \|\Theta^m\|_{0,h}^2\right] + C\|u_h\|_h^2$$

Following [1], we have

$$\frac{1}{\Delta\tau_m}\left[\|\Theta^{m+1}\|_{0,h}^2 - \|\Theta^m\|_{0,h}^2\right] \leq \frac{2}{C}\left[\Gamma^m(\Delta\tau_m, h)\right]^2 - \frac{2}{C}\|u_h\|_h^2,$$

then

$$\frac{1}{\Delta\tau_m}\left[\|\Theta^{m+1}\|_{0,h}^2 - \|\Theta^m\|_{0,h}^2\right] \leq \frac{2}{C}\left[\Gamma^m(\Delta\tau_m, h)\right]^2,$$

multiplying by $\Delta\tau_m$ and summing up, thus we obtain

$$\|\Theta^M\|_{0,h}^2 \leq \|\Theta^0\|_{0,h}^2 + \frac{1}{C}\sum_{m=0}^{M-1}(\Delta\tau_m)\left[\Gamma^m(\Delta\tau_m, h)\right]^2. \tag{78}$$

It remains to estimate the sum on the right-hand side of (78). Following [1], we get

$$\left[\Gamma^m(\Delta\tau_m, h)\right]^2 \leq 3\left\{\left[\Gamma_1^m(\Delta\tau_m, h)\right]^2 + h^2\,(C_4 + C_{21})^2\right\} \tag{79}$$

$$\begin{aligned}\left[\Gamma_1^m(\Delta\tau_m, h)\right]^2 &\leq 2\left[\frac{1}{(\Delta\tau_m)^2}\left\{\int_{\tau_m}^{\tau_{m+1}}\left\|(P-I)\frac{\partial v(s)}{\partial s}\right\|_0 ds\right\}^2 + \left\{\int_{\tau_m}^{\tau_{m+1}}\left\|\frac{\partial^2 v(s)}{\partial s^2}\right\|_0 ds\right\}^2\right],\\ &\leq 2\left[\frac{1}{\Delta\tau_m}\int_{\tau_m}^{\tau_{m+1}}\left\|(P-I)\frac{\partial v(s)}{\partial s}\right\|_0^2 ds + (\Delta\tau_m)\int_{\tau_m}^{\tau_{m+1}}\left\|\frac{\partial^2 v(s)}{\partial s^2}\right\|_0^2 ds\right],\end{aligned}$$

and therefore,

$$\sum_{m=0}^{M-1}(\Delta\tau_m)\left[\Gamma^m(\Delta\tau_m, h)\right]^2 = 2\left[\int_0^T\left\|(P-I)\frac{\partial v(s)}{\partial s}\right\|_0^2 ds + (\Delta\tau)^2\int_0^T\left\|\frac{\partial^2 v(s)}{\partial s^2}\right\|_0^2 ds\right],$$

where $\Delta\tau = \max_{0,\dots,M-1} |\Delta\tau_m|$. Using the fact that,

$$\left\| (P-I)\,\frac{\partial v(s)}{\partial s} \right\|_0 \le h \left| \frac{\partial v(s)}{\partial s} \right|_1,$$

and so we arrive at

$$\sum_{m=0}^{M-1} (\Delta\tau_m)\,\left[\Gamma^m(\Delta\tau_m, h)\right]^2 \le 2\left[h^2 \left\| \frac{\partial v(s)}{\partial s} \right\|^2_{L^2(0,T;H^1(\Omega))} + (\Delta\tau)^2 \left\| \frac{\partial^2 v(s)}{\partial s^2} \right\|^2_{L^2(0,T;L^2(\Omega))} \right]. \tag{80}$$

Putting (78)–(80) together, we finally get the estimate

$$\|\Theta^M\|^2_{0,h} - \|\Theta^0\|^2_{0,h} \le C\,(h^2 + (\Delta\tau)^2). \tag{81}$$

