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# Convergence of the mimetic finite difference and fitted mimetic finite difference method for options pricing

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ABSTRACT

We present in this paper two novel numerical spatial discretization techniques based on the mimetic finite difference method for a degenerated partial differential equation (PDE) in one dimension. This PDE is well known as the Black-Scholes PDE which govern option pricing. To handle the degeneracy of the PDE, a novel fitted mimetic finite difference scheme is proposed together with the standard mimetic finite difference method. The temporal discretization is performing using standard implicit scheme. Furthermore rigorous convergence proofs in appropriate normed spaces are proposed. We validate the theoretical results by presenting numerical results and simulations. Those numerical experiments show that our two novel schemes outperform the standard finite difference method and the standard fitted finite volume method in terms of accuracy.



## 1. Introduction

Derivative security pricing is very critical in the financial industry. In particular, an option is one such derivative security which is a traded contract that gives to its owner the right (not obligatory) to buy (call) or sell (put) a quantity of assets at a fixed price (strike price) on (European option) or before (American option) a given date (maturity/expiry date) [10]. Black and Scholes (1972) in their seminal paper [1] obtained a closed form solution for the value of the European option ($V$) which they showed was governed by the following second order parabolic differential equation with respect to time ($t$) and the price of the underlying stock ($s$)

$$LV := \frac{\partial V(s,t)}{\partial t} + rs\frac{\partial V(s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} - rV(s,t) = 0, \tag{1}$$

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where $L$ is the differential operator associated to the Black-Scholes equation defined by

$$L := \frac{\partial}{\partial t} + rs\frac{\partial}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} - rI, \tag{2}$$

for $(s,t) \in \Omega \times [0,T)$,with associated boundary and final (or payoff) conditions

$$\begin{aligned} V(0,t) &= g_1(t),\ t \in [0,T) \\ V(S_{\max},t) &= g_2(t),\ t \in [0,T) \\ V(s,T) &= \max(K-s,0),\ x \in \overline{\Omega} \end{aligned}$$

where $\Omega = (0, S_{\max}) \subset \mathbb{R}$, $\sigma > 0$ denotes the volatility of the asset, $K \ll S_{\max}$ is the strike, $T > 0$ is the expiry date, $r$ the interest rate and $I$ represents the identity operator. The functions $g_1$ and $g_2$ are assumed to satisfy certain compatibility conditions (please see [2,3,9] for more information). Investment managers are interested in mitigating risk by offsetting any potential losses or gains in the position they take on an investment. They use sensitivity measures of the option called Greeks. The sensitivity measure with respect to the underlying asset (assuming all other variables remains constant) is known as the Delta $(\Delta) = \frac{\partial V}{\partial s}$. There is also the second derivative with respect to the underlying asset and is given by Gamma $(\Gamma) = \frac{\partial^2 V}{\partial s^2}$. The analytical solutions only exist for simple cases where the coefficients are constant, so numerical methods have become popular as realistic tools for their approximations. Finite difference methods are amongst the commonly used standard discretization methods for spatial dicretization [9]. The Black-Scholes operator is known to be a convection-diffusion operator [2,3,9]. For small volatility or asset price, the Black-Scholes operator becomes a convection-dominated operator [2,3,9,11]. This causes numerical oscillations when a standard finite difference method is used and also significantly affects the accuracy of hedging parameters [2,3]. To overcome this challenge, the standard upwind finite difference scheme is used and its stability depends on the associated *Courant-Friedrichs-Lewy* (CFL) condition being satisfied [9]. This alternative method is influenced by the biased directional flow of the convection-diffusion problem and overcomes the instability of the central differencing scheme of the standard finite difference scheme by giving numerically stable results. Furthermore, the initial condition of the PDE has a discontinuity in its first derivative when the stock price is equal to the strike $K$ and the Black Scholes operator is degenerated at $S = 0$. These discontinuity and degeneracy have an adverse impact on the accuracy when the standard finite difference method or the standard upwind finite difference are used (see [20], chapter 26) as the monotonicity is not ensured. Therefore, for the Black-Scholes PDE spatial discretization, it is important to build methods suitable to handle the degeneracy at $S = 0$ and the discontinuity at $S = K$. A fitted finite volume method for one dimensional Black Scholes PDE is proposed in [12], and the rigorous convergence proof is provided in [13]. Although this fitted finite volume method is stable, it is only order 1 with respect to asset price variables.

Here, we propose the so called fitted Mimetic finite difference method (MFDM) to handle the degeneracy of the Black Scholes PDE at $S = 0$. For non degenerated PDE, the standard MFDM is a high-quality (second order and above [4,5,14–16]) spatial discretization technique which follows from the well-known support operator method (SOM) [4,5]. The SOM's are known to guarantee stability on general grids [4,5,14–16] in high dimension. Also, the standard MFDM tends to preserve important properties of the underlying continuum problem such as conservation laws, solution symmetries and fundamental properties of vector and tensor calculus (such as divergence, gradient and curls). This makes the MFDMs far superior to the standard finite difference methods. However, the standard MFDM can not handle the degeneracy at $S = 0$. To overcome this drawback, we proposed a novel scheme by combining the standard fitted method [12,13] with the standard MFDM. Indeed the standard fitted method is used to handle the degeneracy of the PDE in the region where the stocks price approach zero (degeneracy region). In the region, where the PDE in not degenerated, we apply the standard MFDM method. The novel numerical technique from this combination is called fitted MFDM method and will obviously improve the accuracy of the current fitted finite volume in the literature, since most of the approximations of second order accuracy in space are involved.

The paper is organised as follow. In Section 2, we present the theoretical framework for the well posedness result and mimetic finite difference method. In Section 3, we present two spatial discretization techniques (Mimetic finite difference (MFD) and Fitted Mimetic Finite difference (FMFD)) both for elliptic PDEs and our Black Scholes PDE where the advection term is approximated using the first order upwind technique. We further prove the existence and uniqueness of the semi-discrete solutions corresponding to our two novel schemes. In Section 4, we perform the full discretization by using the implicit Euler method for time discretization and prove the convergence of the fully discretized schemes. We conclude the paper by presenting in Section 5 some numerical experiments to sustain our theoretical results and summarising our findings. Those numerical experiments show that our two novel schemes outperform the standard finite difference method and the fitted finite volume method [12] in terms of accuracy.

## 2. Theoretical framework

In this paper, the following notations will be used. For $\Omega \subset \mathbb{R}$ and $1 \le p \le \infty$, we have $L^p(\Omega) = \{v : \left(\int_\Omega |v(x)|dx\right)^{1/p} < \infty\}$ is the space of all $p$-power integrable functions on $\Omega$ with the usual modification of $p = \infty$. Then the inner product for $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. We equip $L^p(\Omega)$ with the norm $\|\cdot\|_{0,p}$. For some $l = 0,1,2,\ldots,$ we let $W_p^l$ be the Sobolev space with norm $\|\cdot\|_{l,p}$ and semi-norm $|\cdot|_{l,p}$. Then for the special case where $p = 2$, we denote the associated Sobolev space (and norm respectively) by $H^l(\Omega)$ ($\|\cdot\|_l$ respectively). Now let $C^l(\Omega)$ (respectively, $C^l(\overline{\Omega})$) be the set of functions with derivatives up to order $k$ continuous on $\Omega$ (respectively $\overline{\Omega}$). We denote by $H^l_0(\Omega) = \{v \in H^l(\Omega) : Tv = v|_{\partial\Omega} = 0\}$, where $T$ is the trace operator $T : H^l(\Omega) \to L^p(\partial\Omega)$. For any Hilbert space $H$ of classes of functions defined on $\Omega$, we denote by $L^p((0,T);H)$ the space defined by

$$L^p((0,T);H) = \{v(\cdot,t) : v(\cdot,t) \in H \ a.e \text{ in } (0,T) : \|v(\cdot,t)\|_H \in L^p((0,T))\}. \tag{3}$$

which is equipped with the norm

$$\|v\|_{L^p((0,T);H)} = \left(\int_0^T \|v(\cdot,t)\|_H^p dt\right)^{1/p} \quad 1 \le p < \infty.$$

where $\|\cdot\|_H$ is the natural norm on $H$. Since (1) is known to be degenerated, we introduce a weighted inner product on $L^2_w(\Omega)$ by $(u,v)_w := \int_0^{S_{\max}} x^2 uv dx$ with a corresponding weighted $L^2$-norm

$$\|v\|_{0,w} := \sqrt{(v,v)_w} = \left(\int_0^{S_{\max}} x^2 v^2 dx\right)^{1/2}.$$

Hence the space of all weighted square-integrable functions is defined as

$$L^2_w(\Omega) := \{v : \|v\|_{0,w} < \infty\}.$$

It is very clear with the use of standard arguments that the pair $(L^2_w(\Omega), (\cdot,\cdot)_w)$ is a Hilbert space (for example [7]). Then we can finally define the following weighted Sobolev space

$$H^1_{0,w}(\Omega) := \left\{v : v \in L^2(\Omega), v' \in L^2_w(\Omega) \text{ and } v(S_{\max}) = 0\right\},$$

where $v'$ denotes the weak partial derivative of $v$ w.r.t. $x$. Then we define the weighted inner product on $H^1_{0,w}(\Omega)$ by $(\cdot,\cdot)_H := (\cdot,\cdot) + (\cdot,\cdot)_w$, which is equipped with the norm

$$\|v\|_{1,w} = \left[\|v\|^2_{L^2(\Omega)} + \|v'\|^2_{0,w}\right]^{1/2} = \left[(x^2 v', v') + (v,v)\right]^{1/2}.$$

*2.1. The continuous problem*

Here, we transform the inhomogeneous problem (1) to the corresponding problem with homogeneous boundary conditions. This is achieved by adding $f(s,t) = -LV_0$ to both sides of (1), by introducing a new variable $u = V - V_0$, with $u^* = e^{-\beta t}(V_0 - V^*)$ and

$$V_0(s,t) = g_1(t) + \frac{g_2(t) - g_1(t)}{S_{\max}} s, \quad \beta = \sigma^2. \tag{4}$$

Please see [2,3,11,12] and the references therein for more details on the transformation of (1) which becomes

$$-\frac{\partial u}{\partial t} - \frac{\partial}{\partial s}\left[as^2 \frac{\partial u}{\partial s} + bsu\right] + cu = f(s,t) \tag{5}$$

where $a = \frac{1}{2}\sigma^2$, $b = r - \sigma^2$, $c = r + b = 2r - \beta, \beta := \sigma^2$, and the boundary and final conditions now satisfy the following compatibility conditions

$$u(s,T) = u^*(s); \quad u(0,t) = g_3(0) - V_0(0,T) = 0 = g_3(S_{\max}) - V_0(S_{\max},T) = u(S_{\max},t). \tag{6}$$

**Remark 1.** Note that without loss of generality, we are considering here only the simplest model where the coefficients are constant. For American put options, a nonlinear term called a penalty should be added in (5).

