Educational Studies in Mathematics (2023) 113:389–404
https://doi.org/10.1007/s10649-023-10225-0

On the origin, characteristics, and usefulness of instrumental and relational understanding

Rune Herheim[1]

Accepted: 13 March 2023 / Published online: 27 April 2023
© The Author(s) 2023

Abstract
In this paper, the aim is to make a theoretical contribution by focusing on the origin, characteristics, and potential values of the concepts of instrumental and relational understanding. Five characteristics are identified to make it easier to operationalise the concepts and use them as an analytical framework. There is also a focus on how the concepts are closely related to two rationales for learning, the instrumental and the social rationale. The potential values of the concepts and their rationales are shown by using them to suggest a revision of the van Hiele framework and to analyse three empirical examples concerning young students' understanding of triangles and a cylinder.

Keywords Instrumental and relational understanding · Rationales for learning · Discourse · Van Hiele revised · Geometry

1 Introduction

This paper concerns instrumental and relational understanding as educational concepts. The main purpose is to clarify and elaborate on their content, how they were introduced and whom to give credit for them, what they relate to and build on, and what their value and usefulness can be. One reason for doing this is that it seems to be unknown to many that the man who came up with the concepts was Stieg Mellin-Olsen. Another reason is that the depth and complexity of the concepts connected to Mellin-Olsen's emphasis on them as results of different rationales for learning have deteriorated and almost vanished. The aspects of value and usefulness are addressed by showing how the concepts can be used to contribute to a revision of the van Hiele levels for geometrical thinking and to analyse and reflect upon empirical data on students' understanding of polygons. The focus of the paper can therefore be summarised with the two following research questions: *What characterises the concepts of instrumental and relational understanding*? *And what are the potential values of these concepts*?

✉ Rune Herheim
rune.herheim@hvl.no

1 Department of Language, Literature, Mathematics and Interpreting, Faculty of Education, Arts and Sports, Western Norway University of Applied Sciences, Bergen, Norway

The works of Sfard and Barwell have been an important point of departure for this paper. Sfard (2007), as part of her commognitive framework, presented the view that thinking and communicating are inseparable processes and "that learning mathematics is tantamount to modifying and extending one's discourse" (p. 567). From this perspective, students' mathematical thinking can be investigated by observing their talk and actions; it is in fact "only understood in the context of demands and patterns of the overall communicative activity" (Sfard & Kieran, 2001, p. 47). Barwell (2016) presented a Bakhtinian, dialogic approach. He argued that mathematical understanding emerges through "dialogic relations between multiple discourses, voices and languages in mathematics classroom interaction" and that "students do not follow a linear path from informal to formal mathematical discourse" (p. 331). It is a move from informal and everyday language towards using more formal and conventional mathematical language. Developing mathematical understanding can therefore be seen as dynamic and nonlinear processes of developing mathematical language and discourse.

The term *discourse* is used in this paper to holistically describe the actions and practices in the classroom, and what kind of thinking, language, questions, and answers are accepted and expected to take place (Mellin-Olsen, 1996). The ways teachers and students communicate, the kind of questions they ask, and the language they use strongly influence which discourses develop in a class (Boaler & Brodie, 2004). This discursive approach is an important basis for an in-depth understanding of the concepts of instrumental and relational understanding and the suggested revision of the van Hiele framework.

2 Instrumental and relational understanding—as results of different rationales for learning

In the early 1970s, Stieg Mellin-Olsen and Richard R. Skemp started their collaboration (e.g., Skemp & Mellin-Olsen, 1973), and at some point, Mellin-Olsen introduced the concepts of instrumental and relational understanding to Skemp. They did several publications about these concepts (e.g., Mellin-Olsen, 1975, 1981; Skemp, 1976). Skemp's, 1976 article is by far the most well-known, and there, he credits Mellin-Olsen for the two concepts: "It was brought to my attention some years ago by Stieg Mellin-Olsen" (Skemp, 1976, p. 20). In that article, Skemp presented very well the easiest accessible part of Mellin-Olsen's thinking, namely the characteristics of the two concepts and reasons for and against teaching for instrumental and relational understanding.

According to Mellin-Olsen and Skemp, instrumental understanding concerns students' rule-based approach with a focus on how to do something. Initially, Mellin-Olsen (1975) related this to Piaget's concept of figurative learning (e.g., Piaget & Inhelder, 1973), remembering disconnected facts and procedures. Skemp (1976) argued that instrumental understanding describes students' ability to know how to follow mathematical procedures and rules without knowing much about why. Mellin-Olsen (1981, 1984) described the foundation for the instrumental approach as a conformist emphasis on the teacher showing what needs to be done and the students producing answers quickly. It is about adhering systematically over time to rules and procedures without awareness or reason (Skemp, 1976). As Skemp (1979) wrote: "Instrumental understanding in a mathematical situation consists of recognizing a task as one of a particular class for which one already knows a rule" (p. 259). Instrumental understanding can be described, by rephrasing Dewey (1933), as "learning by doing *without* reflection." These negative characteristics mainly come to the fore when the instrumental approach is the dominant one. Instrumental mathematics is still to be regarded as a form of understanding, with strengths such as immediate and more apparent rewards (correct answers) and it is usually easier to understand and usually provides answers quicker than relational mathematics (Skemp, 1976).

