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Integral and Weighted Composition Operators on Fock-type Spaces

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Abstract

We study various structures of general Volterra-type integral and weighted composition operators acting between two Fock-type spaces $\mathcal{F}_\varphi^p$ and $\mathcal{F}_\varphi^q$, where φ is a radial function growing faster than the function $z \to |z|^2/2$. The main results show that the unboundedness of the Laplacian of φ provides interesting results on the topological and spectral structures of the operators in contrast to their actions on Fock spaces, where the Laplacian of the weight function is bounded. We further describe the invertible and unitary weighted composition operators. Finally, we show the spaces support no supercyclic weighted composition operator with respect to the pointwise convergence topology and hence with the weak and strong topologies.

Keywords Fock-type spaces · Schatten class · Invertible · Unitary · Volterra-type integral · Weighted composition operators · Supercyclic

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1 Introduction

It is known that generalized Volterra-type integral and weighted composition operators share some similar structures in some spaces of holomorphic functions. Both have been extensively studied on Fock spaces, where the Laplacian of the weight function is bounded; see for instance [5, 6, 11, 13]. In contrast, it seems rather not much is known about the structures of both classes of operators on Fock-type spaces $\mathcal{F}_\varphi^p$ when the Laplacian of the weight function φ becomes unbounded over the complex plane $\mathbb{C}$. It is the main purpose of this work to investigate this case and show how the unboundedness of the Laplacian plays a decisive role in determining the operators' basic structures.

We begin by introducing the weight function φ and the Fock-type spaces $\mathcal{F}_\varphi^p$ where our work takes place. We consider a twice continuously differentiable function $\varphi : [0, \infty) \to [0, \infty)$, and for each point z in $\mathbb{C}$, we extend it by setting $\varphi(z) = \varphi(|z|)$. We further assume that its Laplacian $\Delta\varphi$ is positive and $\tau(z) \simeq 1$ whenever $0 \leq |z| < 1$ and $\tau(z) \simeq (\Delta\varphi(|z|))^{-1/2}$, otherwise, where τ is a radial differentiable function satisfying the conditions

$$\lim_{r\to\infty} \tau(r) = \lim_{r\to\infty} \tau'(r) = 0.$$

In addition, we require that either there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty} \tau'(r)\log(\tau(r)^{-1}) = 0.$$

Here, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

We note that there are many examples of weights φ that satisfy the conditions above. The power functions as $\varphi_m(r) = r^m,\ m > 2$, the exponential type functions $\varphi_\alpha(r) = e^{\alpha r},\ \ \alpha > 0$, and $\varphi_\beta(r) = e^{e^{\beta r}}$, $\beta > 0$, are all primary examples.

Let $0 < p < \infty$, and φ and τ satisfy all the above mentioned admissibility conditions. We define the Fock-type spaces $\mathcal{F}_\varphi^p$ as spaces consisting of all entire functions f on $\mathbb{C}$ for which

$$\|f\|_p^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dm(z) < \infty,$$

where m denotes the usual Lebesgue area measure on $\mathbb{C}$. Several operators in $\mathcal{F}_\varphi^p$ had been extensively studied in the past; see for example, the embedding and Volterra-type integral operators [2, 10], multiplication and the differential operators [10], and composition operator [8, 9, 15]. The results in [8, 9] revealed the composition operator manifests poorer basic structures when it acts between two different Fock-type spaces in contrast to its action between classical Fock spaces. This happened due to the fast growth of the Laplacian of φ. In this work, we study the general Volterra-type integral and weighted composition operators, and plan to enlighten how the fast growth of the Laplacian of φ significantly changes their basic structures as well.

The rest of the manuscript is organized as follows. In Sect. 2, we consider problems related to bounded, compact, and Schatten $\mathcal{S}_p$ class general Volterra-type integral operator. Theorems 2.1 and 2.3 provide answers to these problems. Section 3 is concerned with weighted composition operator $W_{(u,\psi)}$. Corollary 3.1 identifies the bounded and compact $W_{(u,\psi)}$. Then, we consider the operators Schatten $\mathcal{S}_p$ class problem where a complete answer is given in Theorem 3.2. We further characterize the invertible and unitary properties of $W_{(u,\psi)}$ on $\mathcal{F}_\varphi^2$ as stated in Theorem 3.4. We end this section with Proposition 3.5 which shows that no weighted composition operator is supercyclic with respect to the pointwise convergence topology and hence with the weak and strong(norm) topologies on the Fock-type spaces. In Sect. 4, we present the proofs of the results. For the sake of simplicity and generality, we may present the proof of Theorem 2.3 at latter stage than all the other proofs.

2 The General Volterra-type Integral Operator $V_{(g,\psi)}$

For pairs of holomorphic functions (g, ψ), the induced general Volterra-type integral operator is defined by

$$V_{(g,\psi)} f(z) = \int_0^z f(\psi(w))g'(w)dw.$$

There exists a lot of literature about $V_{(g,\psi)}$ which is difficult to give a proper and complete review now. Thus, we may limit ourselves to the works most relevant for this work and refer readers to [12, 13] and the references therein. One of the main reasons the operators are worthy of study stems from their applications in linear isometries [3].

Now, we are prepared to state our first main result about $V_{(g,\psi)}$. We express the results in terms of the function $M_{(g,\psi)}$ defined by

$$M_{(g,\psi)}(z) := |g'(z)|(1 + \varphi'(z))^{-1} e^{\varphi(\psi(z))-\varphi(z)}.$$

Theorem 2.1 *Let (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$ and $0 < p, q < \infty$.*

(i) If $p = q$, then $V_{(g,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded if and only if $M_{(g,\psi)}$ is uniformly bounded over $\mathbb{C}$, and compact if and only if $M_{(g,\psi)}(z) \to 0$ as $|z| \to \infty$.

(ii) If $p < q$, then $V_{(g,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded (compact) if and only if $M_{(g,\psi)}(z) \to 0$ as $|z| \to \infty$.

(iii) If $p > q$, then $V_{(g,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded (compact) if and only if $M_{(g,\psi)}$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{C}, dm)$.

As noted earlier, various properties of $V_{(g,\psi)}$ acting between classical Fock spaces were extensively studied in [12, 13]. The statement in part (ii) of Theorem 2.1 provides one remarkable difference with the corresponding results. Unlike the classical case, boundedness and compactness are equivalent when the operator acts between $\mathcal{F}_\varphi^p$ and $\mathcal{F}_\varphi^q$ whenever $p \neq q$. As it will be seen in the proof, this property stems from the fast growth of the Laplacian of φ. On the other hand, this growth of φ obviously makes it possible that more pairs of symbols (g, ψ) are admissible for bounded(compact) $V_{(g,\psi)}$ than the classical case. Thus, the operator enjoys a richer operator-theoretic structure on the spaces $\mathcal{F}_\varphi^p$.

Note that Theorem 2.1 excludes the trivial cases where either ψ or g is a constant. The operator $V_{(g,\psi)}$ reduces to zero whenever g is a constant. On the other hand, if $\psi = \alpha$ is a constant, then an application of the estimate in (3.1) gives

$$\|V_{(g,\psi)} f\|_q \simeq |f(\alpha)| \|u\|_q.$$

In this case, $V_{(g,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q,\ p \le q$ is bounded if and only if $u \in \mathcal{F}_\varphi^q$. Such trivial cases will be excluded in the rest of the results as well.

The Volterra-type integral operator V_g is recovered upon choosing $\psi(z) = z$ in $V_{(g,\psi)}$. Thus, the following is a special case of Theorem 2.1.

Corollary 2.2 *Let g be a nonconstant entire function on $\mathbb{C}$ and $0 < p, q < \infty$. Then,*

(i) If $p = q$, then $V_g : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded if and only if $|g|/(1 + \varphi')$ is uniformly bounded over $\mathbb{C}$, and compact if and only if $|g'(z)|/(1 + \varphi'(z)) \to 0$ as $|z| \to \infty$.

(ii) If $p < q$, then $V_g : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded (compact) if and only if $|g'(z)|/(1 + \varphi'(z)) \to 0$ as $|z| \to \infty$.

(iii) If $p > q$, then $V_g : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded (compact) if and only if $g'/(1 + \varphi')$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{C}, dm)$.

The first two parts in the corollary simplify the first part of Theorem 3 in [2]. In contrast to the corresponding results on the classical Fock spaces, part (ii) asserts that boundedness and compactness are equivalent. Observe that the corollary shows a richer structure of V_g compared to its action on the classical setting, where this operator is bounded only for polynomial symbols of degree at most two [13].

We now turn our attention to special class of compact $V_{(g,\psi)}$ on the Hilbert space $\mathcal{F}_\varphi^2$, namely the Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class, and plan to prove the following.