By taking $v^0 = I_h\, v_0$, we have $\|\Theta^0\|^2_{0,h} = 0$ and we get the error estimate

$$\|\Theta^m\|_{0,h} \le C_9\,(h + \Delta\tau) \tag{82}$$

which is actually

$$\|I_h\, v(\tau_m) - v_h^m\|_{0,h} \le C_9\,(h + \Delta\tau), \tag{83}$$

combining Eqs. (60) and (83) we get the proof

$$\begin{aligned} \|v(\tau_m) - v_h^m\|_{0,h} &\le \|v(\tau_m) - I_h\, v(\tau_m)\|_{0,h} + \|I_h\, v(\tau_m) - v_h^m\|_{0,h} \\ &\le C_{31}\, h + C_9\,(h + \Delta\tau), \\ &\le C\,(h + \Delta\tau). \quad \blacksquare \end{aligned} \tag{84}$$

5. Numerical experiments

To check the robustness of the method discussed in the previous sections, some numerical experiments will be carried out in this section. All computations were performed in Matlab 13. The final time in all our simulations is $T = 1$. For the temporal error the exact or reference solution is the numerical solution with the smaller time step $\Delta t = 1/800$ and mesh subdivision $N_1 = 70$, $N_2 = 70$, $N_3 = 2$. For the spatial error we used the numerical solution on the mesh with $N_1 = 24 = N_2$, $N_3 = 24$ and $\Delta t = 1/250$ as the reference solution. Note we have projected the numerical solutions on coarse grids in this reference grid in order to compute our spatial errors. For the test, we choose $\theta = 1$ and $\mu = 0$ as it has been done in [8]. We have chosen the ramp payoff final condition given by

$$v(T, x, y, z) = \max(0, \max(x, y) - E),\ \ (x, y, z) \in I_x \times I_y \times I_z, \tag{85}$$

where $E < \max(x, y)$ denotes the exercise price (strike price) of the options and the four boundary conditions are given by

$$\begin{aligned} &v(\tau, 0, y, z) = 0,\ \ v(\tau, x, 0, z) = 0, \\ &v(\tau, X, y, z) = \max(X, y) - E,\ \ v(\tau, x, Y, z) = \max(x, Y) - E. \end{aligned} \tag{86}$$

The domain where we compare the solution is $\Omega = I_x \times I_y \times I_z = [0, x_{\max}] \times [0, y_{\max}] \times [\zeta, z_{\max}]$. We have calculated the errors of the numerical solutions using the discrete norm defined on the left-hand side of estimate in Theorem 4.1 given by

$$\left\| v(\tau_m) - v_h^m \right\|_{0,h} = \left(\sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} \sum_{k=1}^{N_3-1} l_{i,j,k} \times \left(v\left(\tau^m, x_i, y_j, z_k\right) - v^m_{i,j,k}\right)^2 \right)^{1/2}. \tag{87}$$

Table 1 is the spatial computed errors ($\theta = 1$) of the fitted finite volume at time $\Delta t = 1/250$ on the computational domain $[0, 3] \times [0, 1] \times [\zeta, 0.36]$ with parameters values $r = 0.1$, $\mu = 0$, $\sigma = 0.6$, $\xi = 1$, $\zeta = 0.01$, $E = 1.5$, $\rho_{1,2} = 0.3$, $\rho_{1,3} = 0.34$, $\rho_{2,3} = 0.35$ and different mesh size N_1, N_2, N_3

Table 1
Spatial errors on the computational domain $[0,3]\times[0,1]\times[\zeta,0.36]$ with parameters values $r=0.1$, $\mu=0$, $\sigma=0.6$, $\xi=1$, $\zeta=0.01$, $E=1.5$, $\rho_{1,2}=0.3$, $\rho_{1,3}=0.34$, $\rho_{2,3}=0.35$ and different mesh size N_1, N_2, N_3.