Relational understanding concerns understanding structures, searching for patterns, and relating new concepts to previous understanding. Initially, Mellin-Olsen linked this to Piaget's concept of operative learning, an interrelated understanding that goes beyond rote learning. The focus is on seeing mathematical connections and relations, understanding why rules and procedures are the way they are, why they work, and when it is sensible to use them. It is about understanding how mathematical structures are connected and how mathematical concepts are related. Skemp (1976, p. 23) pointed out four advantages of a relational understanding of mathematics: more adaptable to new tasks, easier to remember, more effective as a goal, and easier to structure and extend.

So, both instrumental and relational understanding have their advantages, but the instrumental one is more short-term, rigid, and context-dependent. A key part of Mellin-Olsen's (1981) thinking was to elaborate on possible reasons why instrumental teaching had such a dominant role. This led him to focus on the underlying premises of instrumental and relational understanding. He argued that these understandings are the results of some deeper structures that he termed *rationales for learning*. The rationale that had been dominating the teaching of mathematics he called the I-rationale, the instrumental rationale. The I-rationale means that students learn because the subject matter is part of the school, and the school is important for their future. It is quantitative in the sense that it is about the number of tasks, which operations can be done and at what speed they can be done. "Instrumental understanding can thus be seen as a symptom of some deeper structure, instrumentalism" (Mellin-Olsen, 1981, p. 351), and he defined instrumentalism as a learning strategy, as a rationale for learning. To add to the complexity, Mellin-Olsen (1981, 1984) argued that instrumentalism can generate both instrumental and relational understanding. What he meant was that even though the dominating driving force is that the students do things because they have to, because it is expected in the school setting, and because it makes life easier for them, they can still develop relational understanding.

To establish another rationale for learning, Mellin-Olsen emphasised the social dimension of learning. In 1977, he wrote a book about learning as a social process and in his 1981 article in *Educational Studies in Mathematics*, he referred to Bateson (1973) when he emphasised that what students say and do "cannot be studied and interpreted independently of the context in which it takes place" (p. 353). He, therefore, introduced the S-rationale, the social rationale. This rationale means that students' learning is dependent on the social network they are a part of and they learn because they find the topic important, because it makes sense to learn it. It is qualitative in the sense that students find the content interesting because it can help them achieve things that go beyond their exams. These rationales can help increase the awareness about what leads students towards instrumental or relational understanding and unravel some of the complexity connected to how different understandings can be intertwined and shifting.

As part of taking into consideration the social dimension, Mellin-Olsen (1981, 1984, 1991, 1996) argued for how instrumental and relational understanding can be regarded as ways of being, as discourses. Ways of being in the classroom are about the teacher's and students' focuses, what they regard as understanding, what they say and do, and what questions they ask, and together this constitutes the classroom discourse. An instrumental discourse, which Mellin-Olsen (1996) termed as the exercise discourse in mathematics, is characterised by focusing on the number of tasks done, answers, staying on track, and not falling behind because all the topics must be covered before the exam. A relational discourse is based on a willingness to understand why something is the case and is characterized by dialogic processes through which understanding is created and the logic of an explanation is examined critically.

According to Mellin-Olsen (1977, pp. 74–75), the discourse concept is important because it highlights how instrumental understandings are not just things students say and do, they are ways of being and parts of a larger whole, at a system level. It goes deeper, to the roots, to the rationales for learning that students have. The participants, both students and teachers, reproduce it whether they want to or not, whether consciously or not. To achieve a change, one must address the character of the communication structures of the system, the ways of being in the classrooms, not just what is said and how it is said, but investigating the school culture that teachers and students are part of.

2.1 Prototype thinking

Prototype thinking is one approach to better understand what instrumental understanding can be. Having difficulties making sense of shapes that are different from what you previously have encountered, for example, the idea that only equilateral shapes are genuine triangles, can be regarded as instrumental understanding because of a lack of searching for patterns and relating new concepts to previous understanding. Rosch (1973) introduced the concept of *natural prototypes* as a way of understanding how categories or concepts can be developed and dominated by particular examples. There are forms and shapes such as squares and equilateral triangles that are more perceptually salient than others, and they thus attract more attention and are more easily remembered. Hershkowitz (1989) called them "super examples" that are much more popular than all others. Prototype understanding is strongly based on the visual aspect, Rosch (1973) argued. Instead of basing an understanding on definitions, key properties, and necessary and sufficient conditions for a shape to be categorised as a triangle, a prototype understanding is based on what emerges as the most typical member within the concept. It can vary how dependent students are on prototype thinking because it can be more prevalent for some shapes than others, and it can depend on the context and which examples are being discussed.

Rosch (1973) went on by saying that concepts for which natural prototypes play a key role are easier to learn than concepts in which a distorted, peripheral prototype plays a key part. Even when the natural prototypes are not central to a category, they are easier to learn. Rosch documented in her study that natural prototypes like perfect squares, circles, and equilateral triangles with approximately 1 square inch area were learnt faster than peripheral examples. These aspects about things being easier and faster to learn are in line with the advantages of instrumental understanding pointed out by Skemp (1976). Rosch (1973) did, however, also find that "for the triangle, it appeared that any three-sided, straight-line figure was an equally good 'triangle prototype', and that all such figures were superior to the curved and freehand figures" (p. 347). This claim is somewhat impaired because her straight-line transformations of the prototype equilateral triangle essentially only generated three scalene triangles with small differences, without rotations or any efficacious transformations. For further reading about the prototype phenomenon and prototypicality, see Hershkowitz (1989) and Kaur (2015).