**Theorem 1.** *Let us consider the following problem: Find $u(t) \in H^1_{0,w}(\Omega)$, satisfying the final and boundary conditions (6), such that for all $v \in H^1_{0,w}(\Omega)$*

$$\begin{cases} \left(-\frac{du}{dt}, v\right) + A(u,v;t) = (f,v), \ \text{a.e in } (0,T), \\ u(T) = u^*, \end{cases} \tag{7}$$

*where the bilinear operator A is given by*

$$A(u,v;t) = (as^2 \nabla u + bsu, \nabla v) + (cu, v).$$

*The problem (7) is the variational form associated to (5), and furthermore (7) has a unique solution.*

**Proof.** See for example [2,3]. □

### 2.2. Support operator method and continuum operators for elliptic problems

In this section, we consider the so called Support Operator Method (SOM). Indeed most partial differential equations can be formulated in terms of invariant differential operators of gradient $\nabla$, divergence $\nabla\cdot$ and curl $\nabla\times$. The SOM takes advantage of this and gives an approach for spatial discretisation by constructing discrete analogs of these differential operators [5,6]. The continuum operators tend to satisfy important differential and integral identities. The SOM helps to construct the discrete operators which satisfy discrete versions of those important differential and integral identities. Generally, conservation laws, adjoint relationships and solution symmetries are some properties which we would like the discrete operators to mimic. For illustration, let us consider the following diffusion equation

$$-\nabla\cdot(K\nabla u(x)) = \mathbf{F}, \qquad x\in\Omega\subset\mathbb{R}^n,\ n\in\mathbb{N}. \tag{8}$$

where $K>0$[1] is a bounded invertible matrix function of $x$ which is symmetric positive definite (spd) and $\mathbf{F}$ could be a forcing function. For the operator $\mathbf{A}:H\to H$, defined by

$$\mathbf{A}u = -\nabla\cdot(K\nabla u(x)),\quad x\in\Omega, \tag{9}$$

the following properties hold

$$(\mathbf{A}u,v)_H = (u,\mathbf{A}v)_H,\quad (\mathbf{A}u,u)_H > 0, \tag{10}$$

Then (1) becomes

$$\mathbf{A}u = \mathbf{F}. \tag{11}$$

Now (8) can be re-rewritten as the following first-order system

$$\begin{cases}\nabla\cdot w = \mathbf{F}\\ w = -K\nabla u,\end{cases} \tag{12}$$

which is equivalent to

$$w - \mathbf{G}u = 0,\ \ \mathbf{D}w = \mathbf{F}, \tag{13}$$

where the operators $\mathbf{G}$ and $\mathbf{D}$ are defined as

$$\begin{cases}\mathbf{G}u = -K\nabla u \ \text{ on }\ \Omega\\ \mathbf{D}w = \nabla w \ \text{ on }\ \Omega\end{cases} \tag{14}$$

Note that (13) is the new formulation of (8) with continuous gradient $\nabla$ and divergence $\nabla\cdot$ operators that we would like our discrete operators to mimic. Note that from (14) we have

$$\mathbf{A} = \mathbf{DG}. \tag{15}$$

Let $H=L^2(\Omega)$ be the space of scalar functions $u$ that are smooth on the $\Omega$ equipped with inner product

$$(u,v)_H = \int_\Omega uv\,d\Omega,\quad u,v\in H, \tag{16}$$

and $\mathbf{H}=\left(L^2(\Omega)\right)^n$ equipped with inner product

$$(w,z)_{\mathbf{H}} = \int_\Omega (K^{-1}w,z)\,d\Omega,\quad w,z\in\mathbf{H}=\left(L^2(\Omega)\right)^n;\ n\in\mathbb{N}. \tag{17}$$

The inner product (17) is weighted by the inverse of $K$ and $(\cdot,\cdot)$ is the standard inner product of $\mathbb{R}^n$. Furthermore, since $K$ is spd, so is $K^{-1}$. Note that the following properties are fulfilled ([5,6])

$$(\mathbf{D}w,u)_H = (w,\mathbf{G}u)_{\mathbf{H}} \tag{18}$$

$$\mathbf{D}w,1)_H = 0, \tag{19}$$

$$\mathbf{A} = \mathbf{DG} = \mathbf{DD}^* \tag{20}$$

$$\mathbf{A} = \mathbf{A}^* > 0, \tag{21}$$

where 1 is the constant function with value 1, $\mathbf{A}^*$and $\mathbf{D}^*$ stand for the adjoints of the operators $\mathbf{A}$ and $\mathbf{D}$.

The properties (18)–(20) are the important properties of the continuum operators that we want our discrete operators in the next section to mimic.

[1] A material property tensor in engineering sciences.

## 3. Semi-discrete problem and mimetic method for elliptic problems

The goal here is to build a mimetic finite difference method to discretise the diffusion part of our continuous problem (5). As we just mentioned, the corresponding discrete operators will mimic the properties (18)–(20). The domain $\Omega = [0, S_{\max}]$ is divided into $N+1$ non-overlapping intervals $\mathcal{T} = (I_i)_{0\leq i\leq N}$, such that $I_i = (x_i, x_{i+1})$, $i = 0, 1, \ldots, N$, with $0 = x_0 < x_1 < \cdots < x_{N+1} = S_{\max}$ and set $h_i = x_{i+1} - x_i$, $h_{N+1} = 0$, $h = \max_{0\leq i\leq (N+1)} h_i$. Now for $i = 0, 1, \ldots, N+1$, we set $l_i = x_{i+1/2} - x_{i-1/2}$, and define the following mid-points $x_{i-\frac{1}{2}} = \frac{x_i + x_{i-1}}{2}$ and $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$ for $i = 1, \ldots, N$. We also set $x_{-\frac{1}{2}} = x_0$ and $x_{N+\frac{3}{2}} = x_{N+1}$[2]. Note the family $(\Omega_i)_{0\leq i\leq N+1}$ is another partition of $\Omega$ with $\Omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, that we will call dual partition of $(I_i)_{0\leq i\leq N}$.

We define $V_h := span\{\phi_i\}_{i=0}^{N+1} \cap H^1_{0,w}(\Omega)$ as the set of continuous piecewise with respect to the partition $\{I_i\}_{i=0}^{N}$ such that $v_h = \sum_{i=0}^{N+1} v_h(x_i)\phi_{x_i}$, $\quad v_h \in V_h$ with $\phi_{x_i}(x_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker symbol. The interpolation operator $I_h : C(\overline{\Omega}) \longrightarrow V_h$ is defined by

$$I_h v(x_i) := v(x_i);\;\; i = 0, 1, \ldots, N+1 \;\text{ then }\; I_h v = v_h = \sum_{i=0}^{N+1} v(x_i)\phi_{x_i}.$$

For the purpose of errors analysis we define the following appropriate norms and semi-norms on $V_h$ as

$$\|v_h\|_{0,h} := \sqrt{(v_h, v_h)_h} = \left(\sum_{i=0}^{N+1} l_i v_h(x_i)^2\right)^{1/2}, \tag{22}$$

for the discrete $L^2(\Omega)$ norm, and

$$\|v_h\|_{1,\mathcal{T}} := \left(\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}(v_{i+1} - v_i)^2\right)^{1/2}. \tag{23}$$

For weighted discrete $H^1-$ semi-norm, and

$$\|v_h\|^2_{1,h} = \|v_h\|^2_{1,\mathcal{T}} + \|v_h\|^2_{0,h}. \tag{24}$$

for the weighted discrete $H^1_{0,w}(\Omega)-$norm on $V_h$. Indeed it is easy to show that $||.||_{1,\mathcal{T}}$ is a semi-norm in $V_h$ since $\frac{x^2_{i+\frac{1}{2}}}{h_i} > 0$.

### 3.1. Discrete inner products

We define discrete analogs of the two continuous inner products (16) and (17) which effectively is selecting a quadrature rule on each cell to approximate the integrals. Let $HC$ and $\mathbf{HC}$ be the sets of functions of $H = L^2(\Omega)$ or $\mathbf{H} = \left(L^2(\Omega)\right)^n$ $n = 1$ respectively, constants in the partition $\mathcal{T}$ ($(I_i)$ for $HC$ and $(\Omega_i)$ for $\mathbf{HC}$). The discrete $L^2(\Omega)$ norm defined in $HC$ is given by

$$(U, V)_{HC} = \sum_{i=0}^{N+1} U_i V_i l_i \qquad U, V \in HC. \tag{25}$$

Since the discrete information for fluxes are located at the cell centers, we adopt the midpoint rule for the inner product (17). Hence the discrete $\left(L^2(\Omega)\right)^n$, $n = 1$ norm in $\mathbf{HC}$ is defined as

$$(W, Z)_{\mathbf{HC}} = \sum_{i=0}^{N+1} \frac{W_{i+1/2} Z_{i+1/2}}{K_{i+\frac{1}{2}}} h_i, \quad W, Z \in \mathbf{HC}, \tag{26}$$

where $K_{i+\frac{1}{2}}$ is defined before at the cell $i + \frac{1}{2}$. For $K_{i+\frac{1}{2}} \neq 0$, the discrete inner product (26) is well defined.

Note that in $HC$ and $\mathbf{HC}$, we also have the following standard inner products

$$[U, V]_{HC} = \sum_{i=0}^{N+1} U_i V_i, \qquad U, V \in HC, \tag{27}$$

and

$$[W, Z]_{\mathbf{HC}} = \sum_{i=0}^{N+1} W_{i+1/2} Z_{i+1/2} h_i. \tag{28}$$

Note that in $HC$ and $\mathbf{HC}$ the two inner products are linked by

$$(U, V)_{HC} = [\mathcal{M}U, V]_{HC}, \qquad (W, Z)_{\mathbf{HC}} = [\mathcal{S}W, Z]_{\mathbf{HC}}, \tag{29}$$

where $\mathcal{M}$ and $\mathcal{S}$ are coefficients. More details can be found in [5,6].

[2] Function evaluation at $x_{-\frac{1}{2}}$ or $x_{N+\frac{3}{2}}$ is understood as evaluation at $x_0 = 0$ or at $x_{N+1} = S_{\max}$

### 3.2. The discrete divergence and discrete flux

In this section, we discretize the divergence operator **D**. We denote by $\mathcal{D}$ the discrete divergence operator. On the interior of the domain $\mathcal{D}$ is

$$(\mathcal{D}W)_i = \frac{W_{i+1/2} - W_{i-1/2}}{l_i}, \;\; i = 0, 1, \ldots, N+1, \tag{30}$$

We can easily check that

$$(\mathcal{D}W, 1)_{HC} = 0, \tag{31}$$

which is the divergence property of the discrete divergence $\mathcal{D}$, which mimic the continuous divergence. For $n = 1$, $\mathbf{G} = -K\frac{d}{dx}$. Let us determine the discrete version of **G** denoted by $\mathcal{G}$ that mimic the continuous version properties as we have already mentioned. Indeed $\mathcal{G}$ should fulfil the following propriety

$$(\mathcal{D}W, U)_{HC} = (W, \mathcal{G}U)_{\mathbf{HC}}. \tag{32}$$

We then expand (32) as below

$$\left(\sum_{i=0}^{N+1} U_i(\mathcal{D}W)_i\right) l_i = \sum_{i=0}^{N+1} \frac{W_{i+1/2}(\mathcal{G}U)_{i+1/2}}{K_{i+\frac{1}{2}}} h_i. \tag{33}$$

and further expanding (33) we have that

$$\sum_{i=0}^{N+1}\left(U_i - \left[\frac{h_i}{K_{i+\frac{1}{2}}}\right](\mathcal{G}U)_{i+\frac{1}{2}}\right) W_{i+\frac{1}{2}} - \sum_{i=0}^{N+1} U_i W_{i-\frac{1}{2}} = 0$$

so

$$\sum_{i=0}^{N+1}\left(U_i - \left[\frac{h_i}{K_{i+\frac{1}{2}}}\right](\mathcal{G}U)_{i+\frac{1}{2}}\right) W_{i+\frac{1}{2}} - \sum_{i=1}^{N+1} U_i W_{i-\frac{1}{2}} - U_0 W_{-1/2} = 0 \tag{34}$$

Now, re-indexing any terms with $i - \frac{1}{2}$ to $i + \frac{1}{2}$, using the fact that $h_{N+1} = 0$, we have

$$\sum_{i=0}^{N}\left(-(U_{i+1} - U_i) - (\mathcal{G}U)_{i+\frac{1}{2}}\left[\frac{h_i}{K_{i+\frac{1}{2}}}\right]\right) W_{i+\frac{1}{2}} + U_{N+1} W_{N+\frac{3}{2}} - U_0 W_{-1/2} = 0. \tag{35}$$

Now, this is done to fully concentrate the fluxes at the $i^{th}-$node, to enhance the mimicking property at each $i^{th}-$node. We have that (35) holds for all $U$ in $HC$ such that $U_0 = U_{N+1} = 0$. Hence we can determine $\mathcal{G}$ by solving for $(\mathcal{G}U)_{i+1/2}$ which gives

$$(\mathcal{G}U)_{i+1/2} = -\left(\left[\frac{h_i}{K_{i+\frac{1}{2}}}\right]\right)^{-1} (U_{i+1} - U_i), \quad i = 0, \ldots, N. \tag{36}$$

Let $\mathcal{A}_h$ denote the discrete diffusion operator obtained by forming the composition of the discrete divergence and gradient operator $\mathcal{D}$ and $\mathcal{G}$ respectively. By construction, $\mathcal{D} : \mathbf{HC} \to HC$ and $\mathcal{G} : HC \to \mathbf{HC}$ is given by $\mathcal{A}_h : HC \to HC$ with $\mathcal{A}_h = \mathcal{D}\mathcal{G}$.