Theorem 2.3 *Let $1 < p < \infty$ and (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$ such that $V_{(g,\psi)}$ is bounded on $\mathcal{F}_\varphi^2$. Then, $V_{(g,\psi)}$ belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class if and only if $M_{(g,\psi)}$ belongs to $L^p(\mathbb{C}, dm_\psi)$, where*

$$dm_\psi(z) = \Delta\varphi(|\psi(z)|) dm(z).$$

The Schatten class membership of $V_{(g,\psi)}$ on the classical Fock space was investigated in [12, 13]. Interestingly, comparing the result there with Theorem 2.3, we conclude $V_{(g,\psi)}$ has a richer structure on $\mathcal{F}_\varphi^2$ even if the integral factor $\Delta\varphi(|\psi(z)|) \to \infty$ as $|z| \to \infty$.

3 The Weighted Composition Operator

For a pair of holomorphic functions (u, ψ), we define the induced weighted composition operator $W_{(u,\psi)}$ by $W_{(u,\psi)} f = M_u C_\psi f$, where $C_\psi f = f \circ \psi$ and $M_u(f) = uf$ are, respectively, the composition and multiplication operators. As indicated earlier, in [8, 9], we studied the operator C_ψ acting between the spaces $\mathcal{F}_\varphi^p$ and $\mathcal{F}_\varphi^q$. In contrast to the classical setting, for $p \neq q$, it was proved that C_ψ is bounded if and only if it is compact. This happens due to the fast growth of the Laplacian of the radial weight function φ. A natural question is whether a similar phenomenon happens with the weighted composition operator $W_{(u,\psi)}$, bypassing any possible interplay between the multiplier function u and ψ. As Corollary 3.1 shows, this is indeed the case.

A useful tool in the study of integral operators is Littlewood–Paley-type description of the underlying spaces. For Fock-type spaces $\mathcal{F}_\varphi^p$, this was done in [2] and reads as

$$\|f\|_p^p \simeq |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p (1 + \varphi'(z))^{-p} e^{-p\varphi(z)} dm(z), \tag{3.1}$$

for any entire function f and $0 < p < \infty$. An immediate consequence of (3.1) is that by simply replacing the function $|g'|/(1 + \varphi')$ by the entire weight function u in the proof of Theorem 2.1, we deduce the following interesting corollary, where all the results are expressed in terms of the function

$$m_{(u,\psi)}(z) := |u(z)| e^{\varphi(\psi(z)) - \varphi(z)}.$$

Corollary 3.1 *Let (u, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$ and $0 < p, q < \infty$. Then,*

(i) If $p = q$, then $W_{(u,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded if and only if $m_{(u,\psi)}$ is uniformly bounded over $\mathbb{C}$, and compact if and only if $m_{(u,\psi)}(z) \to 0$ as $|z| \to \infty$.

(ii) If $p < q$, then $W_{(u,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded (compact) if and only if $m_{(u,\psi)}(z) \to 0$ as $|z| \to \infty$.

(iii) If $p > q$, then $W_{(u,\psi)} : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded (compact) if and only if $m_{(u,\psi)}$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{C}, dm)$.

We remark that the discussion following Theorem 2.1 applies for the weighted composition operators as well. It is also worth noting that in [5], the result analogous to Corollary 3.1 in the classical Fock spaces was simplified further to give an explicit expression for the multiplier function u. The simplification relied heavily upon the explicit expression of the reproducing kernel function. The lack of such expression for the kernel in our current setting makes it difficult to give analogous formula for u.

We can now go further and describe all weighted composition operators which belong to the Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class.

Theorem 3.2 *Let (u, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$, $0 < p < \infty$, and $W_{(u,\psi)}$ is bounded on $\mathcal{F}_\varphi^2$. Then, $W_{(u,\psi)}$ belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class if and only if $m_{(u,\psi)}$ belongs to $L^p(\mathbb{C}, dm_\psi)$, where*

$$dm_\psi(z) = \Delta\varphi(|\psi(z)|) dm(z).$$

Unlike Corollary 3.1, where its proof can be directly obtained by simply replacing $g'/(1+\psi')$ by the weight function u in the proof of Theorem 2.1, the proofs of Theorems 2.3 and 3.2 do not necessarily follow one from the other. As it will be seen in the next section, the conclusion in Lemma 4.2 fails to hold with a compact $V_{(g,\psi)}$ while it does for a compact $W_{(u,\psi)}$.

We remark that Berezin-type integral transforms have been also among the forms used to describe Schatten class membership of these class of operators on various spaces of functions. The following proposition shows the corresponding Berezin-type integral transform needs to be integrated against area measure weighted with the unbounded Laplacian of φ.

Proposition 3.3 *Let (u,ψ) be pair of nonconstant entire functions on $\mathbb{C}$, $0<p<\infty$, and $W_{(u,\psi)}$ is compact on $\mathcal{F}_\varphi^2$. Then,*

(i) If $2\le p<\infty$ and $W_{(u,\psi)}$ belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class, then

$$\int_{\mathbb{C}} \|W_{(u,\psi)}k_{\psi(z)}\|_2^p \Delta\varphi(|\psi(z)|)dm(z)<\infty, \tag{3.2}$$

where $k_{\psi(z)}=K_{\psi(z)}/\|K_{\psi(z)}\|_2$ is the normalized reproducing kernel function at the point $\psi(z)$.

(ii) If $0<p<2$ and (3.2) holds, then $W_{(u,\psi)}$ belongs to the $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class.

The analogous of these conditions on the classical setting have been proved to be necessary and sufficient, see for instance [9]. The conditions are likely to be both necessary and sufficient in $\mathcal{F}_\varphi^2$ as well but remains to be verified. However, note that the already obtained condition in Theorem 3.2 is simpler to apply than conditions based on Berezin-type integral transforms.

3.1 Normal, Unitary and Invertible $W_{(u,\psi)}$

The normal, unitary and invertible weighted composition operators on the classical setting were characterized in [5]. The characterization there used effectively the explicit expression of the reproducing kernel function again. In this section, we explore these properties on the space $\mathcal{F}_\varphi^2$. We manage to represent the multiplier function u in terms of the kernel function. However, the lack of an explicit and workable expression for the kernel function still makes it difficult to simplify the representation further.

Theorem 3.4 *Let (u,ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. Then,*

(i) $W_{(u,\psi)}$ is a bounded invertible operator on $\mathcal{F}_\varphi^2$ if and only if $\psi(z)=az+b$, $|a|=1$ and there exists a positive constant C such that

$$\frac{1}{C}\le m_{(u,\psi)}(z)\le C \tag{3.3}$$

for all $z\in\mathbb{C}$. In this case, $W^{-1}_{(u,\psi)}$ is itself a weighted composition operator with symbol $\big(\frac{1}{u(\psi^{-1})},\psi^{-1}\big)$.

(ii) *$W_{(u,\psi)}$ is a unitary operator on $\mathcal{F}_\varphi^2$ if and only if $\psi(z) = az + b$, $|a| = 1$, $W_{(u,\psi)}^* = W_{(u,\psi)}^{-1} = W_{\left(\frac{1}{u(\psi^{-1})}, \psi^{-1}\right)}$, where*

$$u = \frac{\|1\|_2^{-2}}{\overline{u(0)} K_b \circ \psi}, \tag{3.4}$$

and $\|1\|_2$ refers to the norm of the constant function 1.

(iii) *Let $W_{(u,\psi)}$ be a bounded normal operator on $\mathcal{F}_\varphi^2$ and hence $\psi(z) = az + b$ with $|a| \le 1$. If either $a \ne 1$ or $a = 1$ and $b = 0$, then*

$$u = \frac{u(z_0) K_{z_0}}{K_{z_0} \circ \psi}, \tag{3.5}$$

where z_0 is the fixed point of ψ and $u(z_0) = u(0)\|1\|_2^2 K_{z_0}(b)$.

In the context of part (iii), if $z_0 = 0$, then as it will be seen in the proof, K_0 is a constant function. Relation (3.5) implies u is also a constant function. In this case, $W_{(u,\psi)}$ is normal if and only if the corresponding composition operator C_ψ is normal on $\mathcal{F}_\varphi^2$. The latter obviously holds since it is diagnoseable with respect to the standard orthonormal basis $z^m/\|z^m\|_2$, $m \ge 0$ in $\mathcal{F}_\varphi^2$.

In the rest of this section, we give some results on the linear dynamics of weighted composition operators on $\mathcal{F}_\varphi^p$ for $1 \le p < \infty$. It turns out that like the classical setting, no supercyclic weighted composition operator is supported on the spaces $\mathcal{F}_\varphi^p$ even in the case when the spaces are structured with weaker topologies.