Lakatos (1963) presented one of the strongest examples of the opposite of a prototype approach. He described, through the voice of the student Gamma, how investigating extremes and odd examples is the core for gaining a relational understanding of a concept:

> If we want to learn about anything really deep, we have to study it not in its 'normal', regular, usual form, but in its critical state, in fever, in passion. If you want to know the normal healthy body, study it when it is abnormal, when it is ill. If you want to know functions, study their singularities. If you want to know ordinary polyhedra, study their lunatic fringe. This is how one can carry mathematical analysis into the very heart of the subject. (Lakatos, 1963, p. 25)

According to Lakatos, if students are to gain a flexible, in-depth understanding of triangles, they must go beyond just studying the normal and regular ones, the prototype equilateral triangles with the base at the bottom. They must study abnormal triangles, triangles that are rotated, skewed, skinny, or fat, because that is how they can start getting "into the very heart of" triangles. The work by Watson and colleagues on "learner generated examples" (e.g., Watson & Shipman, 2008) shows some of the power of using extreme examples. To develop a relational understanding of triangles, there must be a classroom culture in which students are encouraged to try out, to use Lakatos' terminology, "monster" triangles. Are there then still three sides, three vertices, and a closed area? What are the key properties that make something a triangle, and what are properties that can vary? If prototypes are accompanied by other examples like this, then the prototype actually can play a valuable role in students' understanding.

2.2 Instrumental and relational understanding summarised by five key characteristics

So then, *what characterises the concepts of instrumental and relational understanding*? In Table 1, Mellin-Olsen and Skemp's writings are summarised and structured into five key characteristics of an instrumental and relational understanding of mathematics.

These characteristics have the potential to be used as an analytical framework for studies focusing on students' mathematical talk and understanding. The fourth characteristic can be used to elaborate on how they can be used as analytical distinctions: Rigid explanations can be identified as being closely related to and dependent on rules, as not adaptive to the context, and phrases such as "must" and "have to" are frequently used. The rigidity

Table 1 Key characteristics of instrumental and relational understanding

	Instrumental understanding: students	Relational understanding: students
1	Treat new ideas as disconnected facts and unrelated to previous ideas	Relate new ideas to previous ones, and integrate them into conceptual systems
2	Memorise facts and carry out procedures/rules without focusing on how or why	Look for patterns and underlying features to express how and why
3	Practise an exercise discourse, accept explanations as given without questioning	Practise dialogic discourse, examine the logic of explanations critically
4	Express explanations rigidly—e.g., rely on visual prototypes	Express explanations flexibly
5	Do not reflect on the processes of developing an understanding	Reflect on the processes of developing an understanding

is also connected to prototypes where shapes, even if they are mathematically equivalent to a triangle, for example, are not accepted as triangles if they do not look similar enough to the prototype. Flexible explanations are adaptive to the context and phrases such as “might be” and “can be” are often used in the discussion of properties. A flexible understanding of concepts is about assisting “students in developing more robust, dynamic concept images than the traditional prototypical, static images that tend to prevent inclusive definition” (Sinclair et al., 2017).

The fifth characteristic differs from the first four as it addresses meta-understanding and is closely related to the I- and S-rationales. If the I-rationale is dominating and a teacher wants to establish the S-rationale, an important part of the process will be students’ reflections on what relational understanding means and the process of developing such an understanding. Mellin-Olsen (1981) argued that students over time will develop metaknowledge about mathematics and school, “and this metaknowledge is the base for the kind of learning strategies the learner activates when he faces a new learning situation” (p. 354).

When the concepts of instrumental and relational understanding are discussed in this paper, they are not considered as two disjoint, discrete, dialectic, or binary relations, rather they are considered as understandings and discourses that can co-exist and overlap as part of a continuum. The extremes are quite distinct and valuable for clarifying core aspects of the theory, but real-life classroom situations consist of discourses that are shifting, situated, and messy. Students can alternate between instrumental and relational approaches during a lesson, even during working with a single task. Although elements from instrumental understanding can be valuable (cf. Skemp, 1976; Star, 2005), Mellin-Olsen (1984) underlined that it becomes a problem for the students if the I-rationale and instrumental understanding is too dominant. Mellin-Olsen (1996) pointed out the potential consequences of dominant instrumental discourse, such as rote, superficial understanding and negative attitudes towards mathematics.

The concepts of relational and instrumental understanding can be compared to several other pairs of concepts. A comprehensive discussion is beyond the scope of this paper, but Hiebert and Lefevre’s (1986) distinction between conceptual and procedural knowledge deserves to be mentioned. They built on Ausubel’s (1968) meaningful and rote learning, and like Mellin-Olsen and Skemp, they refer to Piaget’s operative and figurative thinking (Piaget & Inhelder, 1973). Hiebert and Lefevre’s distinction between conceptual and procedural knowledge is valuable since it provides detailed cognitive-based insights into the differences and similarities between the two types of knowledge. It is however a largely individual-centred and cognitive approach, with a specific emphasis on knowledge. The relational and instrumental understandings include a strong emphasis on how such understandings must be understood as fundamental ways of being in the classrooms and how different rationales can lead to different understandings.

In the following two sections, some of *the potential value of instrumental and relational understanding* is addressed, first as part of a suggested revision of the van Hiele framework and then as lenses for analysing students’ utterances about geometry.