Time subdivision	250		
Mesh subdivision	12 × 12 × 12	6 × 6 × 6	3 × 3 × 3
Error of fitted finite volume method	2.10 E-03	5.90 E-03	1.04 E-02

Table 2
Spatial errors on the computational domain $[0,3]\times[0,1]\times[\zeta,1]$ with parameters value $r=0.1$, $\mu=0$, $\sigma=1$, $\xi=1$, $\zeta=0.01$, $E=1.5$, $\rho_{1,2}=0.3$, $\rho_{1,3}=0.34$, $\rho_{2,3}=0.35$ and different mesh size N_1, N_2, N_3.

Time subdivision	250		
Mesh subdivision	12 × 12 × 12	6 × 6 × 6	3 × 3 × 3
Error of fitted finite volume method	5.00 E-03	1.31 E-02	2.14 E-02

Table 3
Temporal errors with the time steps $\Delta t=1/400; 1/200; 1/100; 1/50$ on the computational domain $[0,3]\times[0,3]\times[\zeta,0.1]$ with parameters values $r=0.5$, $\mu=0$, $\sigma=0.5$, $\xi=1$, $\zeta=0.01$, $E=1$, $\rho_{1,2}=0.3$, $\rho_{1,3}=0.34$, $\rho_{2,3}=0.35$ and mesh sizes $N_1=70, N_2=70, N_3=2$.

Mesh subdivision	$(N_1=70)\times(N_2=70)\times(N_3=2)$			
Time subdivision	400	200	100	50
Error of fitted finite volume method	0.00506	0.00914	0.0142	0.0232

Table 2 is the spatial computed errors of the discrete norm ($\theta=1$) of the fitted finite volume at time $\Delta t=2/250$ on the computational domain $[0,3]\times[0,1]\times[\zeta,1]$ with parameters value $r=0.1$, $\mu=0$, $\sigma=1$, $\xi=1$, $\zeta=0.01$, $E=1.5$, $\rho_{1,2}=0.3$, $\rho_{1,3}=0.34$, $\rho_{2,3}=0.35$ and different mesh size N_1, N_2, N_3.

Table 3 is the temporal computed errors for ($\theta=1$) of the fitted finite volume using the time steps $\Delta t=1/400; 1/200; 1/100; 1/50$ on the computational domain $[0,3]\times[0,3]\times[\zeta,0.1]$ with parameters values $r=0.5$, $\mu=0$, $\sigma=0.2$, $\xi=1$, $\zeta=0.01$, $E=1$, $\rho_{1,2}=0.3$, $\rho_{1,3}=0.34$, $\rho_{2,3}=0.35$ and mesh sizes $N_1=70, N_2=70, N_3=2$.

Tables 1–3 clearly show the convergence in time and space of the fitted scheme since the errors decrease when the temporal stepsize or the spatial stepsize decreases.

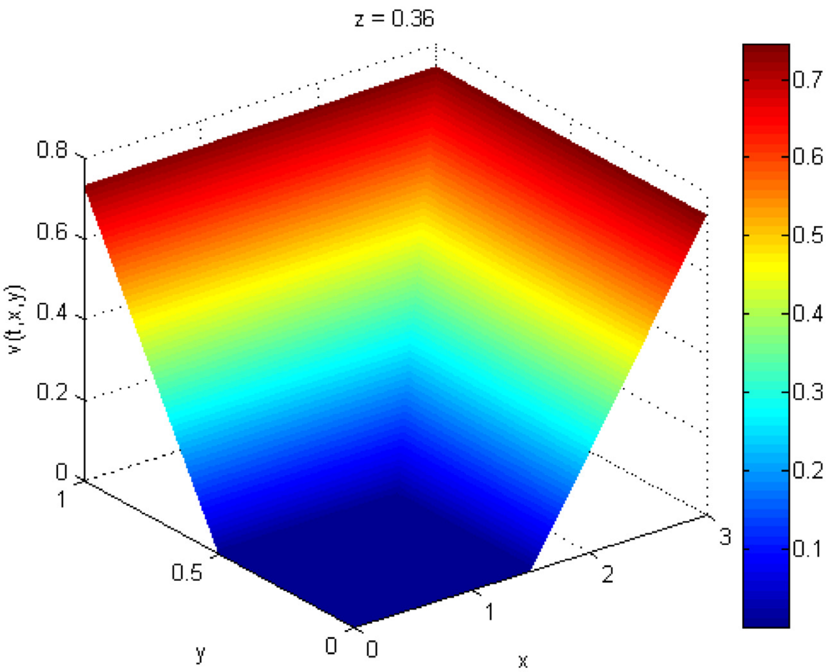
By fitting the data in Fig. 1(b) in log scale, we have obtained 1.1 and 1.05 as slopes, so the spatial computed rate of convergence for the method are very close to $\mathcal{O}$(h). By fitting the data in Fig. 1(c) in log scale, we have obtained 0.85 as order of temporal convergence. So our empirical rates of convergence are $\mathcal{O}(\Delta t)$ in time and $\mathcal{O}(h)$ in space, which is in agreement with our theoretical result.

