J. Math. Anal. Appl. 515 (2022) 126387

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Regular Articles

# Closed range weighted composition operators and dynamical sampling

Tesfa Mengestie

*Mathematics Section, Western Norway University of Applied Sciences, Klingenbergvegen 8, N-5414 Stord, Norway*

ARTICLE INFO

*Article history:*
Received 19 February 2022
Available online 3 June 2022
Submitted by Z. Cuckovic

*Keywords:*
Fock space
Closed range
Weighted composition operator
Invertible and Fredholm
Dynamical sampling
Frame and Riesz bases

ABSTRACT

We solve the closed range problem for weighted composition operators on Fock spaces. The result equivalently characterizes when the operators are bounded from below. We give several applications of the main result related to the operators invertibility, Fredholm, and dynamical sampling structures from frame perspectives. We prove there exists no vector in the Fock space for which its orbit under the weighted composition operator represents a frame family. Furthermore, it is shown that a weighted composition operator preserves frames if and only if it preserves the stronger Riesz basis property. Similar results are provided for the adjoint operator.



## 1. Introduction

For entire functions $u$ and $\psi$, we define the weighted composition operator $W_{(u,\psi)}$ by $W_{(u,\psi)}f = uf(\psi) = M_u C_\psi$ where $M_u$ and $C_\psi$ are respectively the multiplication and composition operators. The study of $W_{(u,\psi)}$ traces back to 1964 with works related to isometries on Hardy spaces [7,8]. Since then, various aspects of the operator on several spaces of holomorphic functions have been studied; see for example [10–13,15] and the references given therein. In this note, we take the study further and solve the closed range problem for $W_{(u,\psi)}$ on the Fock spaces $\mathcal{F}_p$. Recall that for $1 \le p < \infty$, the spaces $\mathcal{F}_p$ consist of all entire functions $f$ on the complex plane $\mathbb{C}$ for which

$$\|f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|e^{-\frac{p}{2}|z|^2} dA(z) < \infty$$

where $dA$ denotes the Lebesgue area measure on $\mathbb{C}$.

*E-mail address:* Tesfa.Mengestie@hvl.no.

https://doi.org/10.1016/j.jmaa.2022.126387

Following the works in [10,15], the operator $W_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if $u$ belongs to the space $\mathcal{F}_q$ and

$$\sup_{z\in\mathbb{C}} |u(z)|e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} < \infty. \tag{1.1}$$

It was further proved in [10] that condition (1.1) implies $\psi(z) = az + b,\ |a| \le 1$ and whenever $|a| = 1$, the multiplier function has the form

$$u = u(0)K_{-\overline{a}b} \tag{1.2}$$

where $K_w(z) = e^{\overline{w}z}$ is the reproducing kernel function. The operator is compact if and only if $|a| < 1$ and

$$\lim_{|z|\to\infty} |u(z)|e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} = 0.$$

Furthermore, if $p > q$, then $W_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if it is compact. This and the representation in (1.2) will be of further use to us in the rest of our consideration.

This note has two main parts. In the first part, we study the closed range problem for $W_{(u,\psi)}$ when it acts between Fock spaces. Theorem 1.2 provides a complete answer to this problem. We further use the result to identify the operators invertible and Fredholm structures. In the second part, we study some more applications of the result related to the operators dynamical sampling and frame preserving properties. We prove, in Theorem 2.2, that there exists no vector in the Fock space $\mathcal{F}_2$ for which its orbit under the weighted composition operator represents a frame family. On the other hand, the operator preserves frames if and only if it preserves the stronger Riesz basis property, and this happens if and only if the operator has a closed range. Similar results are provided for the adjoint of the operator in Theorem 2.2.

For better exposition and since the particular result is needed to prove the more general result, we may first dispose the case of the composition operator.