**Remark 2.** As you can observe, the mimetic finite difference method depends of the discrete inner products in (25) and (26). These discrete inner products are indeed the approximations of the integrals of continuous functions on the grids. Here we have used the rectangle rule, trapezoidal rule or any other method can be also used. Of course, the accuracy of the mimetic method depends of the these discrete inner products.

### 3.3. Mimetic finite difference scheme for Black Scholes PDE

The goal here is to discretize the Black Scholes PDE (5). The mimetic finite method will be used for diffusion part while first order upwind-finite difference scheme for the convection term.

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \mathbf{D}w - \dfrac{\partial}{\partial s}[bsu] + cu = f(t) \\ \mathbf{G}u := w = -K\dfrac{\partial u}{\partial s}, \\ \mathbf{A} = \mathbf{DG}, \quad K = as^2. \end{cases} \tag{37}$$

We partition $I := (0, S_{\max})$ into $N+1$ sub-intervals as we did previously for elliptic problems with dual partition, then we have that,

$$U_i(t) \approx U(s_i, t), \; h_i = s_{i+1} - s_i, \; i = 0, 1, \ldots, N.$$

Set $w_{i+\frac{1}{2}} := \frac{w_{i+1}+w_i}{2}$, $w_{i-\frac{1}{2}} := \frac{w_i+w_{i-1}}{2}$ we can easily see that $w_{i+\frac{1}{2}} \approx w(s_{i+\frac{1}{2}}, t), \quad i = 0, 1, ., N$. Then the discrete mimetic operators (prime and derived) are given by

$$(\mathcal{D}w)_i = \frac{w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}}{l_i}, \quad \text{for } i = 0, \dots, N+1. \tag{38}$$

and

$$(\mathcal{G}U)_{i-1/2} = -\left(\left[\frac{h_{i-1}}{K_{i-\frac{1}{2}}}\right]\right)^{-1}(U_i - U_{i-1}),\ i = 1, \dots, N+1 \tag{39}$$

Now, we have that the discrete operator $\widehat{\mathcal{A}}_h$ is given by

$$\widehat{\mathcal{A}}_h U_h[i] = (\mathcal{D}\mathcal{G})U_h[i] = \left(\frac{-\left[\frac{K_{i+\frac{1}{2}}}{h_i}\right](U_{i+1} - U_i) + \left[\frac{K_{i-\frac{1}{2}}}{h_{i-1}}\right](U_i - U_{i-1})}{l_i}\right),\ i = 1, \dots, N, \tag{40}$$

or

$$\widehat{\mathcal{A}}_h U_h[i] = \alpha_i U_{i+1} + \beta_i U_i + \gamma_i U_{i-1},\ i = 1, \dots, N, \tag{41}$$

where

$$\alpha_i = \left[-\frac{K_{i+\frac{1}{2}}}{h_i l_i}\right], \quad \beta_i = \left[\frac{K_{i+\frac{1}{2}}}{h_i l_i} + \frac{K_{i-\frac{1}{2}}}{l_i h_{i-1}}\right], \quad \gamma_i = -\left[\frac{K_{i-\frac{1}{2}}}{l_i h_{i-1}}\right]. \tag{42}$$

Applying the first order upwind finite difference for the convective term in sense of finite volume method yields

$$-\frac{\partial}{\partial s}(bsu) = \frac{\partial}{\partial s}(-sbu) = \frac{-bs_{i+1/2}U_{i+1} + bs_{i-1/2}U_i}{h_i}, \tag{43}$$

then

$$\widehat{\mathcal{B}}_h U_h[i] = \Gamma_i U_{i+1} + \Upsilon_i U_i, \tag{44}$$

where

$$\Gamma_i = \frac{-bs_{i+1/2}}{h_i}, \quad \Upsilon_i = \frac{bs_{i-1/2}}{h_i};\ i = 1, \dots, N \tag{45}$$

Now from (41) and (44), we set

$$\widehat{\mathcal{C}}_h U_h[i] = \widehat{\mathcal{A}}_h U_h[i] + \widehat{\mathcal{B}}_h U_h[i] = (\alpha_i + \Gamma_i)U_{i+1} + (\beta_i + \Upsilon_i)U_i + \gamma_i U_{i-1},\ i = 1, \dots, N. \tag{46}$$

The semi - discrete of (37) is given by

$$\begin{cases} -\frac{dU_h}{dt} + \widehat{\mathcal{C}}_h U_h = f_h(t), & t \in [0, T], \\ U_h(T) = U_h^* \end{cases} \tag{47}$$

### *3.4. Fitted mimetic finite difference scheme*

The Black-Scholes differential operator is known to be degenerate towards the boundary and hence special techniques are required to handle the degeneracy [2,3,8,12]. In [2,12], the authors proposed a so-called fitted scheme to tackle the degeneracy of the PDE. In this section near $s = 0$ ($i = 1$), the sum of diffusion and convective flux is approximated using the fitted scheme and far from $s = 0$ ($i > 1$), the diffusion flux and the convective flux will be approximated as in the previous section using respectively the standard mimetic finite difference and the upwind finite difference schemes. This combination will yield our novel scheme called Fitted Mimetic Finite Difference Scheme. As the case ($i > 1$) is already covered in the previous section, we will only focus on the case ($i = 1$). Indeed we need to approximate the flux at $s_{1/2}$ with fitted finite volume method to handle the degeneracy of the Black-Scholes differential operator. Note that to find a new approximation at $(DW)_1$, we require the fluxes at $s_{\frac{1}{2}}$ and $s_{\frac{3}{2}}$, i.e. $(\mathcal{G}U)_{\frac{1}{2}}$ and $(\mathcal{G}U)_{\frac{3}{2}}$ respectively. Let us now integrate the Black Sholes PDE (37) across $s_{\frac{1}{2}}$ and $s_{\frac{3}{2}}$,

$$-\int_{s_{\frac{1}{2}}}^{s_{\frac{3}{2}}} \frac{\partial u}{\partial t} ds - \int_{s_{\frac{1}{2}}}^{s_{\frac{3}{2}}} \frac{\partial}{\partial s}\left[as^2\frac{\partial u}{\partial s} + bsu\right] ds + \int_{s_{\frac{1}{2}}}^{s_{\frac{3}{2}}} (cu - f(s,t))ds = 0, \tag{48}$$

and using the midpoint rule, we have that

$$-l_1\frac{dU_1}{dt} - \left[as^2\frac{\partial u}{\partial s} + bsu\right]_{s_{1/2}}^{s_{3/2}} + (cU_1 - f(s,t))l_1 = 0, \tag{49}$$

which gives

$$-\frac{dU_1}{dt}-\frac{1}{l_1}\left[as^2\frac{\partial u}{\partial s}+bsu\right]_{s_{1/2}}^{s_{3/2}}+(cU_1-f(s,t))=0. \tag{50}$$

Then from (50) we have that

$$-\frac{dU_1}{dt}-\frac{1}{l_1}\left[s_{3/2}\Phi(u)|_{s_{3/2}}-s_{1/2}\Phi(u)|_{s_{1/2}}\right]+(cU_1-f(s,t))=0. \tag{51}$$

where $\Phi(u)$ is defined by

$$\Phi(u):=as\frac{\partial u}{\partial s}+bu. \tag{52}$$

Note that the problem is not at $s_{3/2}$ and using (39), $s_{3/2}\Phi(u)|_{s_{3/2}}$ can be approximated as

$$\begin{aligned}s_{3/2}\Phi(u)|_{s_{3/2}} &\approx (-\mathcal{G}U)_{3/2}+bs_{3/2}U_1\\ &=\left[\frac{K_{3/2}}{h_1}\right]U_2+\left[bs_{3/2}-\frac{K_{3/2}}{h_1}\right]U_1.\end{aligned}$$

Let us approximate $s_{1/2}\Phi(u)|_{s_{1/2}}$ using the fitted technique. As in [2,3], we consider the following two-point boundary value problem

$$(asv'+bv)'=C_1,\quad s\in(0,s_1) \tag{53}$$

$$v(0)=U_0,\quad v(s_1)=U_1, \tag{54}$$

where $C_1$ is an unknown constant to be determined. Integrating (53) once, we have that

$$asv'+bv=C_1s+C_2$$

Now, using the condition $v(0)=U_0$, we find that $C_2=bU_0$ and hence

$$\Phi(v)=asv'+bv=C_1s+bU_0. \tag{55}$$

Following [2,12], we have

$$(\Phi(v))|_{s_{1/2}}=\left(asv'+bv\right)_{s_{1/2}}=\frac{1}{2}[(a+b)U_1-(a-b)U_0], \tag{56}$$

Then (55) reduces to

$$v=U_0+(U_1-U_0)s/s_1,\quad s\in[0,s_1]. \tag{57}$$

Remember that $-\left[\frac{\partial}{\partial s}\left(as^2\frac{\partial u}{\partial s}+bsu\right)-cu\right]=-\mathbf{D}\Phi(u)+cu$, so the following approximation can be used where the divergence operator $\mathbf{D}$ is approximated by $\mathcal{D}$

$$\begin{aligned}-\left[\frac{\partial}{\partial s}\left(as^2\frac{\partial u}{\partial s}+bsu\right)-cu\right]\Big|_{s_1} &\approx -\frac{s_{3/2}(\Phi(u))|_{s_{3/2}}-s_{1/2}(\Phi(u))|_{s_{1/2}}}{l_1}+cU_1\\ &=-\frac{\left[\frac{K_{3/2}}{h_1}\right]U_2+\left[bs_{3/2}-\frac{K_{3/2}}{h_1}\right]U_1-\frac{s_{1/2}}{2}[(a+b)U_1-(a-b)U_0]}{l_1}+cU_1\\ &=-\left[\frac{K_{3/2}}{h_1l_1}\right]U_2+\left[\frac{K_{3/2}}{h_1l_1}+\frac{as_{1/2}}{2l_1}+\frac{b}{l_1}\left(s_{3/2}+\frac{s_{1/2}}{2}\right)+c\right]U_1-\frac{(a-b)s_{1/2}}{2l_1}U_0.\end{aligned} \tag{58}$$