3.2 Weak and τ_{pt}-supercyclic Weighted Composition Operators

The various linear dynamical structures of $W_{(u,\psi)}$ on the classical Fock spaces are now well-understood including convex-cyclicity [7]. Whether the unboundedness of the Laplacian of φ has an effect on the dynamical structures as it does for boundedness, compactness and Schatten class membership is another noteworthy problem to investigate. In this section, we plan to study the supercyclicity structure with respect to the pointwise, weak and strong(norm) topologies in the spaces. It would be interesting to know whether the cyclicity and convex-cyclicity results on the classical setting can be extended to the spaces $\mathcal{F}_\varphi^p$ as well.

We may begin by recalling some definitions related to the iterates of an operator. A bounded linear operator T on a separable Banach space $\mathcal{H}$ is said to be hypercyclic if there exists a vector f in $\mathcal{H}$ for which the orbit, $\mathrm{Orb}(T, f) = \{f, Tf, T^2 f, T^3 f, \ldots\}$ is dense in $\mathcal{H}$. Such an f is called a hypercyclic vector for T. The operator is supercyclic with vector f if the projective orbit,

$$\mathrm{Projorb}(T, f) = \left\{\lambda T^n f, \ \ \lambda \in \mathbb{C}, \ n = 0, 1, 2, \ldots\right\}$$

is dense. For a comprehensive account of the theory of dynamics of continuous and linear operators, we refer to the monographs [1, 4]. The weak and τ_{pt}-supercyclicities are defined by simply replacing the norm topology above by these respective topologies on the space. Clearly, weak supercyclicity is a stronger property than τ_{pt}-supercyclicity. More generally, the next diagram exhibits the relations among the various forms of cyclicities for bounded operators.

$$
\begin{tikzcd}
\text{Hypercyclic} \arrow[r, shift left] & \text{Supercyclic} \arrow[l, shift left, "/" marking] \arrow[r, shift left] & \text{Weakly supercyclic} \arrow[l, shift left, "/" marking] \arrow[dl, shift left] \\
& \tau_{pt}-\text{supercyclic} \arrow[ur, shift left, "/" marking] &
\end{tikzcd}
$$

We will prove the following result.

Proposition 3.5 *Let $1 \leq p < \infty$ and (u, ψ) be a pair of entire functions on $\mathbb{C}$ such that $W_{(u,\psi)}$ is bounded on $\mathcal{F}_\varphi^p$. Then, $W_{(u,\psi)}$ cannot be supercyclic on $\mathcal{F}_\varphi^p$ with respect to the pointwise convergence topology.*

As illustrated in the diagram above, pointwise topology is weaker than the weak and strong topologies on $\mathcal{F}_\varphi^p$ and hence the space supports no supercyclic (weakly) weighted composition operators. The analogous statement on the classical Fock spaces was proved in [7]. It turns out that the same conclusion remains enforce for $\mathcal{F}_\varphi^p$ and hence the unboundedness of the Laplacian has no effect in this regard. The questions when the operator admits cyclic and convex-cyclic dynamical structures and how these are related to the fast growth of the Laplacian of φ remain open for further investigation.

4 Proof of the Results

Before we pass to the proofs of the main results, we need some preliminary observations and backgrounds. We may begin by proving a few basic lemmas which provide useful information about the growth and form of the symbol for the composition operator. The lemmas will be used to prove our main results in the sequel.

Lemma 4.1 *Let (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. If $M_{(g,\psi)}$ is bounded on $\mathbb{C}$, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.*

Proof It follows from the assumption that

$$M_\infty(g', |z|) \lesssim \frac{1 + \varphi'(z)}{e^{\varphi(\psi(z))-\varphi(z)}}, \tag{4.1}$$

where $M_\infty(g', |z|)$ is the integral mean of the function g'. From (4.1), definition of φ, and the fact that $M_\infty(g', |z|)$ is a nondecreasing function of $|z|$, we get

$$\limsup_{|z|\to\infty} \varphi(\psi(z)) - \varphi(z) \leq 0. \tag{4.2}$$

If not, one would find a sequence z_j such that $|z_j| \to \infty$ as $j \to \infty$,

$$\limsup_{|z_j|\to\infty} \varphi(\psi(z_j)) - \varphi(z_j) > 0,$$

and hence

$$M_\infty(g', |z_j|) \lesssim \frac{1+\varphi'(z_j)}{e^{\varphi(\psi(z_j))-\varphi(z_j)}}, \tag{4.3}$$

which gives a contradiction since the right-hand side expression in (4.3) tends to zero when $|z_j| \to \infty$, while g' is a nonzero function and $M_\infty(g', |z|)$ is a nondecreasing function of $|z|$. By (4.2) and since ψ is an entire function with its own power expansion, we deduce $\psi(z) = az + b$ with $|a| \le 1$.

Lemma 4.2 *Let (u, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. Then,*

(i) If $m_{(u,\psi)}$ is bounded on $\mathbb{C}$, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ such that $|a| \le 1$.
(ii) If $m_{(u,\psi)}(z) \to 0$ as $|z| \to \infty$, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ such that $|a| < 1$.

We remark that part (ii) of Lemma 4.2 does not necessarily hold if we replace $m_{(u,\psi)}$ by $M_{(g,\psi)}$ unless $|g'(z)|/(1+\varphi'(z)) > 0$ as $|z| \to \infty$. Consequently, by Corollary 3.1 and Theorem 2.1, while compactness of $W_{(u,\psi)}$ implies $\psi(z) = az + b$ with $|a| < 1$, compactness of $V_{(g,\psi)}$ does not necessarily imply $|a| < 1$.

Proof Part (i) follows by arguing as in the previous lemma. Thus, we set $\psi(z) = az+b$ with $|a| \le 1$ and proceed to show that $|a| < 1$ in part (ii). Aiming to argue in the contrary, assume that $|a| = 1$. Then, we can choose a sequence of numbers z_k such that $|z_k| \to \infty$ $\Re(az_kb) \ge 0$, and $|u(z_k)| \ne 0$ as $k \to \infty$. Now, for sufficiently large $|z_k|$

$$|u(z_k)|e^{\varphi(az_k+b)-\varphi(z_k)} \ge |u(z_k)|e^{\varphi(\sqrt{|z_k|^2+|b^2})-\varphi(z_k)} \ge |u(z_k)| > 0,$$

which gives a contradiction since u is a nonzero entire function.

Next, we recall a few basic properties of the spaces $\mathcal{F}_\varphi^p$. The spaces were studied by several authors in the past. We refer the reader to [2, 9, 10] for a more thorough exposition. By Proposition A and Corollary 8 of [2], for a sufficiently large positive number R, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $F_{(w,R)}$ such that when z belongs to $D(w, R\tau(w))$,

$$|F_{(w,R)}(z)|e^{-\varphi(z)} \simeq 1, \tag{4.4}$$

where $D(a, r)$ denotes the Euclidean disk centered at a and radius $r > 0$. Furthermore, the functions $F_{(w,R)}$ belong to $\mathcal{F}_\varphi^p$ for all p with norms estimated by

$$\|F_{(w,R)}\|_p^p \simeq \tau(w)^2, \quad \eta(R) \le |w|. \tag{4.5}$$

An explicit expression for the kernel function in $\mathcal{F}_\varphi^2$ is still an interesting open problem. However, an asymptotic estimation of the norm

$$\|K_w\|_2^2 \simeq \tau(w)^{-2}e^{2\varphi(w)} \tag{4.6}$$

holds for all $w \in \mathbb{C}$.

For subharmonic functions φ and f, it also holds a local pointwise estimate

$$|f(z)|^p e^{-\beta\varphi(z)} \lesssim \frac{1}{\sigma^2\tau(z)^2}\int_{D(z,\sigma\tau(z))}|f(w)|^p e^{-\beta\varphi(w)}dm(w) \tag{4.7}$$

for all finite exponent p, any real number β, and a small positive number σ: see Lemma 7 of [2] for more details.

We end this section by recording the following covering lemma which will be needed in the proof of Theorem 3.2.

Lemma 4.3 *Let $t:\mathbb{C}\to(0,\infty)$ be a continuous function which satisfies $|t(z)-t(w)|\le\frac{1}{4}|z-w|$ for all z and w in $\mathbb{C}$. We also assume that $t(z)\to 0$ when $|z|\to\infty$. Then, there exists a sequence of points z_j in $\mathbb{C}$ satisfying the following conditions.*

(i) $z_j \notin D(z_k, t(z_k)), \quad j\neq k;$
(ii) $\mathbb{C}=\bigcup_j D(z_j, t(z_j));$
(iii) $\bigcup_{z\in D(z_j,t(z_j))} D(z,t(z))\subset D(z_j, 3t(z_j));$
(iv) *The sequence $D(z_j, 3t(z_j))$ is a covering of $\mathbb{C}$ with finite multiplicity.*

The lemma was proved in [2] by adopting an approach originally from [14].