6. Conclusion

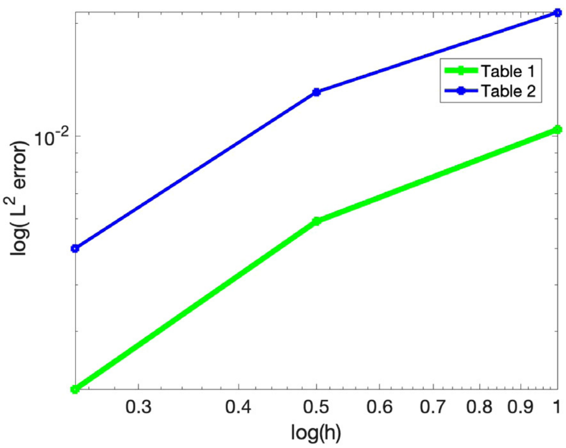
In this paper, we have presented and analyzed the finite volume method with fitted technique for the valuation of options on two dimensional assets with stochastic volatility. We have seen that the method can also be formulated using the finite-element method. Consistency of the spatial discretization and an upper bound of errors estimates of orders $O(h+\Delta\tau)$ for the fully discretization scheme have been established. Numerical experiments have been performed to confirm the temporal and spatial convergence of the fitted scheme.

Declaration of competing interest

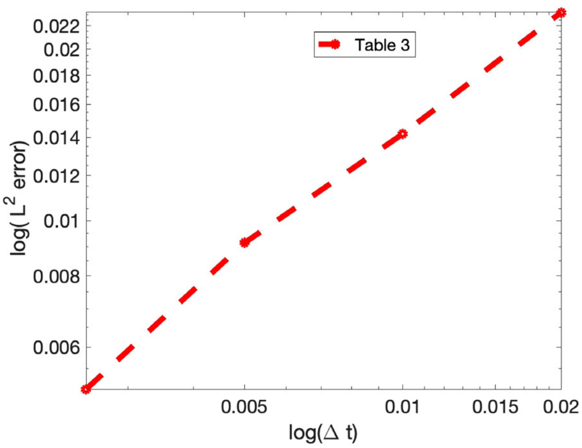
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.



(a)



(b)



(c)

Fig. 1. (a) shows the plot of the numerical solution for fixed value of $z = 0.36$ with final t time $T = 1$ with the value parameters $r = 0.1$, $\mu = 0$, $\sigma = 0.6$, $\xi = 1$, $\zeta = 0.01$, $E = 1.5$, $\rho_{1,2} = 0.3$, $\rho_{1,3} = 0.34$ and $\rho_{2,3} = 0.35$ under the computational domain $[0, 3] \times [0, 1] \times [\zeta, 0.36]$. The spatial convergence is presented in log-scale at (b) with more details in Tables 1 and 2. Graph (c) shows the temporal convergence with more details in Table 3.

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