Combining with the mimetic finite difference approximation at $s_i$, $i>1$ yields our novel scheme called the fitted mimetic finite difference method. Let us set $U^h=(U_1,U_2,\dots,U_N)$, we therefore need to solve in the case of fitted mimetic finite the following system

$$\begin{cases}-\dfrac{dU^h}{dt}+\mathcal{C}^hU^h=f_h^1(t), & t\in[0,T],\\ U^h(T)=U^{h*},\end{cases} \tag{59}$$

where with (46), we have

$$\begin{cases}\mathcal{C}^hU^h[1]=-\left[\dfrac{K_{3/2}}{h_1l_1}\right]U_2+\left[\dfrac{K_{3/2}}{h_1l_1}+\dfrac{as_{1/2}}{2l_1}+\dfrac{b}{l_1}\left(s_{3/2}+\dfrac{s_{1/2}}{2}\right)+c\right]U_1,\\ \mathcal{C}^hU^h[i]=\widehat{\mathcal{C}}^hU^h[i]\quad i>1,\end{cases} \tag{60}$$

### 3.5. Discrete representation of the exact solution on mimetic grid

Consider the mimetic grids as we have defined in our previous section. The goal is to provide the discrete representation of the exact solution of (5) on $\mathcal{T}$, useful in our errors analysis. Indeed remember that

$$-\frac{\partial u(s,t)}{\partial t} - \frac{\partial}{\partial s}\left(as^2\frac{\partial u(s,t)}{\partial s} + bsu(s,t)\right) + cu(s,t) = f(s,t). \tag{61}$$

Then in terms of the continuous mimetic operators, we have that

$$-\frac{\partial u(s,t)}{\partial t} + \mathbf{D}(\mathbf{G}u(x,t) - bsu(s,t)) + cu(s,t) = f(s,t). \tag{62}$$

Let us denote

$$\Phi(u(s,t),s,t) = -\mathbf{G}u(s,t) + bsu(s,t) = \frac{\partial}{\partial s}(as^2u) + bsu(s,t), \tag{63}$$

so

$$-\frac{\partial u(s,t)}{\partial t} - \mathbf{D}(\Phi(u,t)) + cu(s,t) = f(s,t). \tag{64}$$

Integrating (62) within the control volume $\Omega = (s_{i-1/2}, s_{i+1/2})$ yields

$$-\dot{u}(s_i,t)l_i - \left[\Phi(u(s_{i+\frac{1}{2}},t),s_{i+\frac{1}{2}},t) - \Phi(u(s_{i-\frac{1}{2}},t),s_{i-\frac{1}{2}},t)\right] + cl_iu(s_i,t) = f(s_i,t)l_i. \tag{65}$$

Then, multiplying (65) with an arbitrary real number $v_i$, and summing the results, we get

$$-\sum_{i=1}^{N}\dot{u}(s_i,t)l_iv_i - \sum_{i=1}^{N}\left[\Phi(u(s_{i+\frac{1}{2}},t),s_{i+\frac{1}{2}},t) - \Phi(u(s_{i-\frac{1}{2}},t),s_{i-\frac{1}{2}},t)\right]v_i + \sum_{i=1}^{N}cl_iu(s_i,t)v_i = \sum_{i=1}^{N}f(s_i,t)l_iv_i. \tag{66}$$

Let us define, for any $v \in C(\overline{\Omega})$, a lumping operator $\mathcal{P}_h : C(\overline{\Omega}) \to L^\infty(\Omega)$ by

$$\mathcal{P}_hv|_{\Omega_i} := v(s_i), i = 1,\dots,N.$$

If $v$ satisfies the homogeneous Dirichlet boundary conditions, we have $\mathcal{P}_hv|_{\Omega_0} = \mathcal{P}_hv|_{\Omega_{N+1}} = 0$. Hence using the lumping operator we have

$$(-\dot{u}(t),v)_h - \mathcal{A}(u(t),v;t) = (f(t),v)_h, \qquad \forall v \in C(\overline{\Omega}),\ \text{with } v(0) = v(S_{max}) = 0, \tag{67}$$

where

$$\mathcal{A}(u(t),v;t) := \sum_{i=1}^{N}\left[\Phi(u(s_{i+\frac{1}{2}},t),s_{i+\frac{1}{2}},t) - \Phi(u(s_{i-\frac{1}{2}},t),s_{i-\frac{1}{2}},t)\right]\mathcal{P}_hv(s_i) + (cu(t),v)_h, \tag{68}$$

and

$$(w,v)_h = (\mathcal{P}_hw,\mathcal{P}_hv) = \sum_{i=1}^{N}w_iv_il_i, \quad w,v \in C(\overline{\Omega}).$$

To be more precise, (67) is the representation of lumping operator applied to the exact solution

#### 3.5.1. Variational form of mimetic method

In this section, we will establish the key result which implies the unique solvabilty of the mimetic method in the discrete operators. From (65) using the mimetic approximation yields

$$\begin{cases} -\dfrac{dU_i}{dt}l_i - (\mathcal{D}[\Phi_h(U_h)])_i + cU_i(t)l_i = f_i(t)l_i, & i = 0,1,\dots,N+1, \\ u(s_i) \approx U_i, \end{cases} \tag{69}$$

where

$$\Phi_h(U_h,t)|_{s_{i+1/2}} \approx -(\mathcal{G}U)_{i+1/2} + bs_{i+1/2}U_{i+1} = \left[\frac{K_{i+1/2}}{h_i}\right](U_{i+1} - U_i) + bs_{i+1/2}U_{i+1}, \tag{70}$$

$$\Phi_h(U_h,t)|_{s_{i-1/2}} \approx -(\mathcal{G}U)_{i-1/2} + bs_{i-1/2}U_i = \left[\frac{K_{i-1/2}}{h_{i-1}}\right](U_i - U_{i-1}) + bs_{i-1/2}U_i. \tag{71}$$

Then multiplying (69) by arbitrary numbers $v_i$ and summing over $\Omega_i$, we have

$$-\sum_{i=1}^{N} l_i \frac{dU_i}{dt} v_i - \sum_{i=1}^{N} \Big[ (\mathcal{G}U)_{i+\frac{1}{2}} - (\mathcal{G}U)_{i-\frac{1}{2}} + bs_{i+1/2}U_{i+1} - bs_{i-1/2}U_{i-1} \Big] v_i + \sum_{i=1}^{N} l_i (cU_i) v_i = \sum_{i=1}^{N} l_i f_i(t) v_i. \tag{72}$$

We write (72) in variational form as

$$-(\dot{U}_h, v_h)_h + \mathbf{a_h}(U_h, v_h; t) = (f(t), v_h)_h, \qquad \forall \ U_h, v_h \in V_h. \tag{73}$$

where

$$\mathbf{a_h}(U_h(t), v_h; t) := \sum_{i=1}^{N} [\Phi_h(U_h, s, t)]_{s_{i-\frac{1}{2}}}^{s_{i+\frac{1}{2}}} \mathcal{P}_h v_h(s_i) + (cu(t), v_h)_h, \tag{74}$$

$$[\Phi_h(U_h, s, t)]_{s_{i-\frac{1}{2}}}^{s_{i+\frac{1}{2}}} = \Phi_h(U_h, t)|_{s_{i+1/2}} - \Phi_h(U_h, t)|_{s_{i-1/2}}. \tag{75}$$

In this sequel of this work, we will used the following mesh regularity

**Assumption 1.** We assume that there exists a constant $c_0 > 0$ such that

$$c_0^{-1} h_{i+1} \le h_i \le c_0 h_{i+1}, \ \ i = 0, 1, \ldots, N. \tag{76}$$

Furthermore, as the paper is long, we assume without loss of generality that $b = r - \sigma^2 \geq 0$[3].

Note that condition (76) in Assumption 1 implies that

$$c_0^{-1} l_{i+1} \le l_i \le c_0 l_{i+1}, \ \ i = 0, 1, \ldots, N. \tag{77}$$

**Theorem 2.** *Under the regularity of the mesh Assumption 1, there exists a constant $C > 0$ independent of $h$ such that, for all $v_h \in V_h$, we have*

$$\mathbf{a_h}(v_h, v_h) = \mathbf{b_h^1}(v_h, v_h) + \mathbf{b_h^2}(v_h, v_h) \ge C \|v_h\|_{1,h}^2 \tag{78}$$

*where*

$$\mathbf{b_h^1}(v_h, v_h) = -\sum_{i=1}^{N} \Big[ (\mathcal{G}v)_{i+\frac{1}{2}} - (\mathcal{G}v)_{i-\frac{1}{2}} \Big] v_i \tag{79}$$

$$\mathbf{b_h^2}(v_h, v_h) = -\sum_{i=1}^{N} \Big[ bs_{i+1/2} v_{i+1} - bs_{i-\frac{1}{2}} v_i \Big] v_i + \sum_{i=1}^{N} l_i (c v_i) v_i. \tag{80}$$

**Proof.** Throughout the proof, $C$ will represent a positive constant independent of $h$ which may change from line to line. We firstly prove that

$$\mathbf{b_h^1}(v_h, v_h) \ge C \|v_h\|_{1,\mathcal{T}} \tag{81}$$

Then from (79), we have that,

$$\begin{aligned}
\mathbf{b_h^1}(v_h, v_h) &= -\sum_{i=1}^{N} \Big[ (\mathcal{G}v)_{i+\frac{1}{2}} - (\mathcal{G}v)_{i-\frac{1}{2}} \Big] v_i \\
&= -\sum_{i=1}^{N} \left[ \frac{a s_{i+\frac{1}{2}}^2}{h_i} (v_{i+1} - v_i) - \frac{a s_{i-\frac{1}{2}}^2}{h_{i-1}} (v_i - v_{i-1}) \right] v_i \\
&= \sum_{i=1}^{N} \left[ -\frac{a s_{i+\frac{1}{2}}^2}{h_i} v_{i+1} v_i + \left( \frac{a s_{i+\frac{1}{2}}^2}{h_i} + \frac{a s_{i-\frac{1}{2}}^2}{h_{i-1}} \right) v_i^2 - \frac{a s_{i-\frac{1}{2}}^2}{h_{i-1}} v_{i-1} v_i \right].
\end{aligned}$$

Remember that $v_0 = v_{N+1} = 0$. Now we use the following expansions

$$-v_{i+1} v_i = \frac{1}{2} \big( (v_{i+1} - v_i)^2 - v_{i+1}^2 - v_i^2 \big) \tag{82}$$

[3] Indeed we can easily generalise following [18].

and

$$-v_i v_{i-1} = \frac{1}{2}\left((v_i - v_{i-1})^2 - v_i^2 - v_{i-1}^2\right) \tag{83}$$

$$\begin{aligned}
\mathbf{b_h^1}(v_h, v_h) &= \sum_{i=1}^{N}\left[\frac{as^2_{i+\frac{1}{2}}}{2h_i}\left((v_{i+1}-v_i)^2 - v_{i+1}^2 - v_i^2\right) + \left(\frac{as^2_{i+\frac{1}{2}}}{h_i} + \frac{as^2_{i-\frac{1}{2}}}{h_{i-1}}\right)v_i^2\right] \\
&\quad + \sum_{i=1}^{N}\left[\frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}\left((v_i - v_{i-1})^2 - v_i^2 - v_{i-1}^2\right)\right] \\
&= \sum_{i=1}^{N}\left[\frac{as^2_{i+\frac{1}{2}}}{2h_i}(v_{i+1}-v_i)^2 + \frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}(v_i - v_{i-1})^2 + \left(\frac{as^2_{i+\frac{1}{2}}}{2h_i} + \frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}\right)v_i^2\right] \\
&\quad + \sum_{i=1}^{N}\left[-\frac{as^2_{i+\frac{1}{2}}}{2h_i}v_{i+1}^2 - \frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}v_{i-1}^2\right].
\end{aligned}$$