4.1 Proof of Theorem 2.1

The notion of embedding has proved to be useful tool in the study of several operators. We plan to use it here too. Thus, aiming to reformulate our results in terms of appropriate embedding maps, we may first set a pullback measure

$$\mu_{(g,\psi,q)}(E)=\int_{\psi^{-1}(E)}\left(\frac{|g'(z)|}{1+\varphi'(z)}\right)^q e^{-q\varphi(z)}dm(z) \tag{4.8}$$

for every Borel subset E of $\mathbb{C}$, and note that (3.1) gives for each f in $\mathcal{F}_\varphi^p$

$$\begin{aligned}\|V_{(g,\psi)}f\|_q^q &\simeq \int_{\mathbb{C}}\frac{|g'(z)|^q}{(1+\varphi'(z))^q}|f(\psi(z))|^q e^{-q\varphi(z)}dm(z)\\ &=\int_{\mathbb{C}}|f(z)|^q d\mu_{(g,\psi,q)}(z),\end{aligned} \tag{4.9}$$

where

$$d\mu_{(g,\psi,q)}(z)=\frac{|g'(\psi^{-1}(z))|^q e^{-q\varphi(\psi^{-1}(z))}}{(1+\varphi'(\psi^{-1}(z)))^q}dm(\psi^{-1}(z)).$$

It follows that $V_{(g,\psi)}:\mathcal{F}_\varphi^p\to\mathcal{F}_\varphi^q$ is bounded (compact) if and only if the embedding map $I_d:\mathcal{F}_\varphi^p\to L^q(\mathbb{C},\mu_{(g,\psi,q)})$ is bounded (compact), respectively. We proceed to use this equivalent reformulation to prove the assertions in Theorem 2.1.

Part (i) and ***(ii)***: If $p \leq q$, setting

$$S(w) := \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} e^{q\varphi(z)} d\mu_{(g,\psi,q)}(z)$$

for some $\delta > 0$, and taking (4.8) into account, we may rewrite $S(w)$ as

$$\begin{aligned} S(w) &= \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} e^{q\varphi(z)} d\mu_{(g,\psi,q)}(z) \\ &= \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} \frac{|g'(\psi^{-1}(z))|^q e^{q\varphi(z)}}{(1+\varphi'(\psi^{-1}(z)))^q} e^{-q\varphi(\psi^{-1}(z))} dm(\psi^{-1}(z)) \\ &= \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} M^q_{(g,\psi)}(\psi^{-1}(z)) dm(\psi^{-1}(z)). \end{aligned} \tag{4.10}$$

Now, by [2, Theorem 1] and (4.10), the embedding map $I_d : \mathcal{F}^p_\varphi \to L^q(\mathbb{C}, \mu_{(g,\psi,q)})$ is bounded if and only if S is uniformly bounded on the complex plane. In other words, $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is bounded if and only if

$$\sup_{w\in\mathbb{C}} S(w) = \sup_{w\in\mathbb{C}} \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} M^q_{(g,\psi)}(\psi^{-1}(z)) dm(\psi^{-1}(z)) < \infty. \tag{4.11}$$

Similarly, $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is compact if and only if

$$\lim_{|w|\to\infty} S(w) = \lim_{|w|\to\infty} \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} M^q_{(g,\psi)}(\psi^{-1}(z)) dm(\psi^{-1}(z)) = 0. \tag{4.12}$$

Thus, we plan to show that the corresponding conditions for $p \leq q$ in Theorem 2.1 are equivalent to conditions (4.11) and (4.12). First note that by [2, Lemma 20]

$$\varphi'(w) + 1 \simeq \varphi'(z) + 1 \tag{4.13}$$

for all z in the disk $D(w, \delta\tau(w))$. Furthermore, since $|g'e^{\varphi(\psi)}|^p$ is subharmonic, applying the pointwise estimate in (4.7) and relation (4.10), we have

$$\tau(w)^{\frac{2p-2q}{p}} M^q_{(g,\psi)}(w) \lesssim \frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} M^q_{(g,\psi)}(\psi^{-1}(z)) dm(\psi^{-1}(z)) \tag{4.14}$$

for all $w \in \mathbb{C}$ and some $\delta > 0$.

(i) We prove when $p = q$. Assume that $M_{(g,\psi)}$ is finite. Then by Lemma 4.1, $\psi(z) = az + b, 0 < |a| \leq 1$ and hence $\psi^{-1}(z) = \frac{z-b}{a}$. It follows that

$$\sup_{w\in\mathbb{C}} \frac{1}{\tau^2(w)} \int_{D(w,\delta\tau(w))} M^p_{(g,\psi)}(\psi^{-1}(z)) dm(\psi^{-1}(z))$$

$$\leq \sup_{w\in\mathbb{C}} \frac{1}{\tau^2(w)} \sup_{z\in D(w,\delta\tau(w))} M^p_{(g,\psi)}(\psi^{-1}(z)) \int_{D(w,\delta\tau(w))} dm(\psi^{-1}(z))$$
$$= |a|^2 \sup_{w\in\mathbb{C}} \sup_{z\in D(w,\delta\tau(w))} M^p_{(g,\psi)}(\psi^{-1}(z)) \leq |a|^2 \sup_{w\in\mathbb{C}} M^p_{(g,\psi)}(w) < \infty$$

and (4.11) holds as asserted.

Conversely, if (4.11) holds, then the relation in (4.14) obviously implies $M_{(g,\psi)}$ is finite.

For compactness, arguing as in the boundedness part, we easily see that (4.12) holds if and only if $M_{(g,\psi)}(z) \to 0$ as $|z| \to \infty$.

(ii) Let $p < q$ and suppose that (4.11) holds. We need to show that $M_{(g,\psi)}(z) \to 0$ as $|z| \to \infty$. It follows from (4.14) and the admissibility condition for τ

$$M_{(g,\psi)}(w) \lesssim \tau^{\frac{2q-2p}{qp}}(w)\Big(\sup_{w\in\mathbb{C}} S(w)\Big)^{1/q} \lesssim \tau^{\frac{2q-2p}{pq}}(w) \to 0 \quad \text{as } |w| \to \infty.$$

Next, by assuming that $M_{(g,\psi)}(w) \to 0$ as $|w| \to \infty$, we proceed to show (4.12) holds. Observe that the assumption and (3.1) imply $g \in \mathcal{F}^q_\varphi$ since

$$\int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+\varphi'(z))^q} e^{-q\varphi(z)} dm(z) \leq \sup_{z\in\mathbb{C}} \big(M^q_{(g,\psi)}(z)\big) \int_{\mathbb{C}} e^{-q\varphi(\psi(z))} dm(z) < \infty. \quad (4.15)$$

It follows that there exists a positive constant C such that

$$\frac{1}{\tau^{\frac{2q}{p}}(w)} \int_{D(w,\delta\tau(w))} M^q_{(g,\psi)}(\psi^{-1}(z)) dm(\psi^{-1}(z)) = C|a|^2 \frac{\tau^2(w)}{\tau^{\frac{2q}{p}}(w)} M^q_{(g,\psi)}(\psi^{-1}(w))$$
$$= C|a|^2 \tau^{\frac{2p-2q}{p}}(w) M^q_{(g,\psi)}(\psi^{-1}(w)).$$

To this end, taking into account (4.15) which shows that the weight function $|g'|/(1+\varphi')$ grows at a rate slower than the rate at which the function $e^{-\varphi}$ decays and the definition of τ, we deduce

$$\tau^{\frac{2p-2q}{p}}(w) M^q_{(g,\psi)}(\psi^{-1}(w)) \to 0$$

as $|w| \to \infty$ and hence (4.12) follows.