3 A suggestion for a revision of the van Hiele framework

The van Hiele framework, with five levels for concretising students’ geometrical thinking, has played an important role in improving mathematics curriculum and teaching (Clements et al., 1999). According to Sinclair and Moss (2012) and Swoboda and Vighi (2016), it continues to be the dominant theoretical account of students' learning about shape. A short-cut version of the levels is as follows. At the first level, *visualisation*, shapes are viewed by their appearance without analysing their properties, in line with Rosch's (1973) emphasis on how a prototype understanding is strongly based on the visual aspect. At the second level, *analysis*, the properties of the shapes are taken into consideration. At the third level, *abstraction*, the relationships between types of shapes can be recognised based on their properties. At the fourth level, *deduction*, students can do deductive proofs and at the fifth level, *rigor*, students can understand mathematical systems and different kinds of proofs.

Several studies have applied van Hieles' framework, including quantitative studies like the one by Gunčaga et al. (2017), which addressed year four students' abilities to recognise shapes. My focus in this paper is, however, on qualitative studies. The most common qualitative approach for investigating the early understanding of geometry is one-to-one interviews. Three such studies with 4–6 year old children were done by Clements et al. (1999), Tsamir et al. (2008), and Dağlı and Halat (2016). In these studies, the researchers presented figures and investigated the children's abilities to recognise and name shapes. There was a focus on prototypes and non-prototypes, intuitive and non-intuitive examples, and distinguishing valid examples from nonexamples. The findings across the three studies are in line with Swoboda and Vighi's (2016) summary of findings from research on early geometrical thinking: The children were, to a large extent, able to recognise prototype examples, but they experienced difficulties when trying to identify non-intuitive or non-prototype examples such as triangles in different sizes, types, and orientation. In their verbal explanations, they relied primarily on visual aspects such as pointy, round, and skinny to distinguish shapes, the first level according to van Hiele. They were only able to recognise simple properties to some extent such as "three sides" and "equal length", the second level in the van Hiele theory. These three studies serve as examples of the important but also varying role the van Hiele framework has played in mathematics education research. Clements et al. (1999) argued that "the van Hiele theory does not adequately describe young children's conceptions" (p. 194), and they went on to nuance the first visualisation level and contradicted the claim that children can only be on one level at a time. Tsamir et al. (2008) on the other hand, based their whole study on the van Hiele framework. Dağlı and Halat (2016) are somewhere in between, where they applied the van Hiele framework but acknowledged that there are objections to it from other researchers.

Sinclair et al. (2016) pointed out that research questioning the van Hiele levels began to appear during the '90s, but already in the '80s researchers like Burger and Shaughnessy (1986) identified a need for revision. But what exactly is it that has made researchers criticise the van Hiele framework? Much of the answer can be found in how strongly the van Hieles (1984) claimed that one cannot go on to the next level before one masters the previous levels, that "each level has its own linguistic symbols," and that "two people who reason at two different levels cannot understand each other" (p. 246). Burger and Shaughnessy (1986) documented the difficulties of assigning students that are in transition between levels and how students' understanding can be at different levels for different shapes. They questioned the Piaget-inspired sequential and discrete nature of the van Hiele levels because the students in their research "oscillated from one level to another on the same task" (p. 45). Students who showed level 1 understanding had flashes of level 2. Burger and Shaughnessy argued therefore that "the levels appear to be dynamic rather than static and of a more continuous nature than their discrete descriptions" (p. 45).

The discursive move in the theory about instrumental and relational understanding and I- and S-rationales, including the focus on seeing the differences in students' language and understanding as continuums, has laid the foundation for a revision of the van Hiele framework. I suggest a discourse-based van Hiele model with a dynamic continuous scale to describe and analyse students' thinking and talking. This revision is also based on Sfard's (2007) fusion of thinking and communicating, and her documentation of how learning mathematics is equivalent to modifying and extending discourses, as well as Barwell's (2016) emphasis on how students' informal and formal explanations develop together without following discrete and separate levels. Barwell argued that there is not a linear path from informal to formal discourse for the students, "rather, they work with the teacher to expand the repertoire of possible ways to make meaning in mathematics" (p. 331) and they develop their "repertoires of ways of talking about mathematics" (p. 333). Viewing geometrical understanding as being of a more continuous and dynamic nature can help overcome the limitations with discrete levels. Sinclair and Moss (2012) as well viewed geometrical understanding as dynamic and as a form of communication which contrasts the emphasis on rigid levels and manipulation of mental structures. Viewing geometrical understanding "as a form of communication entails that this thinking arises as a result of interactions" (Sinclair & Moss, 2012, p. 30) and not as a transition from one level to the next as part of students' natural development.

Many researchers have used the van Hiele levels as an analytical framework and by that provided insights into students' geometrical thinking. But as Kaur (2015) argued, the conflicting results from research on students' understanding of geometry indicate that difficulties occur when the framework is operationalised. Revising the van Hiele framework into a more dynamic, rather than sequential model, and seeing what students say and do as the unit of analysis, makes it easier to operationalise the model as an analytical framework. In the revised version, I propose that the development of geometrical understanding be seen as continuous and dynamic changes, and as the development of language and communication. The van Hiele levels thus become levels of discourse, and according to Kaur (2015), this makes it possible "to make claims about students' thinking in terms of how students communicate" (p. 409).

4 Instrumental and relational understanding together with I- and S-rationales as analytical lenses

In this section, the concepts of instrumental and relational understanding together with the I- and the S-rationales are used as lenses to analyse year two (6–7 years) and year six (10–11 years) students' utterances during large group discussions about geometry, drawn from my research data. The two first examples concern the concept of a triangle, the first from the year six class and the second from the year two class. The third example involves the same year two students when they discuss a cylinder.