Now we consider the following expansions

$$-\frac{a}{2}\sum_{i=1}^{N}\frac{s^2_{i-\frac{1}{2}}}{h_{i-1}}v_{i-1}^2 = -\sum_{i=1}^{N}\frac{as^2_{i+\frac{1}{2}}}{2h_i}v_i^2 + \frac{as^2_{N+\frac{1}{2}}}{2h_N}v_N^2, \tag{84}$$

and

$$-\frac{a}{2}\sum_{i=1}^{N}\frac{s^2_{i+\frac{1}{2}}}{h_i}v_{i+1}^2 = -\sum_{i=1}^{N}\frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}v_i^2 + \frac{as^2_{\frac{1}{2}}}{2h_0}v_1^2. \tag{85}$$

Then substituting the above expansions, we have that

$$\begin{aligned}
\mathbf{b_h^1}(v_h, v_h) &= \sum_{i=1}^{N}\left[\frac{as^2_{i+\frac{1}{2}}}{2h_i}(v_{i+1}-v_i)^2 + \frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}(v_i - v_{i-1})^2 + \left(\frac{as^2_{i+\frac{1}{2}}}{2h_i} + \frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}\right)v_i^2\right] \\
&\quad - \sum_{i=1}^{N}\frac{as^2_{i-\frac{1}{2}}}{2h_{i-1}}v_i^2 + \frac{as^2_{\frac{1}{2}}}{2h_0}v_1^2 - \sum_{i=1}^{N}\frac{as^2_{i+\frac{1}{2}}}{2h_i}v_i^2 + \frac{as^2_{N+\frac{1}{2}}}{2h_N}v_N^2 \\
&= \sum_{i=1}^{N}\left[\frac{as^2_{i+\frac{1}{2}}}{2h_{i+\frac{1}{2}}}(v_{i+1}-v_i)^2 + \frac{as^2_{i-\frac{1}{2}}}{2h_{i-\frac{1}{2}}}(v_i - v_{i-1})^2\right] + \frac{as^2_{\frac{1}{2}}}{2h_0}v_1^2 + \frac{as^2_{N+\frac{1}{2}}}{2h_N}v_N^2 \\
&\geq C\|v_h\|^2_{1,\mathcal{T}}.
\end{aligned}$$

Now we need to also prove that,

$$\mathbf{b_h^2}(v_h, v_h) \geq C\|v_h\|^2_{0,h}. \tag{86}$$

Now from (80), we have that,

$$\begin{aligned}
\mathbf{b_h^2}(v_h, v_h) &= -\sum_{i=1}^{N}\left[bs_{i+1/2}v_{i+1} - bs_{i-\frac{1}{2}}v_i\right]v_i + \sum_{i=1}^{N}l_i(cv_i)v_i \\
&= -b\sum_{i=1}^{N}\left[s_{i+1/2}v_{i+1} - s_{i-\frac{1}{2}}v_i\right]v_i, + \sum_{i=1}^{N}l_i(cv_i)v_i. \\
&= -b\sum_{i=1}^{N}s_{i+1/2}v_{i+1}v_i + b\sum_{i=1}^{N}s_{i-1/2}v_i^2 + \sum_{i=1}^{N}l_i(cv_i)v_i.
\end{aligned}$$

Now by using the following substitution

$$-(v_{i+1}v_i) = \frac{1}{2}\left[(v_{i+1}-v_i)^2 - v_{i+1}^2 - v_i^2\right],$$

we have that,

$$\mathbf{b_h^2}(v_h, v_h) = \sum_{i=1}^{N}\left[\frac{bs_{i+\frac{1}{2}}}{2}\left((v_{i+1}-v_i)^2 - v_{i+1}^2 - v_i^2\right)\right] + b\sum_{i=1}^{N}s_{i-\frac{1}{2}}v_i^2 + \sum_{i=1}^{N}cl_iv_i^2$$

$$= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] - \sum_{i=1}^{N}\frac{bs_{i+\frac{1}{2}}}{2}\left[v_{i+1}^2+v_i^2\right] + b\sum_{i=1}^{N}s_{i-\frac{1}{2}}v_i^2 + \sum_{i=1}^{N}cl_iv_i^2$$

Now let us consider the following expansions,

$$-\sum_{i=1}^{N}\frac{bs_{i+\frac{1}{2}}}{2}v_{i+1}^2 = -\frac{b}{2}\left[\sum_{i=1}^{N}s_{i-\frac{1}{2}}v_i^2\right] + \frac{b}{2}s_{\frac{1}{2}}v_1^2. \tag{87}$$

Then

$$\begin{aligned}\mathbf{b_h^2}(v_h,v_h) &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] - \sum_{i=1}^{N}\frac{bs_{i+\frac{1}{2}}}{2}\left[v_{i+1}^2+v_i^2\right] + b\sum_{i=1}^{N}s_{i-\frac{1}{2}}v_i^2 + \sum_{i=1}^{N}cl_iv_i^2,\\ &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] - \sum_{i=1}^{N}\frac{bs_{i+\frac{1}{2}}}{2}v_i^2 + \frac{b}{2}s_{\frac{1}{2}}v_1^2 + \sum_{i=1}^{N}\frac{bs_{i-\frac{1}{2}}}{2}v_i^2 + \sum_{i=1}^{N}cl_iv_i^2\\ &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] + \sum_{i=1}^{N}\frac{b}{2}\left(s_{i-\frac{1}{2}}-s_{i+\frac{1}{2}}\right)v_i^2 + \frac{b}{2}s_{\frac{1}{2}}v_1^2 + \sum_{i=1}^{N}cl_iv_i^2\end{aligned}$$

Remember that $b = r-\sigma^2 \geq 0,\ c = r+b = 2r-\beta,\ \beta := \sigma^2$. So we have

$$\begin{aligned}\mathbf{b_h^2}(v_h,v_h) &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] - \sum_{i=1}^{N}\frac{b}{2}l_iv_i^2 + \frac{b}{2}s_{\frac{1}{2}}v_1^2 + \sum_{i=1}^{N}cl_iv_i^2\\ &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] - \sum_{i=1}^{N}\left(\frac{r-\sigma^2}{2}\right)l_iv_i^2 + \frac{b}{2}s_{\frac{1}{2}}v_1^2 + \sum_{i=1}^{N}(2r-\sigma^2)l_iv_i^2\\ &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1}-v_i)^2\right] + \sum_{i=1}^{N}\left(\frac{3r-\sigma^2}{2}\right)l_iv_i^2 + \frac{bs_{\frac{1}{2}}}{2}v_1^2\\ &\geq C\|v_h\|_{0,h}^2.\end{aligned}$$

That is,

$$\mathbf{b_h^2}(v_h,v_h) \geq C\|v_h\|_{0,h}^2. \tag{88}$$

Then from (78), we have that

$$\mathbf{a_h}(v_h,v_h) = \mathbf{b_h^1}(v_h,v_h) + \mathbf{b_h^2}(v_h,v_h) \geq C_1\|v_h\|_{1,\mathcal{T}}^2 + C_2\|v_h\|_{0,h}^2 = C\|v_h\|_{1,h}^2. \tag{89}$$

□

### 3.5.2. Coercivity of fitted mimetic method

Following similar arguments as in the previous subsection, we will prove the coercivity of the fitted scheme. Now we have that,

$$-(\dot{U}^h,v_h)_h + \mathbf{A_h}(U^h,v_h) = (f(t),v_h)_h,\ \forall v_h \in V_h \subset H_{0,w}^1(\Omega) \tag{90}$$

Note that the difference between the standard mimetic scheme and the fitted mimetic scheme is that the convection flux plus the diffusion flux is approximated at $s_{1/2}$ as

$$\begin{aligned}s_{1/2}\Phi(U_h)|_{s_{1/2}} &= \frac{s_{1/2}}{2}[(a+b)U_1-(a-b)U_0]\\ &= \frac{as_{1/2}}{2}[U_1-U_0] + \frac{bs_{1/2}}{2}[U_1+U_0]\\ &= -(\mathcal{G}U)_{\frac{1}{2}} + \frac{bs_{1/2}}{2}[U_1+U_0].\end{aligned} \tag{91}$$

So for the fitted mimetic scheme, we have

$$\mathbf{A_h}(U^h,v_h) = \mathbf{B_h^1}(U^h,v_h) + \mathbf{B_h^2}(U^h,v_h) \tag{92}$$

$$\mathbf{B_h^1}(U^h,v_h) = -\sum_{i=1}^{N}\left[(\mathcal{G}U)_{i+\frac{1}{2}} - (\mathcal{G}U)_{i-\frac{1}{2}}\right]v_i \ \text{ with } (\mathcal{G}U)_{\frac{1}{2}} = -\frac{as_{1/2}}{2}[U_1-U_0]. \tag{93}$$

$$\mathbf{B_h^2}(U^h, v_h) = -(bs_{3/2}U_2 - \frac{bs_{1/2}}{2}U_1)v_1 - \sum_{i=2}^{N}\left[bs_{i+1/2}U_{i+1} - bs_{i-\frac{1}{2}}U_i\right]v_i + \sum_{i=1}^{N} l_i(cU_i)v_i. \tag{94}$$

**Theorem 3.** *Under Assumption 1, there exists a positive constant $C > 0$, independent of $h$ , such that for all $v_h \in V_h$, we have*

$$\mathbf{A_h}(v_h, v_h) \geq C\|v_h\|_{1,h}^2. \tag{95}$$

**Proof.** Throughout the proof, $C$ will represent a positive constant independent of $h$ which may change from line to line. As for the standard mimetic, we firstly prove that

$$\mathbf{B_h^1}(v_h, v_h) \geq C\|v_h\|_{1,\mathcal{T}}. \tag{96}$$

Note that for fitted mimetic scheme $(\mathcal{G}U)_{\frac{1}{2}} = -\frac{as_{1/2}}{2}[U_1 - U_0]$. For standard mimetic scheme we have seen from (39) that

$$(\mathcal{G}U)_{\frac{1}{2}} = -\frac{as_{1/2}^2}{h_0}[U_1 - U_0] = -\frac{as_{1/2}^2}{2s_{1/2}}[U_1 - U_0] = -\frac{as_{1/2}}{2}[U_1 - U_0]. \tag{97}$$

Therefore the diffusion parts for fitted mimetic and standard mimetic are equal. We therefore have

$$\mathbf{B_h^1}(v_h, v_h) \geq C\|v_h\|_{1,\mathcal{T}}. \tag{98}$$

as for the standard mimetic scheme.