Part (iii): Let $0 < q < p < \infty$. Invoking the reformulation in (4.9) again, $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is bounded (compact), respectively, if and only if the embedding map $I_d : \mathcal{F}^p_\varphi \to L^q(\mu_{(\psi,q)})$ is bounded (compact). By [2, Theorem 1], boundedness or compactness of I_d holds if and only if for some $\delta > 0$, the function

$$\mathcal{G}(z) := \frac{1}{\tau^2(z)} \int_{D(z,\delta\tau(z))} e^{q\varphi(w)} d\mu_{(g,\psi,q)}(w)$$
$$= \frac{1}{\tau^2(z)} \int_{D(z,\delta\tau(z))} M^q_{(g,\psi)}(\psi^{-1}(w)) dm(\psi^{-1}(w))$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dm)$. Thus, we plan to show that the condition in the theorem is equivalent to this reformulation and assume first $M_{(g,\psi)} \in L^{\frac{qp}{p-q}}(\mathbb{C}, dm)$. Applying Hölder's inequality

$$\int_{\mathbb{C}} |\mathcal{G}(z)|^{\frac{p}{p-q}} dm(z) = \int_{\mathbb{C}} \left(\frac{1}{\tau^2(z)} \int_{D(z,\delta\tau(z))} M^q_{(g,\psi)}(\psi^{-1}(w)) dm(\psi^{-1}(w)) \right)^{\frac{p}{p-q}} dm(z)$$
$$\lesssim \int_{\mathbb{C}} \tau^{-2}(z) \int_{D(z,\delta\tau(z))} M^{\frac{pq}{p-q}}_{(g,\psi)}(\psi^{-1}(w)) dm(\psi^{-1}(w)) dm(z) =: \mathcal{G}_1.$$

Since $w \in D(z, \delta\tau(z))$, by [2, Lemma 5], there exists a positive constant c with

$$\frac{1}{c}\tau(w) \le \tau(z) \le c\tau(w). \tag{4.16}$$

Then, for any $\zeta \in D(z, \delta\tau(z))$

$$|\zeta - w| \le |\zeta - z| + |z - w| \le 2\delta\tau(z) \le 2\delta c\tau(w) = \beta\tau(w), \quad \beta := 2\delta c.$$

This shows that $D(z, \delta\tau(z)) \subset D(w, \beta\tau(w))$ which together with Fubini's Theorem and (4.16) again imply

$$\begin{aligned}
\mathcal{G}_1 &= \int_{\mathbb{C}} \tau^{-2}(z) \int_{\mathbb{C}} \chi_{D(z,\delta\tau(z))}(w) M^{\frac{pq}{p-q}}_{(g,\psi)}(\psi^{-1}(w)) dm(\psi^{-1}(w)) dm(z) \\
&\le \int_{\mathbb{C}} M^{\frac{pq}{p-q}}_{(g,\psi)}(\psi^{-1}(w)) \left(\int_{\mathbb{C}} \chi_{D(w,\beta\tau(w))}(z) \tau(z)^{-2} dm(z) \right) dm(\psi^{-1}(w)) \\
&= \int_{\mathbb{C}} M^{\frac{pq}{p-q}}_{(g,\psi)}(\psi^{-1}(w)) \left(\int_{D(w,\beta\tau(w))} \tau(z)^{-2} dm(z) \right) dm(\psi^{-1}(w)) \\
&\simeq \int_{\mathbb{C}} M^{\frac{pq}{p-q}}_{(g,\psi)}(\psi^{-1}(w)) dm(\psi^{-1}(w)) \simeq \int_{\mathbb{C}} M^{\frac{pq}{p-q}}_{(g,\psi)}(z) dm(z) < \infty,
\end{aligned}$$

where the last conclusion follows since $\psi^{-1}(w) = (w - b)/a$ and $a \neq 0$.

To prove the sufficiency assertion, it suffices to show the local behavior of the measure $\mu_{(u,\psi,q)}$ in (4.11). Using the local estimate in (4.7), we have

$$\int_{\mathbb{C}} |\mathcal{G}(z)|^{\frac{p}{p-q}} dm(z) = \int_{\mathbb{C}} \left(\frac{1}{\tau^2(z)} \int_{D(z,\delta\tau(z))} M^q_{(g,\psi)}(\psi^{-1}(w)) dm(\psi^{-1}(w)) \right)^{\frac{p}{p-q}} dm(z)$$
$$\gtrsim \int_{\mathbb{C}} M^{\frac{pq}{p-q}}_{(g,\psi)}(z) dm(z),$$

and completes the proof of Theorem 2.1.

4.2 Proof of Theorem 3.2

Sufficiency. We recall that a compact operator T belongs to the Schatten $\mathcal{S}_p$ class if and only if the positive operator $(T^*T)^{p/2}$ belongs to the trace class $\mathcal{S}_1$. Furthermore, $T \in \mathcal{S}_p$ if and only if $T^* \in S_p$ and $\|T\|_{\mathcal{S}_p} = \|T^*\|_{\mathcal{S}_p}$. Thus, we may first consider the case $p < 2$ and estimate the trace of $\big(W_{(u,\psi)}W^*_{(u,\psi)}\big)^{p/2}$ by

$$\begin{aligned}\mathrm{tr}\big((W_{(u,\psi)}W^*_{(u,\psi)})^{\frac{p}{2}}\big) &= \int_{\mathbb{C}} \Big\langle \big(W_{(u,\psi)}W^*_{(u,\psi)}k_z\big)^{\frac{p}{2}}, k_z\Big\rangle dm(z)\\ &\leq \int_{\mathbb{C}} \Big\langle W_{(u,\psi)}W^*_{(u,\psi)}k_z, k_z\Big\rangle^{\frac{p}{2}} dm(z) = \int_{\mathbb{C}} \|W^*_{(u,\psi)}k_z\|_2^p dm(z), \end{aligned} \tag{4.17}$$

where the inequality holds since $0 < p \leq 2$, $W_{(u,\psi)}W^*_{(u,\psi)}$ is a positive operator, and $k_z = K_z/\|K_z\|_2$ is a unit norm vector, see [16, Proposition 1.31]. On the other hand, by the reproducing property of the kernel function,

$$W^*_{(u,\psi)}K_w(z) = \big\langle W^*_{(u,\psi)}K_w, K_z\big\rangle = \overline{u(w)}K_{\psi(w)}(z) \tag{4.18}$$

from which and estimate (4.6), we get

$$\|W^*_{(u,\psi)}k_w\|_2 \simeq \frac{|u(w)|\tau(w)}{\tau(\psi(w))}e^{\varphi(\psi(w))-\varphi(w)}.$$

This along with (4.17) and compactness of $W_{(u,\psi)}$ imply

$$\begin{aligned}\mathrm{tr}\big(\big(W_{(u,\psi)}W^*_{(u,\psi)}\big)^{\frac{p}{2}}\big) &\leq \int_{\mathbb{C}} \left(\frac{\tau(w)}{\tau(\psi(w))}\right)^p |u(w)|^p e^{p(\varphi(\psi(w))-\varphi(w))} dm(z)\\ &\lesssim \int_{\mathbb{C}} \frac{|u(w)|^p}{\tau(\psi(w))^2} e^{p(\varphi(\psi(w))-\varphi(w))} dm(z) = \int_{\mathbb{C}} m^p_{(u,\psi)}(z)\frac{dm(z)}{\tau^2(\psi(w))} < \infty,\end{aligned}$$

where the last inequality follows by definition $\tau(w)^p \gtrsim \tau^2(w)$ for all $p \leq 2$. Considering this and condition (4.17), we conclude the trace of $(\big(W_{(u,\psi)}W^*_{(u,\psi)}\big)^{\frac{p}{2}}$ is finite.

Suppose now that $p \geq 2$ and recall that a compact map $W_{(u,\psi)}$ belongs to $\mathcal{S}_p$ if and only if the sequence $\big(\|W_{(u,\psi)}e_n\|_2\big)$ belongs to ℓ^p for any orthonormal set $\{e_n\}$ of $\mathcal{F}^2_\varphi$ [16, Theorem 1.33]. We have

$$\sum_{n=1}^{\infty} \|W_{(u,\psi)}e_n\|_2^p = \sum_{n=1}^{\infty}\left(\int_{\mathbb{C}} |e_n(\psi(z))|^2|u(z)|^2 e^{-2\varphi(z)}dm(z)\right)^{\frac{p}{2}}. \tag{4.19}$$

We may first dispose the case when $p = 2$. As it is known, $W_{(u,\psi)}$ belongs to the Hilbert–Schmidt class $\mathcal{S}_2(\mathcal{F}^2_\varphi)$ if and only if

$$\sum_{n=1}^{\infty} \|W_{(u,\psi)}e_n\|_2^2 < \infty.$$

In this case, (4.19) and (4.6) give

$$\begin{aligned}
\sum_{n=1}^{\infty} \|W_{(u,\psi)}e_n\|_2^2 &= \int_{\mathbb{C}} \left(\sum_{n=1}^{\infty} |e_n(\psi(z))|^2 \right) |u(z)|^2 e^{-2\varphi(z)} dm(z) \\
&\simeq \int_{\mathbb{C}} \tau^{-2}(\psi(z)) |u(z)|^2 e^{2\varphi(\psi(z))-2\varphi(z)} dm(z) \\
&= \int_{\mathbb{C}} \tau^{-2}(\psi(z)) m_{(u,\psi)}^2(z) dm(z) < \infty.
\end{aligned}$$