4.1 I-rationale and instrumental understanding

"That's not a triangle!" a student erupts. I had just started teaching mathematics with my year six class, and the topic was polygons. I had said I would draw a triangle on the blackboard, and I drew something like Fig. 1. The student who claimed my drawing was not a triangle immediately got acknowledging nods from many of their classmates. "It is a triangle, but it is upside down!" another student added, laughing.

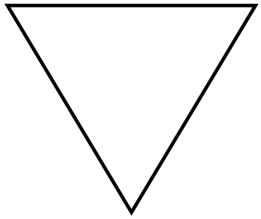

Fig. 1 A triangle?

What makes a twelve-year-old student claim a drawing is not a triangle, even though the teacher has said it is a triangle? And why did so many of the classmates support the student's claim? The other student builds a bridge between the disagreeing students and me by saying that "it is a triangle." She acknowledges that my drawing is a triangle, but she also put into words what was problematic: "it is upside down."

The students' unwillingness to accept a rotated triangle as a triangle can be described as a prototype issue, an instrumental understanding. They rely rigidly on a visual prototype, the equilateral triangle pointing upwards. This hinders them from looking for patterns and properties which could help them identify other shapes that fulfil the requirements for being a triangle. The student who agrees to the figure being a triangle is able to recognise the shape as a triangle, but she also adds that it is upside-down. This indicates that her understanding is somewhere on the continuum between instrumental and relational understanding, between being dependent on prototypes and identifying key properties. A quick look at school textbooks in many different subjects shows that the equilateral triangle pointing upwards is massively overrepresented (Sinclair & Moss, 2012). It surrounds the students and influences their understanding by contributing to the establishment of a triangle prototype, which gives them difficulties making sense of something different from what they previously have encountered.

According to Hershkowitz (1989), students' abilities in identifying shape improve considerably with age. The fact that these year six students still have trouble distinguishing between critical and non-critical properties for triangles indicates that there are more overarching challenges than just the problem with the rotated triangle. Mellin-Olsen (e.g., 1981) emphasised the role of the underlying rationale, the I-rationale, behind instrumental understanding. If the I-rationale has been dominating these students' work with polygons and possibly other mathematical topics as well, then there is not a general quick fix. Work, over time, is required to establish a discourse in which students are expected to talk about the properties of polygons. Establishing such a relational discourse requires that students' rationale for learning be addressed.

4.2 S-rationale and relational understanding

In the following example, the two students Jan and Anne (year two, 6–7 years) answer the question “what is a triangle?”:

Jan	I would say that it is … it can vary … it can be like this (shows with his hands an equilateral triangle pointing upwards) … so that all the edges are of equal length, and then it can be like this (shows with his hands an isosceles triangle pointing upwards). And it is one edge there, one edge there, and one edge there (points to the edges while he talks)	Eg vil seia at det er ein … det kan vera litt varierande … det kan vera sånn (viser med hendene ein likesida trekant med spissen opp) … sånn at alle kantane er like lange, og så kan det vera sånn (viser med hendene ein likebeina trekant med spissen opp). Og det er ein kant der, ein kant der og ein kant der (peikar på kantane medan han snakkar)
Anne	It is kind of a little block with only three edges (other students are talking) … there are triangles of such kind in blocks … so that you just make a small roof, and it has three edges … I mean like this, this, this (points to the edges while she talks)	Det er ein sånn liten kloss med berre tre kantar (andre studentar snakkar) … det finst sånne her trekantar i klossar … sånn at du berre lagar eit lite tak, og den har tre kantar … eg meiner sånn, sånn og sånn vert det (peikar på kantane medan ho snakkar)

When asked what a triangle is, Jan says “it can vary.” By talking about it as something that varies, as well as choosing the wording “it can,” he avoids being bombastic and contributes by that to a flexible understanding of triangles. When Jan starts to be concrete about what a triangle is, he again begins with “it can” and even again a third time. This choice of words can be regarded as flexible, opening, and inviting, and thus valuable for the process of developing a dialogic approach through which relational understanding can be created. Jan does not take a prototypical approach by saying that only one particular shape can be regarded as a triangle. Such explanations can appear static and fixed, disable replies, and bring discussions to an end. Jan, on the other hand, is reflecting and wondering when he explains what a triangle “can be.”

Jan complements his oral explanation with gestures. He uses both hands to draw in the air an equilateral, up-pointing triangle; the prototype triangle. It indicates that he has developed an understanding that the base should be at the bottom and that the edges should be of equal length. However, Jan shows an understanding that goes beyond the prototype; he knows that what triangles look like can vary. He says and shows with his hands that not only is it “so that all the edges are of equal length,” but it can also be so that only two edges are of equal length, like in isosceles triangles. In other words, there are more triangles than the prototype equilateral triangle with the base at the bottom.

Jan underlines how triangles have three edges and points to where the edges are on his air-drawn isosceles triangle. In this single explanation, he is on to several key properties of triangles. When showing two different types of triangles, equilateral and isosceles, he uses the length of the edges to distinguish between them. He presents logical explanations about two types of triangles and about how to differentiate between them. Jan looks for patterns and underlying features and goes in the direction of talking about a conceptual system. He expresses how and why, provides two representations, one with gestures and one verbally, and combines them at the end when he counts the edges while pointing. He is moving towards the third level, abstraction, in the van Hiele framework because he quite flexibly talks about properties and how different properties can give two different types of triangles. This gives grounds for talking about the triangle as an overarching concept with several subcategories. He has just begun year two but is well on his way to developing an in-depth relational understanding of triangles.