Now from (94), we have that,

$$\mathbf{B_h^2}(v_h, v_h) = -(bs_{3/2}v_2 - \frac{bs_{1/2}}{2}v_1)v_1 - \sum_{i=2}^{N}\left[bs_{i+1/2}v_{i+1} - bs_{i-\frac{1}{2}}v_i\right]v_i + \sum_{i=1}^{N} l_i(cv_i)v_i \tag{99}$$

$$= -\frac{bs_{1/2}}{2}v_1v_1 - \sum_{i=1}^{N}\left[bs_{i+1/2}v_{i+1} - bs_{i-\frac{1}{2}}v_i\right]v_i + \sum_{i=1}^{N} l_i(cv_i)v_i. \tag{100}$$

Note that comparing with the standard mimetic scheme, we only have the extra term

$$-\frac{bs_{1/2}}{2}v_1v_1 = -\frac{bs_{1/2}}{2}(v_1)^2$$

. Following line by line what we did for the standard mimetic method, we have

$$\begin{aligned}
\mathbf{B_h^2}(v_h, v_h) &= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1} - v_i)^2\right] - \sum_{i=1}^{N}\frac{bs_{i+\frac{1}{2}}}{2}v_i^2 + \sum_{i=1}^{N}\frac{bs_{i-\frac{1}{2}}}{2}v_i^2 + \sum_{i=1}^{N} cl_iv_i^2 - \frac{bs_{1/2}}{2}v_1v_1 \\
&= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1} - v_i)^2\right] + \sum_{i=1}^{N}\frac{b}{2}\left(s_{i-\frac{1}{2}} - s_{i+\frac{1}{2}}\right)v_i^2 + \sum_{i=1}^{N} cl_iv_i^2 - \frac{bs_{1/2}}{2}v_1^2 \\
&= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1} - v_i)^2\right] - \sum_{i=1}^{N}\frac{b}{2}l_iv_i^2 + \sum_{i=1}^{N} cl_iv_i^2 - \frac{bl_0}{2}v_1^2 \\
&\geq b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1} - v_i)^2\right] - \sum_{i=1}^{N}\frac{b}{2}l_iv_i^2 + \sum_{i=1}^{N} cl_iv_i^2 - \frac{bl_1}{2}v_1^2 \quad (\text{since } -l_0 \geq -l_1) \\
&= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1} - v_i)^2\right] + \sum_{i=2}^{N}\left(c - \frac{b}{2}\right)l_iv_i^2 + (c - b)l_1v_1^2 \\
&= b\sum_{i=1}^{N}\left[\frac{s_{i+\frac{1}{2}}}{2}(v_{i+1} - v_i)^2\right] + \sum_{i=1}^{N}\left(\frac{3r - \sigma^2}{2}\right)l_iv_i^2 + rl_1v_1^2 \\
&\geq C\|v_h\|_{0,h}^2, \quad (\text{since } r - \sigma^2 \geq 0)
\end{aligned}$$

That is,

$$\mathbf{B_h^2}(v_h, v_h) \geq C\|v_h\|_{0,h}^2. \tag{101}$$

Then we have that

$$\mathbf{A_h}(v_h, v_h) \geq C\|v_h\|_{1,h}^2. \tag{102}$$

□

**Remark 3.** Note that using the coercive properties in (78) and (95), with the fact that the linear mapping $v \to (f, v)_h$ and the two bilinear forms $a_h(.)$ and $\mathbf{A}_h(.)$ are continuous in $V_h$ and $V_h \times V_h$ respectively, the existence and uniqueness of the discrete solution $u_h$ is ensured for both the mimetic and fitted mimetic methods in (73) and (90). The proof is done exactly as for the continuous case (see [17, Theorem 1.33]).

*3.5.3. Consistency of the fluxes*

In this section, we prove the consistency of the fluxes. We can now state the following important theorem

**Theorem 4.** *Let $\Phi$ be the operator defined in (69) and $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be such that $\Phi'(w, \cdot, t) \in L^2(\Omega)$ for all $t \in (0, T)$. Let $\Phi_h$ be the approximation of $\Phi$ using mimetic finite difference method (see (70) and (71)) or fitted mimetic method (see (70) and (71)) with $\Phi_h(w_h(s_{\frac{1}{2}}), s_{\frac{1}{2}}, t) = \frac{s_{\frac{1}{2}}}{2}[(a+b)w_1 - (a-b)w_0]$. Then under Assumption 1, there exists a constant $C > 0$ independent of $w$ and $h$ such that the following estimate holds:*

$$|\Phi_h(w_h, s_{i+\frac{1}{2}}, t) - \Phi(w, s_{i+\frac{1}{2}}, t)| \leq C \int_{s_i}^{s_{i+1}} \left[|\Phi'(w, \cdot, t)| + |w'| + |w|\right] ds, \quad i = 0, 1, \dots, N, \tag{103}$$

*where $w_h = I_h w$.*

**Proof.** For the proof, we consider both Mimetic and Fitted-Mimetic cases. For the Mimetic case, the proof presented is for all $i = 0, 1, \dots, N$. In what follows, we consider the case 1 (mimetic). Remember from (69), that for $i = 0, 1, \dots N$, we have

$$-\frac{dU_i}{dt} l_i - \left[\Phi(U)|_{s_{i+1/2}} - \Phi(U)|_{s_{i-1/2}}\right] + (cU_i - f_i(t)) l_i = 0, \tag{104}$$

where

$$\Phi(w, s, t) := as^2 \frac{\partial w}{\partial s} + bsw \tag{105}$$

and

$$\Phi_h(w_h, s_{i+\frac{1}{2}}, t) := as_{i+\frac{1}{2}}^2 \frac{w_{i+1} - w_i}{h_i} + bs_{i+\frac{1}{2}} w_{i+1}. \tag{106}$$

Then from (105) and (106), we have

$$\begin{aligned}
\Phi_h(w_h, s_{i+\frac{1}{2}}, t) - \Phi(w, s_{i+\frac{1}{2}}, t) &= as_{i+\frac{1}{2}}^2 \frac{w_{i+1} - w_i}{h_i} + bs_{i+\frac{1}{2}} w_{i+1} - as_{i+\frac{1}{2}}^2 w'(s_{i+\frac{1}{2}}) - bs_{i+\frac{1}{2}} w(s_{i+\frac{1}{2}}) \\
&= as_{i+\frac{1}{2}}^2 \left[\frac{w_{i+1} - w_i}{h_i} - w'(s_{i+\frac{1}{2}})\right] + bs_{i+\frac{1}{2}} \left[w_{i+1} - w(s_{i+\frac{1}{2}})\right] \\
&=: A_1 + A_2.
\end{aligned}$$

Indeed using the Sobolev embedding theorem, as we are in dimension 1, $H^2(\Omega) \hookrightarrow C^1(\Omega)$, using the Taylor expansion with integral remainder, we have

$$w(z) = w(s_{i+\frac{1}{2}}) + (z - s_{i+\frac{1}{2}}) w'(s_{i+\frac{1}{2}}) + \int_{s_{i+\frac{1}{2}}}^{z} (z - s) w''(s) ds. \tag{107}$$

Then when we set $z = s_{i+1}$, we have that

$$w(s_{i+1}) = w(s_{i+\frac{1}{2}}) + (s_{i+1} - s_{i+\frac{1}{2}}) w'(s_{i+\frac{1}{2}}) + \int_{s_{i+\frac{1}{2}}}^{s_{i+1}} (s_{i+1} - s) w''(s) ds, \tag{108}$$

and also set $z = s_i$, we have that

$$w(s_i) = w(s_{i+\frac{1}{2}}) + (s_i - s_{i+\frac{1}{2}}) w'(s_{i+\frac{1}{2}}) - \int_{s_{i+\frac{1}{2}}}^{s_i} (s_i - s) w''(s) ds. \tag{109}$$

Then we have that

$$\frac{w(s_{i+1}) - w(s_i)}{h_i} = \frac{1}{h_i}\left[(s_{i+1} - s_i) w'(s_{i+\frac{1}{2}}) + \int_{s_i}^{s_{i+\frac{1}{2}}} (s_i - s) w''(s) ds + \int_{s_{i+\frac{1}{2}}}^{s_{i+1}} (s_{i+1} - s) w''(s) ds\right]$$

$$\frac{w(s_{i+1}) - w(s_i)}{h_i} - w'(s_{i+\frac{1}{2}}) = \frac{1}{h_i}\left[\int_{s_i}^{s_{i+\frac{1}{2}}} (s_i - s) w''(s) ds + \int_{s_{i+\frac{1}{2}}}^{s_{i+1}} (s_{i+1} - s) w''(s) ds\right]$$

We use the following inequalities

$$\left|\int_{s_i}^{s_{i+\frac{1}{2}}} (s_i - s) w''(s) ds\right| \leq \frac{h_i}{2} \int_{s_i}^{s_{i+1}} |w''|(s) ds \tag{110}$$

$$\left|\int_{s_{i+\frac{1}{2}}}^{s_i+1} (s_{i+1} - s) w''(s) ds\right| \leq \frac{h_i}{2} \int_{s_i}^{s_{i+1}} |w''|(s) ds. \tag{111}$$

Then we have that,

$$\left|\frac{w(s_{i+1})-w(s_i)}{h_i}-w'(s_{i+\frac{1}{2}})\right| \le \frac{1}{h_i}\left[\frac{h_i}{2}\int_{s_i}^{s_{i+1}}|w''(s)|ds+\frac{h_i}{2}\int_{s_i}^{s_{i+1}}|w''(s)|ds\right]$$
$$\le \int_{s_i}^{s_{i+1}}|w''(s)|ds.$$

Hence we have that

$$as_{i+\frac{1}{2}}^2\left|\frac{w(s_{i+1})-w(s_i)}{2h_i}-w'(s_{i+\frac{1}{2}})\right| \le \int_{s_i}^{s_{i+1}}\frac{as_{i+\frac{1}{2}}^2}{2}|w''(s)|ds. \tag{112}$$

Now we consider the following from Assumption 1, for $s_i \le s \le s_{i+1}$, we have

$$\left(\frac{s_{i+\frac{1}{2}}}{s}\right)^2 \le \left(\frac{s_{i+\frac{1}{2}}}{s_i}\right)^2 = \left(\frac{s_{i+1}+s_i}{2s_i}\right)^2 = \left(1+\frac{h_i}{2s_i}\right)^2 = \left(1+\frac{h_i}{2(s_{i-1}+h_{i-1})}\right)^2$$
$$\le \left(1+\frac{h_i}{2h_{i-1}}\right)^2 \le \left(1+\frac{c_0}{2}\right)^2, \tag{113}$$

then from (113) we have that

$$as_{i+\frac{1}{2}}^2\left|\frac{w(s_{i+1})-w(s_i)}{h_i}-w'(s_{i+\frac{1}{2}})\right| \le \int_{s_i}^{s_{i+1}}\frac{as_{i+\frac{1}{2}}^2}{2s^2}|s^2w''(s)|ds$$
$$\le a\left(1+\frac{c_0}{2}\right)^2\int_{s_i}^{s_{i+1}}|s^2w''(s)|ds.$$

Now from (105), we have that

$$as^2w'' = \Phi'(w,s,t)-(2a+b)sw'-bw, \tag{114}$$

then we have that

$$as_{i+\frac{1}{2}}^2\left|\frac{w(s_{i+1})-w(s_i)}{h_i}-w'(s_{i+\frac{1}{2}})\right| \le \left(1+\frac{c_0}{2}\right)^2\int_{s_i}^{s_{i+1}}|\Phi'(w,s,t)-(2a+b)sw'-bw|ds$$
$$\le \left(1+\frac{c_0}{2}\right)^2\int_{s_i}^{s_{i+1}}\big[|\Phi'(w,s,t)|+|(2a+b)||w'|+|bw|\big]ds$$
$$= \left(1+\frac{c_0}{2}\right)^2\left(\int_{s_i}^{s_{i+1}}\big[|\Phi'(w,s,t)|+|b||w|\big]ds+\int_{s_i}^{s_{i+1}}|(2a+b)||w'|ds\right)$$
$$|A_1| \le C\left(\int_{s_i}^{s_{i+1}}\big[|\Phi'(w,s,t)|+|w'|+|w|\big]ds\right).$$