Now for $p > 2$, applying Hölder's inequality for the integral in (4.19), we deduce

$$\begin{aligned}
\mathcal{I}_n &:= \left(\int_{\mathbb{C}} |e_n(\psi(z))|^2 |u(z)|^2 e^{-2\varphi(z)} dm(z) \right)^{\frac{p}{2}} \\
&\leq \left(\int_{\mathbb{C}} |e_n(\psi(z))|^2 |u(z)|^p e^{-p\varphi(z)} e^{(p-2)\varphi(\psi(z))} dm(z) \right) \\
&\quad \times \left(\int_{\mathbb{C}} |e_n(\psi(z))|^2 e^{-\varphi(\psi(z))} dm(z) \right)^{\frac{p-2}{2}}.
\end{aligned}$$

Upon performing a change of variables

$$\int_{\mathbb{C}} |e_n(\psi(z))|^2 e^{-\varphi(\psi(z))} dm(z) \lesssim \|e_n\|^2 = 1,$$

which implies

$$\mathcal{I}_n \lesssim \int_{\mathbb{C}} |e_n(\psi(z))|^2 |u(z)|^p e^{-p\varphi(z)} e^{(p-2)\varphi(\psi(z))} dm(z).$$

On the other hand, because of the reproducing property of the kernel and Parseval's identity,

$$K_w(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)} \text{ and } \|K_w\|_2^2 = \sum_{n=1}^{\infty} |e_n(w)|^2. \tag{4.20}$$

From (4.20) and the estimate in (4.6), we obtain

$$\sum_{n=1}^{\infty} |e_n(\psi(z))|^2 \simeq \tau^{-2}(\psi(z)) e^{2\varphi(\psi(z))},$$

and hence

$$\sum_{n=1}^{\infty}\|W_{(u,\psi)}e_n\|_2^p \simeq \sum_{n=1}^{\infty}\mathcal{I}_n \lesssim \int_{\mathbb{C}}\frac{|u(z)|^p}{\tau(\psi(z))^2}e^{p(\varphi(\psi(z))-\varphi(z))}dm(z)$$
$$= \int_{\mathbb{C}} m^p_{(u,\psi)}(z)\frac{dm(z)}{\tau^2(\psi(z))} \simeq \int_{\mathbb{C}} m^p_{(u,\psi)}(z)\Delta\varphi(\psi(z))dm(z) < \infty,$$

Therefore, the sufficiency condition follows.

Necessity. Now let us prove the necessity assertion. We may again follow two cases and assume first $0 < p < 2$. Since $W_{(u,\psi)}$ is assumed to be in $\mathcal{S}_p$, the operator $W^*_{(u,\psi)}W_{(u,\psi)}$ belongs to $\mathcal{S}_{p/2}(\mathcal{F}^2_\varphi)$. Then, there exists an orthonormal basis (e_n) for $\mathcal{F}^2_\varphi$ such that $(W_{(u,\psi)})^*W_{(u,\psi)}$ has the canonical decomposition

$$W^*_{(u,\psi)}W_{(u,\psi)}f = \sum_{n=1}^{\infty}\lambda_n\langle f, e_n\rangle e_n, \tag{4.21}$$

where (λ_n) is the sequence of the singular values of the positive operator $W_{(u,\psi)}W^*_{(u,\psi)}$. We recall that the operator $W^*_{(u,\psi)}W_{(u,\psi)}$ with the above decomposition belongs to $\mathcal{S}_p(\mathcal{F}^2_\varphi)$ if and only if

$$\|W^*_{(u,\psi)}W_{(u,\psi)}\|^p_{\mathcal{S}_p} = \sum_{n=1}^{\infty}|\lambda_n|^p < \infty. \tag{4.22}$$

On the other hand, estimate (4.6) yields

$$\int_{\mathbb{C}}\frac{m^p_{(u,\psi)}(z)}{\tau^2(\psi(z))}dm(z) \simeq \int_{\mathbb{C}}\frac{|u(z)|^p}{\tau^2(\psi(z))}e^{p\left(\varphi(\psi(z))-\varphi(z)\right)}\frac{\|K_{\psi(z)}\|_2^2}{\tau^{-2}(\psi(z))e^{2\varphi(\psi(z))}}dm(z)$$
$$= \sum_{n=1}^{\infty}\int_{\mathbb{C}}|u(z)|^p e^{-p\varphi(z)}|e_n(\psi(z))|^2 e^{(p-2)\varphi(\psi(z))}dm(z).$$

Applying Hölder's inequality with exponent $\frac{2}{p} > 1$, we observe that the above sum is bounded by

$$\sum_{n=1}^{\infty}\left(\int_{\mathbb{C}}|u(z)|^2e^{-2\varphi(z)}|e_n(\psi(z))|^2dm(z)\right)^{\frac{p}{2}}$$
$$\times\left(\int_{\mathbb{C}}|e_n(\psi(z))|^2e^{-2\varphi(\psi(z))}dm(z)\right)^{\frac{2-p}{2}}. \tag{4.23}$$

It follows from our assumption $W_{(u,\psi)}$ is a compact operator and hence by Lemma 4.2, $\psi(z) = az + b$ with $0 < |a| < 1$. From this and definition of τ,

$$\int_{\mathbb{C}} |e_n(\psi(z))|^2 e^{-2\varphi(\psi(z))} dm(z) = |a|^2. \tag{4.24}$$

Then, the sum in (4.23) is bounded (up to a constant multiple) by

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\int_{\mathbb{C}} |u(z)|^2 e^{-2\varphi(z)} |e_n(\psi(z))|^2 dm(z) \right)^{\frac{p}{2}} &\simeq \sum_{n=1}^{\infty} \left(\langle W_{(u,\psi)}^* W_{(u,\psi)} e_n, e_n \rangle \right)^{\frac{p}{2}} \\ &= \sum_{n=1}^{\infty} \lambda_n^{p/2} = \| W_{(u,\psi)}^* W_{(u,\psi)} \|_{\mathcal{S}_{p/2}}^{p/2} = \| W_{(u,\psi)} \|_{\mathcal{S}_p}^{p} < \infty.\end{aligned}$$

Suppose now that $2 \leq p < \infty$ and let $W_{(u,\psi)}$ be in $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class. Then, it is compact, and by Lemma 4.2 again, $\psi(z) = az + b, 0 < |a| < 1$. Let e_k be an orthonormal basis in $\mathcal{F}_\varphi^2$. Using the sequence in Lemma 4.3, define an operator T by $Te_k(z) = f_{\psi(z_k)}(z) = F_{\psi(z_k)}(z)/\tau(\psi(z_k))$. Observe that by (4.5), $f_{\psi(z_k)}$ is a sequence of unit norm functions in the spaces and by [2, Proposition 9], T is a bounded operator in $\mathcal{F}_\varphi^2$. Consequently, by [16, Theorem 1.33]

$$\sum_{k=1}^{\infty} \| W_{(u,\psi)} f_{\psi(z_k)} \|_2^p = \sum_{k=1}^{\infty} \| W_{(u,\psi)} T e_k \|_2^p,$$

which together with Lemma 4.3 implies

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{1}{\tau(\psi(z_k))^p} \left(\int_{D(\psi(z_k),\tau(\psi(z_k)))} |u(z)|^2 e^{2\varphi(\psi(z))-2\varphi(z)} dm(z) \right)^{p/2} \\ &\simeq \sum_{k=1}^{\infty} \left(\int_{D(\psi(z_k),\tau(\psi(z_k)))} |u(z)|^2 |f_{(\psi(z_k))(\psi(z))}|^2 e^{-2\varphi(z)} dm(z) \right)^{p/2} \\ &\lesssim \sum_{k=1}^{\infty} \| W_{(u,\psi)} f_{\psi(z_k)} \|_2^p < \infty.\end{aligned}$$

On the other hand, if δ is sufficiently small, using (4.7), we get

$$\begin{aligned}\int_{\mathbb{C}} \frac{m_{(u,\psi)}^p(z)}{\tau(\psi(z))^2} dm(z) &\lesssim \sum_{k=1}^{\infty} \int_{D(\psi(z_k),\delta\tau(\psi(z_k)))} \frac{S_p(z)}{\tau(\psi(z))^p} \frac{dm(z)}{\tau(\psi(z))^2} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{\tau(\psi(z_k))^p} \int_{D(\psi(z_k),\delta\tau(\psi(z_k)))} S_p(z) \frac{dm(z)}{\tau(\psi(z))^2},\end{aligned}$$

where

$$S_p(z) = \left(\int_{D(\psi(z),\delta\tau(\psi(z)))} |u(w)|^2 e^{2\varphi(\psi(w))-2\varphi(w)} dm(w) \right)^{p/2}.$$

It follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{\tau(\psi(z_k))^p} \int_{D(\psi(z_k),\delta\tau(\psi(z_k)))} S_p(z) \frac{dm(z)}{\tau^2(\psi(z))} \\ &\simeq \sum_{k=1}^{\infty} \frac{1}{\tau^p(\psi(z_k))} \left(\int_{D(\psi(z_k),\delta\tau(\psi(z_k)))} |u(w)|^2 e^{2\varphi(\psi(w))-2\varphi(w)} dm(w) \right)^{p/2}, \end{aligned}$$

and completes the proof.