**Theorem 1.1.** *Let $1 \le p, q < \infty$ and $C_\psi : \mathcal{F}_p \to \mathcal{F}_q$ be bounded and hence $\psi(z) = az + b,\ |a| \le 1$. Then $C_\psi$ has a closed range if and only if either $a = 0$ or $|a| = 1$ and $p = q$. The closed range is given by*

$$\mathcal{R}(C_\psi) = \begin{cases} \mathbb{C}, & a = 0 \\ \mathcal{F}_p, & |a| = 1 \ \textit{and}\ p = q. \end{cases} \tag{1.3}$$

Having completely identified the closed range composition operators, the question now is whether there exists an interplay between the multiplier function $u$ and the composition symbol $\psi$ to induce a nontrivial closed range $W_{(u,\psi)}$ whenever $p \ne q$ or $0 < |a| < 1$. Our next main result answers this in the negative.

**Theorem 1.2.** *Let $1 \le p, q < \infty$ and $\psi$ and $u$ be entire functions on $\mathbb{C}$ such that $u$ is not identically zero. If $W_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded and hence $\psi(z) = az + b,\quad |a| \le 1$, then $W_{(u,\psi)}$ has a closed range if and only if either $a = 0$ or $|a| = 1,\ u = u(o)K_{-\overline{a}b}$ with $u(0) \ne 0$, and $p = q$. The closed range is given by*

$$\mathcal{R}(W_{(u,\psi)}) = \begin{cases} \{f(b)u : f \in \mathcal{F}_p\}, & a = 0 \\ \mathcal{F}_p, & |a| = 1,\ u = u(o)K_{-\overline{a}b} \ \textit{with}\ \ u(0) \ne 0 \ \ \textit{and}\ \ p = q. \end{cases} \tag{1.4}$$

As will be explained latter in the proof, the result equivalently characterizes when $W_{(u,\psi)}$ is bounded from below on Fock spaces. As a consequences of Theorem 1.2, a bounded multiplication operator $M_u : \mathcal{F}_p \to \mathcal{F}_q$ has a closed range if and only if $u$ is non-zero and $p = q$. Note that $M_u$ is bounded on $\mathcal{F}_p$ if and only if $u = \alpha$ =constant. Thus, $\mathcal{R}(M_u) = \{\alpha f : f \in \mathcal{F}_p\} = \mathcal{F}_p$ when $\alpha \ne 0$.

Next, we consider another consequence of the result about invertible and Fredholm structures. Note that when $|a| = 1$ and $u(0) \neq 0$, by Theorem 1.2 the operator $W_{(u,\psi)}$ is surjective on $\mathcal{F}_p$ and hence its adjoint is injective. Thus, we may record the following.

**Corollary 1.3.** *Let $1 \leq p < \infty$ and $\psi$ and $u$ be entire functions on $\mathbb{C}$. If $W_{(u,\psi)}$ is bounded on $\mathcal{F}_p$ and hence $\psi(z) = az + b, |a| \leq 1$, then the following statements are equivalent.*

(i) $|a| = 1$ *and* $u = u(o)K_{-\overline{a}b}$ *with* $u(0) \neq 0$*;*
(ii) $W_{(u,\psi)}$ *is invertible. Furthermore, the inverse is itself a weighted composition operator given by*

$$W^{-1}_{(u,\psi)} = W_{(u_1,\psi_1)}$$

*where* $\psi_1(z) = (z - b)/a$ *and* $u_1 = \frac{e^{-|b|^2}}{u(0)} K_b$*;*
(iii) $W_{(u,\psi)}$ *is Fredholm of index zero.*

Observe that since the inverse itself is a weighted composition operator, its boundedness follows from (1.1).
A word on notation: the notion $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

### *1.1. Proof of Theorem 1.1*

The case for a constant $\psi$ is clear. Let us start by assuming that $p \leq q$ and $0 < |a| < 1$. In this case, $C_\psi$ is an injective map and by the Open Mapping Theorem, $C_\psi$ has a closed range if and only if it is bounded from below. That is, there exists $\epsilon > 0$ such that for each $f \in \mathcal{F}_p$

$$\|C_\psi f\|_q \geq \epsilon \|f\|_p.$$

We plan to use this equivalency to verify the claim. Assume the operator has closed range and consider the sequence of functions $f_n(z) = (z - z_0)^n \in \mathcal{F}_p$ where $z_0 = b/(1 - a)$ is the fixed point of $\psi$. Then,

$$C_\psi f_n(z) = (az + b - z_0)^n = a^n f_n(z)$$

and hence

$$\|C_\psi f_n\|_q = |a|^n \|f_n\|_q \geq \epsilon \|f_n\|_p$$

for some $\epsilon > 0$. This together with the assumption $0 < |a| < 1$ and the inclusion property in Fock spaces [17, Theorem 2.10] imply

$$|a|^n = \frac{\|C_\psi f_n\|_q}{\|f_n\|_q} \geq \frac{\epsilon \|f_n\|_p}{\|f_n\|_q} \geq \epsilon \to 0$$

as $n \to \infty$, which is a contradiction.