Anne is thinking in three dimensions when she says that a triangle is "a little block with only three edges." Several of the students' examples were three-dimensional, and according to Swoboda and Vighi (2016), that is valuable because that is how children discover the world. Anne looks for patterns and underlying features when she says that "there are triangles of such kind in blocks," pointing to the fact that two-dimensional figures can be found in three-dimensional figures, or that three-dimensional figures consist of two-dimensional figures. She relates the idea of triangles to her previous experience with 3D shapes. Anne goes on to talk about making "a small roof," so the block she is talking about is probably the triangular block they use as a roof when they build houses with wooden blocks. She alternates effortlessly between thinking in two and three dimensions. The imaginary block Anne talks about is a triangular prism, with two triangular bases. When she explains what a triangle is, she brings up these two bases. She draws attention away from the three rectangular lateral faces and focuses on what is triangular. Even if the block she refers to is three-dimensional, she is clear that there are triangles there by saying "and it has three edges." Like Jan, Anne illustrates the three edges when she finishes her explanation by counting the edges while pointing at them in the air. Her example might be regarded as an example of what Jan introduced.

Anne says, "there are" and she uses the expression "det finst," which can be translated to "there exists," to argue that there actually are triangles in blocks. Using the expression "det finst" suggests that Anne has an inquiring approach that can strengthen a dialogic approach. Discussions that include "det finst" tend to be more wondering, opening, and continuing than "that's the way it is" discussions. The latter discussions are more instrumental in the sense that facts and procedures are regarded as static knowledge. The former illustrates one of the qualities of dialogic processes within relational understanding. Bringing forward a "det finst" focus can contribute to extending the peers' understanding of what triangles can be—there is more than just one type of triangle, there are several triangles in addition to the prototype triangle pointing upwards. It can promote an understanding of triangles as something being used as well. Anne's wording "make" adds to the view of mathematics as something that can be used to develop or create something, as something more than pursuing static prototypes.

Another element in Anne's explanation is the risk aspect. She presents a new idea when she brings up a block in a mathematics discussion about triangles. It stands out from the other students' ideas and is in that respect in line with Lakatos' emphasis on studying something, not in its normal, regular form to achieve in-depth understanding. Going beyond what is usual to talk about, beyond what you think the teacher and peers expect, can be risky because you do not know how it will be received. Anne shows courage when she chooses to present an association like this, and even more so when she decides to continue despite being interrupted by some of the other students in her first thinking pause.

4.3 When I/S rationales and instrumental/relational understandings intertwine

What students say and do is sometimes easy to identify as instrumental or relational understanding and the rationale behind it can be obvious. It is however quite often more difficult to tell what kind of understanding takes place and what kind of rationale is the driving force. The following example is a continuation of the previous example with the same group of year two students. The students are challenged to take a mathematical perspective and discuss a kitchen roll tube (Fig. 2):

Fig. 2 Kitchen roll or spyglass

Anne	It is a round thing, because it is round there (points to one of the bases), and round there (points to the other base), and then it is round around here (points to the "rest")	Det er ein runding, fordi den er rund der (peikar på ei av endeflatene), og rund der (peikar på andre endeflate), så er den rund rundt her (peiker på "resten")
Per	It's a spyglass! Look! (Takes the kitchen roll and shows how it can be used as a spyglass)	Det er ein kikkert! Sjå! (Tek kjøkkenrullen og viser korleis den kan nyttast som kikkert)
Jan	He–he, that is not a shape	He–he, det er ikkje nokon form
Teacher	Doesn't a spyglass have a shape (mishears the student)?	Har ikkje ein kikkert ein form (høyrer feil)?
Jan	Spyglass *is* not a shape	Kikkert *er* ingen form
Teacher	*Is* not a shape (acknowledging tone of voice)	*Er* ingen form (anerkjennande tonefall)
Astrid	It is just a round thing	Det er jo berre ein runding
Anne	No, it's not just a round thing	Nei, det er ikkje berre ein runding
Per	It's a loong round thing!	Det er ein laang runding!

In this excerpt, Anne applies much of the same analytical approach she used when she talked about triangles. She argues, quite enthusiastically, that the roll is a round thing consisting of three sub-shapes, by decomposing the three-dimensional shape into two-dimensional shapes through first pointing out the two bases as round things and then the remaining part as round as well. She identifies through this some underlying features and uses her understanding of 2D shapes to investigate a 3D shape. Per does not follow up on Anne's explanation, rather he grabs the roll and uses it as a spyglass and asks the others to look when he pretends to be a pirate. Jan guides the discussion straight back to mathematics by his emphasis on shape and classification when he says that a spyglass "is not a shape" (and corrects the teacher as part of that process). Astrid tries an easy explanation by stating that it is just a round thing which Anne immediately rejects, and then Per tunes back in and supports Anne's rejection of Astrid's suggestion by adding that it is a "loong round thing."