4.3 Proof of Proposition 3.3

Note that since $W_{(u,\psi)}$ is compact in $\mathcal{F}_\varphi^2$, it admits a Schmidt decomposition, and there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_\varphi^2$ and a sequence of nonnegative numbers $(\lambda_{(n,u,\psi)})$ with $\lambda_{(n,u,\psi)} \to 0$ as $n \to \infty$ such that for all f in $\mathcal{F}_\varphi^2$,

$$W_{(u,\psi)} f = \sum_{n=1}^{\infty} \lambda_{(n,u,\psi)} \langle f, e_n \rangle e_n. \tag{4.25}$$

Then, $W_{(u,\psi)}$ with such a decomposition belongs to $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ if and only if

$$\|W_{(u,\psi)}\|_{\mathcal{S}_p}^p = \sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^p < \infty. \tag{4.26}$$

Applying (4.25), in particular to the kernel function, we obtain the relation

$$\|W_{(u,\psi)} K_{\psi(z)}\|_2^2 = \sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^2 |e_n(\psi(z))|^2,$$

and from which and (4.6) we have

$$\begin{aligned} &\int_{\mathbb{C}} \|W_{(u,\psi)} k_{\psi(z)}\|_2^p \frac{dm(z)}{\tau(\psi(z))^2} \\ &\simeq \int_{\mathbb{C}} \left(\sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^2 |e_n(\psi(z))|^2 \right)^{\frac{p}{2}} \tau(\psi(z))^p e^{-p\varphi(\psi(z))} \frac{dm(z)}{\tau^2(\psi(z))}. \end{aligned} \tag{4.27}$$

We may now consider two different cases depending on the size of the exponent p and proceed first to show the necessity for the case $p > 2$. Applying Hölder's inequality to the sum shows that the left-hand side in (4.27) is bounded by

$$\int_{\mathbb{C}} \sum_{n=1}^{\infty} |\lambda_{(n,g,\psi)}|^p |e_n(\psi(z))|^2 \Big(\sum_{n=1}^{\infty} |e_n(\psi(z))|^2 \Big)^{\frac{p-2}{2}} \tau^p(\psi(z)) e^{-p\varphi(\psi(z))} \frac{dm(z)}{\tau^2(\psi(z))}$$
$$\simeq \sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^p \int_{\mathbb{C}} |e_n(\psi(z))|^2 e^{-2\varphi(\psi(z))} dm(z) \simeq \sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^p = \|W_{(u,\psi)}\|_{\mathcal{S}_p}^p,$$

where the last equality follows from (4.26).
(ii) To prove the sufficiency for $0 < p < 2$, observe that by estimate (4.6)

$$\|W_{(u,\psi)}\|_{\mathcal{S}_p}^p = \sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^p \|e_n\|_2^2 \simeq \sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^p \int_{\mathbb{C}} |e_n(\psi(z))|^2 \frac{\|K_{\psi(z)}\|_2^{-2}}{\tau^2(\psi(z))} dm(z).$$

Since $p < 2$, Hölder's inequality applied with exponent $2/p$ and subsequently invoking relations (4.3) give

$$\|W_{(u,\psi)}\|_{\mathcal{S}_p}^p \le \int_{\mathbb{C}} \Big(\sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^2 |e_n(\psi(z))|^2 \Big)^{\frac{p}{2}} \Big(\sum_{n=1}^{\infty} |e_n(\psi(z))|^2 \Big)^{\frac{2-p}{2}} \frac{\|K_{\psi(z)}\|_2^{-2}}{\tau^2(\psi(z))} dm(z)$$
$$= \int_{\mathbb{C}} \Big(\sum_{n=1}^{\infty} |\lambda_{(n,u,\psi)}|^2 |e_n(\psi(z))|^2 \Big)^{\frac{p}{2}} \frac{\|K_{\psi(z)}\|_2^{-p}}{\tau^2(\psi(z))} dm(z)$$
$$= \int_{\mathbb{C}} \|W_{(u,\psi)} k_{\psi(z)}\|_2^p \frac{dm(z)}{\tau^2(\psi(z))} \simeq \int_{\mathbb{C}} \|W_{(u,\psi)} k_{\psi(z)}\|_2^p \Delta\varphi(|\psi(z)|) dm(z).$$

4.4 Proof of Theorem 3.4

Part (i): Suppose that $W_{(u,\psi)}$ is a bounded invertible operator. Then by Proposition 3.1 and Lemma 4.2, it follows that $\psi(z) = az + b$, $|a| \le 1$. The adjoint operator $W^*_{(u,\psi)}$ is also invertible and hence there exists a positive constant c such that

$$\|W^*_{(u,\psi)} K_z\|_2^2 \ge c \|K_z\|_2^2 \tag{4.28}$$

for all $z \in \mathbb{C}$. By (4.18), we have

$$W^*_{(u,\psi)} K_z = \overline{u(z)} K_{\psi(z)}$$

which together with the estimates (4.6) and (4.28) yield

$$\frac{|u(z)|^2 \|K_{\psi(z)}\|_2^2}{\|K_z\|_2^2} \simeq |u(z)|^2 e^{2\varphi(\psi(z)) - 2\varphi(z)} \frac{\tau^2(z)}{\tau^2(\psi(z))}$$
$$\simeq |u(z)|^2 e^{2\varphi(\psi(z)) - 2\varphi(z)} = m^2_{(u,\psi)}(z) \gtrsim c, \tag{4.29}$$

where we used the growth assumption on φ to compare $\tau(z)$ and $\tau(az+b)$ for $|a| \leq 1$. The relation (4.29) shows u has no zeros in $\mathbb{C}$. Furthermore, since ψ is nonconstant and hence $a \neq 0$, replacing z by $\psi^{-1}(z) = (z-b)/a$ in (4.29) gives

$$\frac{e^{2\varphi(\psi^{-1}(z))-2\varphi(z)}}{|u(\psi^{-1}(z))|^2} = m^2_{(1/u(\psi^{-1}),\psi^{-1})}(z) \lesssim \frac{1}{c} \tag{4.30}$$

for all $z \in \mathbb{C}$. It follows from Corollary 3.1 that the weighted composition operator with symbol $(1/u(\psi^{-1}), \psi^{-1})$ is bounded in $\mathcal{F}^2_\varphi$. Then, Lemma 4.2 implies that $|1/a| \leq 1$ which together with $|a| \leq 1$ gives the assertion $|a| = 1$.

Conversely, assume that $\psi(z) = az + b$, $|a| = 1$ and condition (3.3) holds. By (3.3), the multiplier function u has no zero in $\mathbb{C}$. Applying Corollary 3.1, both $W_{u,\psi}$ and $W_{(1/u(\psi^{-1}),\psi^{-1}))}$ are bounded. A simple computation shows

$$W_{(u,\psi)} W_{(1/u(\psi^{-1}),\psi^{-1})} f = W_{(1/u(\psi^{-1}),\psi^{-1})} W_{(u,\psi)} f = f.$$

for each $f \in \mathcal{F}^2_\varphi$. Thus, $W_{(u,\psi)}$ is invertible and $W^{-1}_{(u,\psi)} = W_{(1/u(\psi^{-1}),\psi^{-1}))}$.