Now to estimate $A_2$, first order Taylor expansion yields

$$w_{i+1}-w(s_{i+\frac{1}{2}}) = \int_{s_{i+\frac{1}{2}}}^{s_{i+1}}w'ds, \tag{115}$$

and then we have

$$bs_{i+\frac{1}{2}}\left|w_{i+1}-w(s_{i+\frac{1}{2}})\right| \le b\left|s_{i+\frac{1}{2}}\int_{s_{i+\frac{1}{2}}}^{s_{i+1}}w'ds\right| \le C\int_{s_i}^{s_{i+1}}|w'|ds$$
$$|A_2| \le C\int_{s_i}^{s_{i+1}}|w'|ds.$$

Therefore we have that

$$|\Phi(w_h,s_{i+\frac{1}{2}},t)-\Phi(w,s_{i+\frac{1}{2}},t)| = |A_1+A_2| \le |A_1|+|A_2|$$
$$\le C\left(\int_{s_i}^{s_{i+1}}\big[|\Phi'(w,s,t)|+|w'|+|w|\big]ds\right).$$

Therefore,

$$|\Phi(w_h,s_{i+\frac{1}{2}},t)-\Phi(w,s_{i+\frac{1}{2}},t)| \le C\left(\int_{s_i}^{s_{i+1}}\big[|\Phi'(w,s,t)|+|w'|+|w|\big]ds\right). \tag{116}$$

Now for the fitted mimetic case, it is sufficient to consider the flux approximation at $i=0$, hence we have that

$$\begin{aligned}\left|\Phi_h(w_h(s_{\frac{1}{2}}),s_{\frac{1}{2}},t)-\Phi(w(s_{\frac{1}{2}}),s_{\frac{1}{2}},t)\right| &= \left|\frac{s_{\frac{1}{2}}}{2}[(a+b)w_1-(a-b)w_0]-as_{\frac{1}{2}}^2w'(s_{\frac{1}{2}})-bs_{\frac{1}{2}}w(s_{\frac{1}{2}})\right|\\ &= \left|\frac{s_{\frac{1}{2}}}{2}\left[(a+b)\left(\frac{w_1-w_0}{h_0}h_0+w_0\right)-(a-b)w_0\right]-as_{\frac{1}{2}}^2w'(s_{\frac{1}{2}})-bs_{\frac{1}{2}}w(s_{\frac{1}{2}})\right|\\ &= \left|s_{\frac{1}{2}}^2(a+b)\left(\frac{w_1-w_0}{h_0}\right)+bs_{\frac{1}{2}}w_0-as_{\frac{1}{2}}^2w'(s_{\frac{1}{2}})-bs_{\frac{1}{2}}w(s_{\frac{1}{2}})\right|,\ \text{since } h_0=2s_{\frac{1}{2}}\\ &= \left|as_{\frac{1}{2}}^2\left(\frac{w_1-w_0}{h_0}-w'(s_{\frac{1}{2}})\right)+bs_{\frac{1}{2}}\left(w_0-w(s_{\frac{1}{2}})\right)+bs_{\frac{1}{2}}^2\left(\frac{w_1-w_0}{h_0}\right)\right|\\ &\le \left|as_{\frac{1}{2}}^2\left(\frac{w_1-w_0}{h_0}-w'(s_{\frac{1}{2}})\right)\right|+\left|bs_{\frac{1}{2}}\left(w_0-w(s_{\frac{1}{2}})\right)\right|+\left|bs_{\frac{1}{2}}^2\left(\frac{w_1-w_0}{h_0}\right)\right|\\ &=: D_{10}+D_{11}+D_{12}.\end{aligned}$$

Now, we estimate $D_{10}$ by using a similar argument as with the general case, that is

$$\left(\frac{w_1-w_0}{h_0}-w'(s_{\frac{1}{2}})\right)=\frac{1}{h_0}\int_{s_0}^{s_{\frac{1}{2}}}w''(s)(s_0-s)ds+\frac{1}{h_0}\int_{s_{\frac{1}{2}}}^{s_1}w''(s)(s_1-s)ds. \tag{117}$$

Hence,

$$D_{10}=as_{\frac{1}{2}}^2\left|\frac{w_1-w_0}{h_0}-w'(s_{\frac{1}{2}})\right|\le\frac{as_{\frac{1}{2}}^2}{2}\int_{s_0}^{s_1}|w''|ds\le C\int_{s_0}^{s_1}\big[|\Phi'(w)|+|w'|+|w|\big]ds.$$

Now we consider the estimate of $D_{11}$

$$D_{11}=bs_{\frac{1}{2}}\left|w_0-w(s_{\frac{1}{2}})\right|\le bs_{\frac{1}{2}}\int_{s_0}^{s_1}|w'|ds\le C\int_{s_0}^{s_1}|w'|ds.$$

Also, the estimate of $D_{12}$

$$D_{12}=bs_{\frac{1}{2}}^2\left|\frac{w_1-w_0}{h_0}\right|\le bs_{\frac{1}{2}}^2\int_{s_0}^{s_1}|w'|ds\le C\int_{s_0}^{s_1}|w'|ds.$$

Therefore,

$$\left|\Phi_h(w_h(s_{\frac{1}{2}}),s_{\frac{1}{2}},t)-\Phi(w(s_{\frac{1}{2}}),s_{\frac{1}{2}},t)\right|\le C\int_{s_0}^{s_1}\big[|\Phi'(w)|+|w'|+|w|\big]ds. \tag{118}$$

□

## 4. Fully discrete schemes

We subdivide the time interval $[0,T]$ in $M$ subdivisions and use the transformation $t:=T-t$. We consider here the implicit scheme for time integration although the general $\theta-$Euler method can be used. The full discretization of (73) with the implicit Euler method can be formulated as : Find a sequence $U_h^1,\dots,U_h^M\in V_h$ such that for $m\in\{0,\dots,M-1\}$

$$\begin{cases}\left(\frac{U_h^{m+1}-U_h^m}{\Delta t},v_h\right)_h+a_h\left(U_h^{m+1},v_h;t_{m+1}\right) = \left(f^{m+1},v_h\right)_h\\ U_h^0\quad\text{given}\end{cases} \tag{119}$$

where the bilinear form $a_h$ is given by (73). Similarly, when the fitted mimetic difference method is applied for the spatial discretization, the full discretization is formulated as follows:

Find a sequence $U_1^h,\dots,U_M^h\in V_h$ such that for $m\in\{0,\dots,M-1\}$

$$\begin{cases}\left(\frac{U_{m+1}^h-U_m^h}{\Delta t},v_h\right)_h+\mathbf{A}_h\left(U_{m+1}^h,v_h;t_{m+1}\right) = \left(f^{m+1},v_h\right)_h\\ U_0^h\quad\text{given}\end{cases} \tag{120}$$

where the bilinear form $\mathbf{A}_h$ is given by (90).

**Theorem 5.** *Let $\zeta_h^m$ be the numerical solution of the fully discretized scheme using the mimetic method (119) ($\zeta_h^m=U_h^m$ for the mimetic method) or the fitted mimetic method (120) ($\zeta_h^m=U_m^h$ for fitted mimetic method). Let $u$ be the unique solution of (66).*

*If* $u \in H^1\left(0,T;H^1(\Omega)\right) \cap H^2\left(0,T;L^2(\Omega)\right)$ *and* $\Phi(u,.,.) \in C(0,T,H^1(\Omega))$, *then there exists a positive constant* $C$, *independent of* $h$, $\Delta t$, $M$, *and* $N$ *such that*

$$\left|\left|u(t_m) - \zeta_h^m\right|\right|_{0,h} \leq C(h+\Delta t). \tag{121}$$

**Remark 4.** Note that the existence and uniqueness of $\zeta_h^{m+1}$ in (119) and (120) is ensured using the well known Lax-Milgram Theorem [17], since for every fixed time $t_{m+1}$ the bilinear forms $(.,.)_h + \Delta t\, a_h(.,.)$ and $(.,.)_h + \Delta t\, \mathbf{A}_h(.,.)$ are $V_h$-coercive.

The proof of Theorm 5 need the following lemma, similar to [18, Lemma 2].

**Lemma 1.** *There exist two constants* $C_3$ *and* $C_4$ *independent of* $h$ *such that the coefficient* $\tau_{i+\frac{1}{2}} = \frac{s_{i+1/2}^2}{h_i}$ *used in (23) for the discrete* $H^1_{0,w}(\Omega)-$*norm and its inverse are bounded as follows:*

$$\tau_{i+\frac{1}{2}} \leq C_3, \qquad \frac{1}{\tau_{i+\frac{1}{2}}} \leq C_4 h_i \qquad\qquad i = 0,\dots,N. \tag{122}$$

Let us prove the Lemma 1.

**Proof.** Indeed, we have

$$\tau_{i+\frac{1}{2}} = \frac{s_{i+1/2}^2}{h_i} = \frac{s_{i+1/2}^2}{s_{i+1}-s_i} = \frac{\frac{s_{i+1/2}^2}{s_{i+1}}}{1-\frac{s_i}{s_{i+1}}} \leq \frac{S_{max}}{1-\frac{s_i}{s_{i+1}}}. \tag{123}$$

By setting $\nu = \frac{s_i}{s_{i+1}}$, $i = 1,\dots,N$, we have $0<\nu<1$, and therefore using the Taylor expansion for (130), we have

$$\frac{1}{1-\nu} = 1+\nu+\mathcal{O}(\nu^2). \tag{124}$$

Then, there exists $C>0$ such that

$$\frac{1}{1-\nu} < 2+C, \tag{125}$$

and finally

$$\tau_{i+\frac{1}{2}} \leq C_3, \quad \text{with} \quad C_3 = S_{\max}(2+C). \tag{126}$$

Let us prove the second estimation. Indeed we have

$$\frac{1}{\tau_{i+\frac{1}{2}}} = \frac{h_i}{s_{i+1/2}^2} \tag{127}$$

It will be sufficient to prove that, there exists a constant $M>0$ such that

$$\frac{1}{s_{i+1/2}} \leq M, \tag{128}$$

to conclude the existence of $C_4>0$ such that $\frac{1}{\tau_{i+\frac{1}{2}}} \leq C_4 h$. Indeed we have

$$\frac{1}{s_{i+\frac{1}{2}}} = \frac{1}{S_{\max}\cdot\frac{s_{i+\frac{1}{2}}}{S_{\max}}} = \frac{1}{S_{\max}} \times \frac{1}{Z_1}, \quad \text{with } Z_i = \frac{s_{i+\frac{1}{2}}}{S_{\max}}. \tag{129}$$

We can rewrite as

$$\frac{1}{1-\xi} = \frac{1}{1-(1-Z_i)} = \frac{1}{Z_1}, \quad \text{where } \xi = 1-Z_1. \tag{130}$$

As $0<\xi<1$, Taylor expansion allows to have

$$\frac{1}{1-\xi} = 1+\xi+\mathcal{O}(\xi^2). \tag{131}$$

So, there exist a constant $M_1>0$ such that

$$\frac{1}{1-\xi} \leq 2+M_1. \tag{132}$$

and therefore

$$\frac{1}{s_{i+1/2}} \leq M, \quad \text{with } M = \frac{2+M_1}{S_{max}}, \tag{133}$$