Anne's approach is relational; she is on her way to seeing that the surface of the roll consists of two circles and something connecting the bases. Jan's focus is on classification and shape, and his interest appears to be in the direction of becoming able to position the roll into a conceptual system with other shapes. The roll does not fit into the shapes that Jan knows as he has not yet learnt about cylinders or prisms with circles as bases. Both show signs of the S-rationale; they appear to be interested in the subject matter and have a relational approach. Per is at first most interested in having fun, although he uses some sort of shape recognition to see the roll as a spyglass. Astrid's attempt to simply regard the roll as a round thing without further questioning corresponds to one of the characteristics of instrumental understanding. Anne disagrees immediately, and that makes Per suggest the roll to be a long round thing. He joins the mathematical focus of the discussion. It is not a mathematically precise description, but he supports Anne's argument to not accept the roll as just a round thing by using informal language.

It is not straightforward to comment on the rationales here. Even if Anne takes a relational approach and Jan directs the discussion straight back to mathematics after Per's spyglass attempt, it might be that they just do what is expected of them. Maybe they just adapt to the accepted discourse and are doing what Mellin-Olsen (1981) described as "demonstrating some knowledge, in order to obtain the teacher's praise and subsequently a good mark or degree" (p. 359). It can often be difficult to tell whether it is the S-rationale, the I-rationale, or maybe the most common, a combination, that drives the students because all three of them can generate relational understanding. And what about Per? If he tunes back in because he suddenly becomes interested in the mathematical properties of the roll, it could be the S-rationale. If it is just because he adapts to the discourse, then it could be the I-rationale.

5 Concluding comments

Star (2014) argued that Mellin-Olsen and Skemp's "notion of instrumental and relational understanding will and perhaps should continue to be widely used by mathematics educators throughout the world for advancing important conversations about mathematical understanding" (p. 307). The main purpose of this paper has been to contribute to that. This has included shedding some light on Mellin-Olsen's role as the source of origin for the concepts and presenting his discursive move through his emphasis on the two rationales for learning, the instrumental and social rationales. I have used Mellin-Olsen and Skemp's publications to generate five key characteristics of instrumental and relational understanding. The hope is that these characteristics can be used as analytical frameworks for studies on students' mathematical thinking and talking.

To concretise some of the potential value of these concepts, I use them to suggest a revision of the van Hiele framework. My inspiration for the revision of the van Hiele framework is heavily based on the discursive move Mellin-Olsen made, together with Sfard's (2007) fusion of thinking and communication and Barwell's (2016) Bakhtinian approach. With support from researchers such as Burger and Shaughnessy (1986), Clements et al. (1999), Sinclair and Moss (2012), and Kaur (2015), I have argued that a van Hiele framework with a discourse-based approach by which students' understanding is regarded as developing along a continuum offers a valuable analytical framework for students' geometrical understanding. In short, it means that researchers can be able to say something about students' mathematical thinking based on what students say and do.

To show some of the analytical potentials of the five characteristics of instrumental and relational understanding, they are used to analyse three examples of students' utterances about triangles and a cylinder. The analysis shows that the year six students did not include a triangle oriented downwards as a triangle. There could be several reasons for this, it could be that they had rigid prototype thinking, and/or that they had not been working enough with properties of shapes and examples other than the prototypes. The year two students were able to extract and point out key properties such as three sides and three vertices, distinguish between triangles based on side lengths, and identify triangles and circles (round things) in 3D shapes by decomposing them. They showed a flexible understanding of triangles that goes beyond the prototype triangle. In line with Lakatos' (1963) recommendations, they go beyond the normal regular form when they discuss equilateral and isosceles triangles and move between two and three dimensions. Addressing non-prototype examples can help students understand key features of shapes, which properties are variant and invariant, and what makes, for example, a triangle into a triangle.

Mellin-Olsen had a natural progression as a mathematics education researcher during the 1970s and '80s. He started by being inspired by Piaget, then moved on to Vygotsky. After that, with his focus on discourse, the social dimension of learning, and how what students say and do is influenced by the voices of others and the discourse, he went in the same direction as Bakhtin (apparently without knowing about Bakhtin's work). Mellin-Olsen emphasised the complexity of learning in that understanding is not an either-or situation—it is a continuum. The whole context, the life situation for students, must be taken into consideration if you want to study and understand their mathematical learning and their rationales for learning. Statements like the following, that students' "behaviour cannot be studied and interpreted independently of the context in which it takes place" (Mellin-Olsen, 1981, p. 353) align with Bakhtinian perspectives and is one of the important take-home messages that Mellin-Olsen gave. Furthermore, he underlined that what students say when discussing mathematics, and how they say it, are not just signs of what their understanding is. It is part of a discourse, a way of being, and if teachers want to address a lack of understanding they must take into consideration their students' rationales.

Acknowledgements I would like to say thank you to Professor Emerita Marit Johnsen-Høines, Professor Tamsin Meaney, and Professor Angelika Bikner-Ahsbahs for critical reading and constructive feedback. I am also grateful for the helpful and detailed comments from the Editor and the anonymous reviewers.

Funding Open access funding provided by Western Norway University Of Applied Sciences. This work is part of the project *Learning about teaching argumentation for critical mathematics education in multilingual classrooms* funded by the Research Council of Norway, grant number/project number: 273404, link: https://prosjektbanken.forskningsradet.no/en/project/FORISS/273404?Kilde=FORISS&distribution=Ar&chart=bar&calcType=funding&Sprak=no&sortBy=date&sortOrder=desc&resultCount=30&offset=30&Portefoljestyrer.1=Portef%C3%B8lje%20Industri%20og%20tjenesten%C3%A6ringer&source=FORISS&projectId=223262. Western Norway University of Applied Sciences has provided research support, including salary, equipment, and reimbursements for attending conferences.