Part (ii): Recall that an operator is unitary if it is invertible and its inverse and adjoint coincides. Assume that $W_{(u,\psi)}$ is unitary. Then by part (i), $W^{-1}_{(u,\psi)} = W_{(1/u(\psi^{-1}),\psi^{-1})} = W^*_{(u,\psi)}$ and for each $f \in \mathcal{F}^2_\varphi$

$$W_{(u,\psi)} W^*_{(u,\psi)} f = W^*_{(u,\psi)} W_{(u,\psi)} f = f.$$

We proceed to show that u has the form in (3.4). Letting $f = K_z$ in the above relation and applying (4.18)

$$W_{(u,\psi)} W^*_{(u,\psi)} K_z = W_{(u,\psi)} \overline{u(z)} K_{\psi(z)} = u\overline{u(z)} K_{\psi(z)} \circ \psi = K_z,$$

from which setting $z = 0$ in particular gives

$$u\overline{u(0)} K_b \circ \psi = K_0. \tag{4.31}$$

Considering the standard orthonormal basis $e_n(z) = z^n/\|z^n\|_2$, $n \geq 0$ where we set $e_0 = 1/\|1\|_2$ and the series representation in (4.20)

$$K_0(w) = \sum_{n=0}^{\infty} e_n(w)\overline{e_n(0)} = \frac{1}{\|1\|_2^2} \tag{4.32}$$

for each $w \in \mathbb{C}$. From which and by (4.31), $K_b \circ \psi$ has no zero. Since u has no zeros either, we conclude

$$u = \frac{K_0}{\overline{u(0)} K_b(\psi)} = \frac{\|1\|_2^{-2}}{\overline{u(0)} K_b \circ \psi}.$$

To prove the converse, assume that u has the form in (3.4) and $\Phi(z) = \|1\|_2^2\overline{u(0)}K_b(z)$. Then for each $f \in \mathcal{F}_\varphi^2$, we have

$$W_{(\Phi,\psi^{-1})}W_{(u,\psi)}f = W_{(\Phi,\psi^{-1})}(uf(\psi)) = \Phi u(\psi^{-1})f(\psi(\psi^{-1}))$$
$$= \|1\|_2^2\overline{u(0)}K_b(z)\frac{\|1\|_2^{-2}}{\overline{u(0)}K_b(\psi(\psi^{-1}))}f = f.$$

Similarly, $W_{(u,\psi)}W_{(\Phi,\psi^{-1})}f = f$. Thus, $W^*_{(u,\psi)} = W^{-1}_{(u,\psi)} = W_{(\Phi,\psi^{-1})}$ and completes the proof of this part.

Part (iii): Recall that an operator T on a Hilbert space is normal if it commutes with its adjoint T^*. The assumption implies $\psi(z) = az + b$ fixes the point $z_0 = b/(1-a)$ when $a \neq 1$ and $z_0 = 0$, or $a = 1$ and $b = 0$. Applying $W^*_{(u,\psi)}$ to K_{z_0}

$$W^*_{(u,\psi)}K_{z_0} = \overline{u(z_0)}K_{\psi(z_0)} = \overline{u(z_0)}K_{z_0},$$

which shows that $\overline{u(z_0)}$ is an eigenvalue of $W^*_{(u,\psi)}$ with eigenvector K_{z_0}. Since $W_{(u,\psi)}$ is normal, K_{z_0} is also an eigenvector of $W_{(u,\psi)}$ with eigenvalue $u(z_0)$ and

$$W_{(u,\psi)}K_{z_0} = u(z)K_{z_0} \circ \psi = u(z_0)K_{z_0}$$

from which we conclude

$$u(z) = \frac{u(z_0)K_{z_0}}{K_{z_0} \circ \psi}.$$

Evaluating both sides of the preceding equation at 0 and applying (4.32) yield $u(z_0) = u(0)\|1\|_2^2 K_{z_0}(b)$, and completes the proof.

4.5 Proof of Proposition 3.5

Let f be any vector in $\mathcal{F}_\varphi^p$. Then, its orbit under $W_{(u,\psi)}$ has elements of the form

$$W^n_{(u,\psi)}f = f(\psi^n)\prod_{j=0}^{n-1} u \circ \psi^j \tag{4.33}$$

for all nonnegative integers n and ψ^0 is the identity map.

Assume on the contrary that there exists a τ_{pt}-supercyclic vector f in $\mathcal{F}_\varphi^2$. It follows that neither the multiplier function u nor the vector f has zeros in $\mathbb{C}$. This is because if any of them vanishes at point w, then (4.33) implies that every element in the projective orbit of f vanishes at w which extends to the closure resulting a contradiction.

Next, since $W_{(u,\psi)}$ is bounded, by Lemma 4.2, we may set $\psi(z) = az + b$, with $|a| \leq 1$. It follows the map ψ fixes the point $z_0 = \frac{b}{1-a}$ for $a \neq 1$ and $z_0 = 0$, or $a = 1$ and $b = 0$. Then, the rest of the proof follows exactly the same arguments used in the proof of Theorem 1.7 in [7].

4.6 Proof of Theorem 2.3

The proofs for $p \geq 2$ and the sufficiency part for $0 < p < 2$ are quite similar to the corresponding proof of Theorem 3.2 by simply replacing $|u|$ by $|g'|/(1+\psi')$ and making some trivial adjustments. Thus, we only need to prove the necessity of the condition for $1 < p < 2$. To this end, observe that an immediate consequence of (4.20) and Parseval's identity is that

$$\frac{\partial}{\partial \overline{w}} K_w(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n'(w)}, \text{ and } \Big\| \frac{\partial}{\partial \overline{w}} K_w \Big\|_2^2 = \sum_{n=1}^{\infty} |e_n'(w)|^2. \tag{4.34}$$

By [2, Lemma 22], we further have

$$\Big\| \frac{\partial}{\partial \overline{w}} K_w \Big\|_2^2 = \sum_{n=1}^{\infty} |e_n'(w)|^2 \simeq (\varphi'(w))^2 \|K_w\|_2^2. \tag{4.35}$$

On the other hand, by the relation in (3.1), the inner product

$$\langle f, g\rangle_D = f(0)\overline{g(0)} + \int_{\mathbb{C}} f'(z)\overline{g'(z)} \frac{e^{-2\varphi(z)}}{(1+\varphi'(z))^2} dm(z) \tag{4.36}$$

gives a norm on $\mathcal{F}_\psi^2$ equivalent to the usual one.

For $1 < p < \infty$, $V_{(g,\psi)}$ belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class if and only if

$$\sum_{n=1}^{\infty} |\langle V_{(g,\psi)} e_n, e_n\rangle_D|^p < \infty.$$

for any orthonormal basis (e_n) in the space with respect to the inner product in (4.36); see [16, Theorem1.27] for more. Since $p > 1$, by Hölder's inequality

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle V_{(g,\psi)} e_n, e_n\rangle_D|^p &\leq \sum_{n=1}^{\infty} \bigg(\int_{\mathbb{C}} |e_n(\psi(z)) e_n'(z)| \frac{|g'(z)| e^{-2\varphi(z)}}{(1+\varphi'(z))^2} dm(z) \bigg)^p \\
&\leq \sum_{n=1}^{\infty} \int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+\varphi'(z))^2} |e_n(\psi(z))|^p |e_n'(z)|^{2-p} e^{-2\varphi(z)} dm(z) \\
&\times \bigg(\int_{\mathbb{C}} \frac{|e_n'(z)|^2}{(1+\varphi'(z))^2} e^{-2\varphi(z)} dm(z) \bigg)^{p-1} \\
&\simeq \int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+\varphi'(z))^2} \sum_{n=1}^{\infty} \big(|e_n(\psi(z))|^p |e_n'(z)|^{2-p} \big) e^{-2\varphi(z)} dm(z)
\end{aligned} \tag{4.37}$$

Similarly, since $p < 2$, applying Hölder's inequality to the sum above again and using (4.6) and (4.35)

$$\sum_{n=1}^{\infty} |e_n(\psi(z))|^p |e_n'(z)|^{2-p} \leq \left(\sum_{n=1}^{\infty} |e_n(\psi(z))|^2\right)^{p/2} \left(\sum_{n=1}^{\infty} |e_n'(z)|^2\right)^{(2-p)/2}$$
$$= \|K_{\psi(z)}\|_2^p \left\| \frac{\partial}{\partial \overline{z}} K_z \right\|_2^{2-p} \simeq \|K_{\psi(z)}\|_2^p \big(\|K_z\|_2 \varphi'(z)\big)^{2-p}$$
$$\simeq \frac{e^{p\varphi(\psi(z))+(2-p)\varphi(z)}(\varphi'^{2-p}(z))}{\tau^p(\psi(z))(\tau^{2-p}(z))}. \tag{4.38}$$

Next, we claim

$$\frac{\varphi'^{2-p}(z)}{\tau(\psi(z))^p(\tau^{2-p}(z))} \lesssim \frac{\varphi'^p(z)}{\tau^2(\psi(z))} \tag{4.39}$$

for all $1 < p < 2$ and sufficiently large z. But this follows rather easily since by definition of τ

$$\varphi'(z))^{2(1-p)} \lesssim \frac{\tau^{2-p}(z)}{\tau^{2-p}(\psi(z))}$$

as $|z| \to \infty$. Setting the estimate in (4.39) and (4.38) in (4.37)

$$\sum_{n=1}^{\infty} |\langle V_{(g,\psi)} e_n, e_n\rangle_D|^p \lesssim \int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+\varphi'(z))^2} \frac{e^{p\varphi(\psi(z))-p\varphi(z)}(\varphi'(z))^p}{\tau^2(\psi(z))} dm(z)$$
$$\lesssim \int_{\mathbb{C}} M^p_{(g,\psi)}(z) \frac{dm(z)}{\tau^2(\psi(z))} \simeq \int_{\mathbb{C}} M^p_{(g,\psi)}(z) \Delta\varphi(|\psi(z)|) dm(z) < \infty$$

and completes the proof.

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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