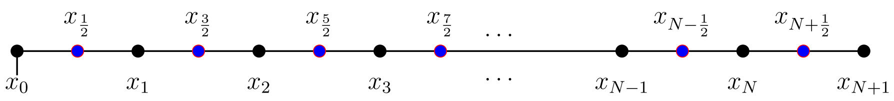


**Fig. 1.** (a) Mimetic grid.

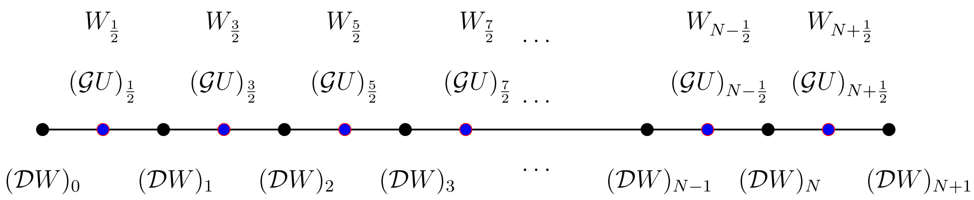


**Fig. 2.** (b) Discrete-mimetic finite difference grid.

**Table 1**
This table showing $L^2-$error for the various spatial discretization methods.

| N | Finite difference | Fitted finite volume | Mimetic FDM | Fitted mimetic FDM |
|---|---|---|---|---|
| 100 | 0.0013 | 7.6163e−04 | 1.7625e−04 | 9.8033e−05 |
| 200 | 6.5418e−04 | 4.8151e−04 | 5.4550e−05 | 2.5078e−05 |
| 500 | 2.6398e−04 | 2.3435e−04 | 1.2412e−05 | 4.0682e−06 |
| 1000 | 1.3238e−04 | 1.2486e−04 | 4.2083e−06 | 1.0218e−06 |
| 2000 | 6.6291e−05 | 6.4397e−05 | 1.4547e−06 | 2.5604e−07 |
| 3000 | 4.4216e−05 | 4.3373e−05 | 7.8569e−07 | 1.1382e−07 |
| 5000 | 2.6540e−05 | 2.6236e−05 | 3.6286e−07 | 4.1031e−08 |
| 10000 | 1.3274e−05 | 1.3198e−05 | 1.2768e−07 | 1.0270e−08 |

$T = 1, r = 0.1, K = 100, M = 1000, \sigma = 0.3, S_{\max} = 3K$.

and the proof of the lemma is completed. □

We can now give more information about the proof of Theorem 5.

**Proof.** Indeed the proof of Theorem 5 follows the same lines as that in [13, Theorem 7]. Indeed we can just use Lemma 1 at the place of [18, Lemma 2] to have a short summary of the proof as in [18, Theorem 3]. □

## 5. Numerical tests and conclusion

### *5.1. Test 1*

We run all numerical simulations on a 8 GB 1600 MHz DDR3, MacBook Pro. We simulate our results using Matlab R2017b. For the simulations, we consider the European put option parameters, $K = 100,\ r = 0.1,\ \sigma = 0.3,\ T = 1,\ S = 3K$. We choose the space to be (0,1000) and time intervals to be $(0, 1)$. They are subdivided into $N = 10,000$ and $M = 1000$ subintervals respectively in space and time. The analytical solution is given in [19].

### *5.2. Test 2*

Here we compare our novel numerical schemes (mimetic and fitted mimetic) with some existing numerical schemes (finite difference scheme and standard fitted finite volume [12]).

Table 1 shows the $L^2$ error for the mimetic finite difference method compared with the Fitted finite volume method [12] for the European put option.

As we can observe from Table 1, the mimetic methods outperforms the standard finite difference and the fitted finite volume methods. In particular the fitted mimetic finite difference scheme presented more accurate results than the standard mimetic scheme. This shows the importance of the fitted scheme to handling the degeneracy of the Black-Scholes differential operator. Furthermore, the CPU time (in seconds) for all the methods giving in Table 2 confirms that our novel methods are robust.

### *5.3. Conclusion*

In this paper, we have proposed two novel schemes (mimetic finite difference and fitted mimetic finite difference method) to approximate the degenerate Black-Scholes equation which governs option pricing. The fitted technique is well

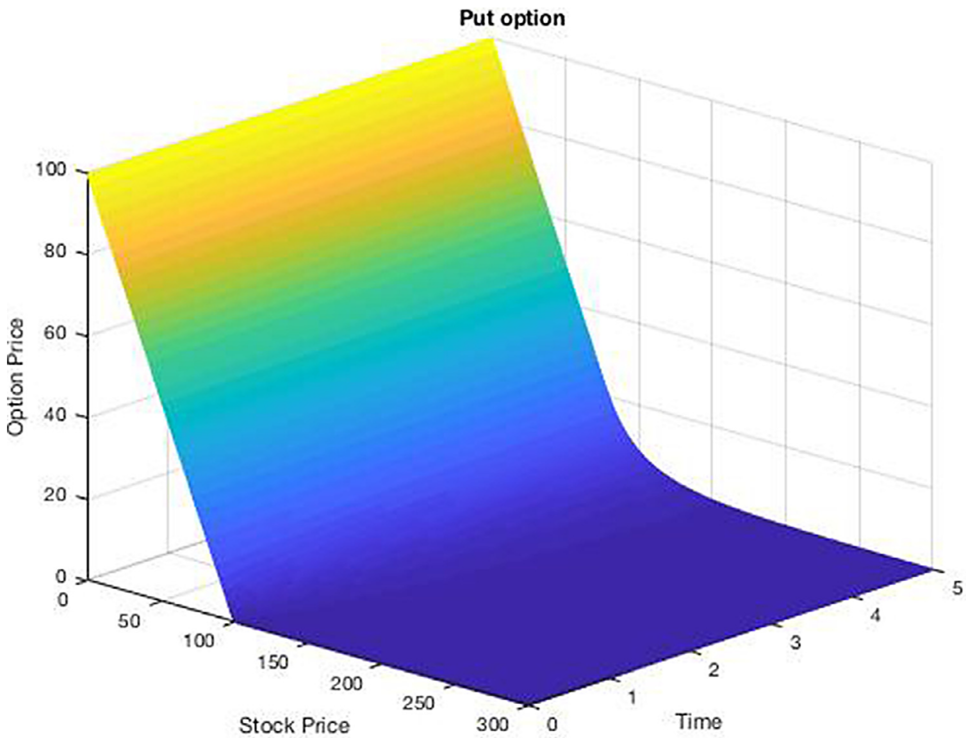


**Fig. 3.** (c) The numerical solution of the Fitted-mimetic method coupled with the implicit time stepping scheme.

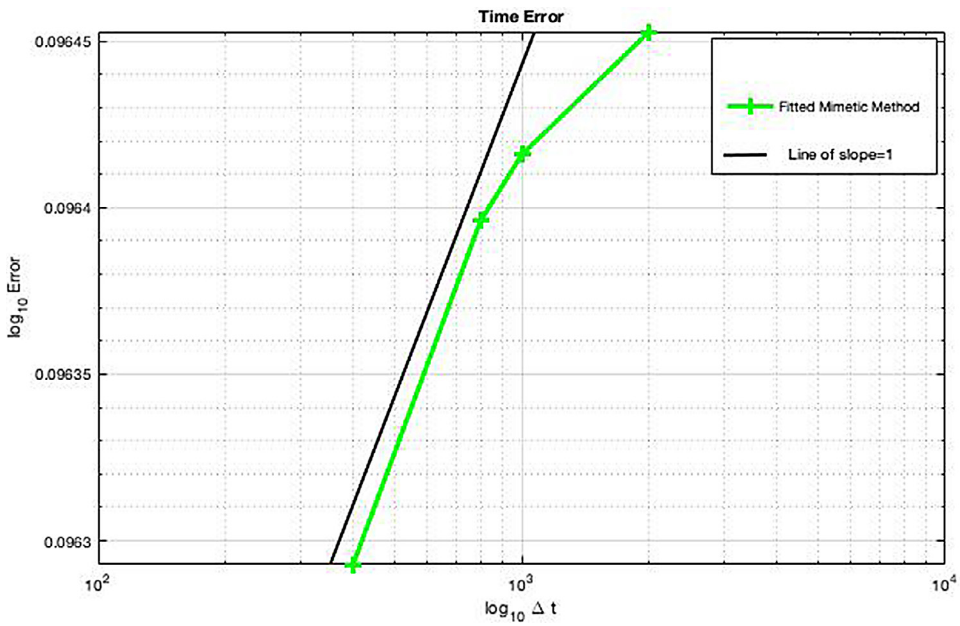


**Fig. 4.** (d) Time error vs the time step. The graph shows that the practical error of convergence is 1, which is in agreement with our theoretical result. Note that the fixed number of subdivisions in space is $N = 10,000$.

known to handle the degeneracy of the PDE. We further established the errors estimate of $\mathcal{O}(h + \Delta t)$ for the two full discrete schemes where the time integration is performed using the implicit Euler scheme. We presented numerical results to confirm the theoretical results and show the efficiency of the methods comparing to the fitted finite volume and finite difference methods.

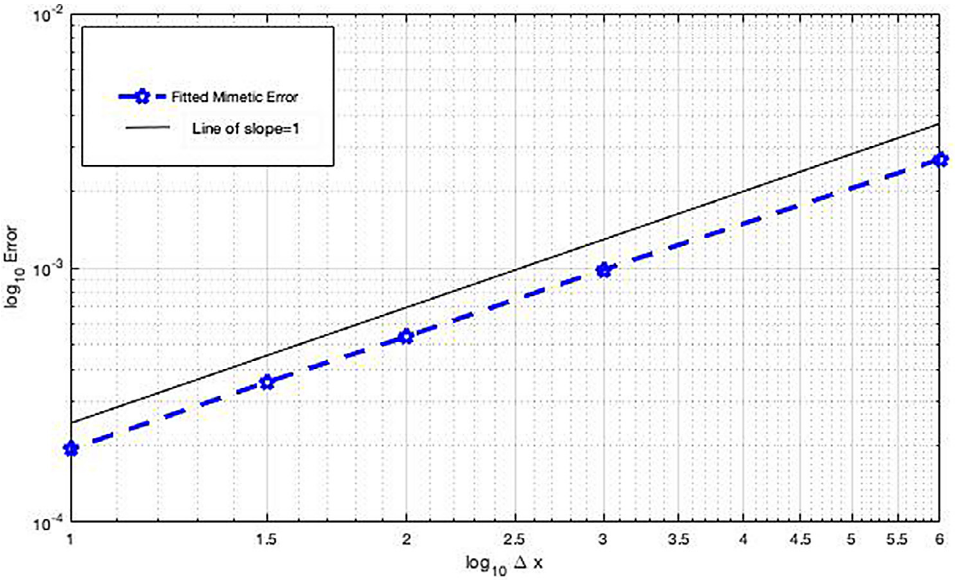


**Fig. 5.** (e) Fitted mimetic space error vs the space step. The graph shows that the error of convergence is 1, which is in agreement with our theoretical result. Note that the fixed number of subdivisions in time is $M = 1000$.

**Table 2**
This table showing CPU time for the various spatial discretization methods.

| N | Finite Difference CPU | Fitted FV CPU | Mimetic FDM CPU | Fitted MFDM CPU |
|---|---|---|---|---|
| 100 | 0.039 | 0.036 | 0.037 | 0.032 |
| 200 | 0.049 | 0.046 | 0.049 | 0.043 |
| 500 | 0.102 | 0.064 | 0.102 | 0.061 |
| 1000 | 0.163 | 0.096 | 0.161 | 0.103 |
| 2000 | 0.312 | 0.257 | 0.310 | 0.185 |
| 3000 | 0.531 | 0.371 | 0.527 | 0.407 |
| 5000 | 1.078 | 1.064 | 1.072 | 0.875 |
| 10000 | 3.996 | 3.019 | 3.264 | 2.791 |

$T = 1, r = 0.1, K = 100, M = 1000, \sigma = 0.3, S_{\max} = 3K$.

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