Data availability The data analysed during the current study are available from the corresponding author upon reasonable request.

Declarations

The data collection is approved by the Norwegian centre for research data, and all personal identifying information is anonymised. The students have signed consent forms.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

Ausubel, D. P. (1968). *Educational psychology, a cognitive view*. Holt.

Barwell, R. (2016). Formal and informal mathematical discourses: Bakhtin and Vygotsky, dialogue and dialectic. *Educational Studies in Mathematics, 92*(3), 331–345. https://doi.org/10.1007/s10649-015-9641-z

Bateson, G. (1973). *Steps to an ecology of mind*. Paladin Books.

Boaler, J., & Brodie, K. (2004). The importance, nature and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 773–781). Ontario Institute for Studies in Education/University of Toronto.

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17*(1), 31–48. https://doi.org/10.5951/jresematheduc.17.1.0031

Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education, 30*(2), 192–212.

Dağlı, Ü. Y., & Halat, E. (2016). Young children's conceptual understanding of triangle. *Eurasia Journal of Mathematics, Science & Technology Education, 12*(2), 189–202. https://doi.org/10.12973/eurasia.2016.1398a

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. D. C. Heath & Co Publishers.

Gunčaga, J., Tkačik, Š., & Žilková, K. (2017). Understanding of selected geometric concepts by pupils of pre-primary and primary level education. *European Journal of Contemporary Education, 6*(3), 497–515. https://doi.org/10.13187/ejced.2017.3.497

Hershkowitz, R. (1989). Visualization in geometry: Two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61–76.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Lawrence Erlbaum.

Kaur, H. (2015). Two aspects of young children's thinking about different types of dynamic triangles. *ZDM-Mathematics Education, 47*(3), 407–420. https://doi.org/10.1007/s11858-014-0658-z

Lakatos, I. (1963). Proofs and Refutations (I). *The British Journal for the Philosophy of Science, 14*(53), 1–25.

Mellin-Olsen, S. (1975). *Instrumentalisme som pedagogisk begrep. Belyst ved undersøkelser av matematikkundervisning* [Instrumentalism as a pedagogical concept. Exemplified with investigations of mathematics teaching]. Universitetet i Bergen.

Mellin-Olsen, S. (1977). *Læring som sosial prosess* [Learning as a social process]. Gyldendal.

Mellin-Olsen, S. (1981). Instrumentalism as an educational concept. *Educational Studies in Mathematics, 12*(3), 351–367. https://doi.org/10.1007/BF00311065

Mellin-Olsen, S. (1984). *Eleven, matematikken og samfunnet. En undervisningslære* [The student, the mathematics, and the society. Learning about teaching]. NKI-forlaget.

Mellin-Olsen, S. (1991). *Hvordan tenker lærere om matematikkundervisning* [How do teachers think about mathematics teaching]? Høgskolen i Bergen.

Mellin-Olsen, S. (1996). Oppgavediskursen i matematikk [The exercise discourse in mathematics]. *Tangenten – Tidsskrift for Matematikkundervisning, 7*(2), 9–15.

Piaget, J., & Inhelder, B. (1973). *Memory and intelligence*. Routledge & Kegan Paul.

Rosch, E. H. (1973). Natural categories. *Cognitive Psychology, 4*(3), 328–350. https://doi.org/10.1016/0010-0285(73)90017-0

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences, 16*(4), 565–613. https://doi.org/10.1080/10508400701525253

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42–76. https://doi.org/10.1207/S15327884MCA0801_04

Sinclair, N., Bussi, M. G. B., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. *ZDM-Mathematics Education, 48*(5), 691–719. https://doi.org/10.1007/s11858-016-0796-6

Sinclair, N., Bussi, M. G. B., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2017). Geometry education, including the use of new technologies: A survey of recent research. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education, ICME-13* (pp. 277–287). Springer. https://doi.org/10.1007/s11858-016-0796-6

Sinclair, N., & Moss, J. (2012). The more it changes, the more it becomes the same: The development of the routine of shape identification in dynamic geometry environment. *International Journal of Educational Research, 51–52*(3), 28–44. https://doi.org/10.1016/j.ijer.2011.12.009

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Skemp, R. (1979). *Intelligence, learning, and action: A foundation for theory and practice in education*. John Wiley & Sons.

Skemp, R. R., & Mellin-Olsen, S. (1973). *Qualitative differences in mathematical thinking, Report to the Nuffield Foundation and the NAVF*. Nuffield Foundation and NAVF.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36*(5), 404–411. https://doi.org/10.2307/30034943

Star, J. (2014). Instrumental and relational understanding in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 304–307). Springer.

Swoboda, E., & Vighi, P. (2016). *Early geometrical thinking in the environment of patterns, mosaics and isometries (ICME-13 Topical Surveys)*. Springer.

Tsamir, P., Tirosh, D., & Levenson, E. (2008). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics, 69*(2), 81–95. https://doi.org/10.1007/s10649-008-9133-5

van Hiele, P. M. (1984). The child's thought and geometry. In D. Fuys, D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele* (pp. 243–252). Brooklyn College, City University of New York.

Watson, A., & Shipman, S. (2008). Using learner generated examples to introduce new concepts. *Educational Studies in Mathematics, 69*(2), 97–109. https://doi.org/10.1007/s10649-008-9142-4

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.