Next, if $p > q$, then $C_\psi : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if it is compact. This holds if and only if $0 < |a| < 1$. It is known that a compact operator can have closed range if and only if its range is finite dimensional. On the other hand, since $a \neq 0$, the operator $C_\psi$ is one-to-one and $\mathcal{F}_p$ is infinite dimensional. Consequently, the range of $C_\psi$ cannot be of finite dimensional and hence not closed.

Let us now turn to the case when $|a| = 1$. For each $f \in \mathcal{F}_p$

$$\|C_\psi f\|_q^q = \frac{q}{2\pi} \int_{\mathbb{C}} |f(az)|^q e^{-\frac{q}{2}|az|^2} = \|f\|_q^q \tag{1.5}$$

from which the sufficiency of the condition follows. Furthermore, since for each $f \in \mathcal{F}_p$, the function $f_a(z) := f(\frac{z}{a}) \in \mathcal{F}_p$, we have the equality in (1.3). It remains to show that $p < q$ cannot be admissible under closed range assumption when $|a| = 1$. To this end, set $g_n(z) = z^n \in \mathcal{F}_p$ and by (1.5)

$$\|C_\psi g_n\|_q = \|g_n\|_q.$$

Using polar integration and Stirling's formula

$$\|g_n\|_p^p = p \int_0^\infty r^{np+1} e^{-pr^2/2} dr = \Big(\frac{1}{p}\Big)^{np/2} \Gamma\Big(\frac{np+2}{2}\Big) \simeq \Big(\frac{n}{e}\Big)^{\frac{np}{2}} \sqrt{n}. \tag{1.6}$$

Now, boundedness from below implies

$$\|C_\psi f_n\|_q = |u(0)| e^{\frac{1}{2}|b|^2} \|f_n\|_q \geq \epsilon \|f_n\|_p$$

for some $\epsilon > 0$. This holds only if

$$\frac{\|f_n\|_q}{\|f_n\|_p} \simeq n^{\frac{1}{2q} - \frac{1}{2p}} \geq \epsilon$$

for all $n \in \mathbb{N}$. A contradiction arises when $n \to \infty$.

*1.2. Proof of Theorem 1.2*

We split the proof into three cases.

*Case 1*: If $\psi$ is a constant, then $W_{(u,\psi)} f = uf(b)$ for each $f \in \mathcal{F}_p$ and

$$\mathcal{R}(W_{(u,\psi)}) = \{f(b)u : f \in \mathcal{F}_p\}.$$

Furthermore, any Cauchy sequence $f_m(b)u \in \mathcal{R}(W_{(u,\psi)})$ converges to the function $cu$ where $c = \lim_{m\to\infty} f_m(b)$. Since $cu \in \mathcal{R}(W_{(u,\psi)})$, the range is closed as asserted.

In the rest of the proof, we assume $a \neq 0$. In this case $\psi$ is a bijective map on $\mathbb{C}$ and since $u$ not identically zero, it follows from the Open Mapping Theorem and uniqueness principle for analytic functions that, $W_{(u,\psi)}$ is an injective map. Therefore, $W_{(u,\psi)}$ has closed range if and only if it is bounded from below. We plan to use this equivalency as needed below again.

*Case 2*: Let us consider the case when $|a| = 1$. Using (1.2), for each $f \in \mathcal{F}_p$

$$\begin{aligned}\|W_{(u,\psi)} f\|_q^q &= \frac{q|u(0)|^q}{2\pi} \int_{\mathbb{C}} |f(az+b)|^q e^{-\frac{q}{2}|az+b|^2} \Big(\frac{|K_{-\overline{a}b}(z)|^q}{e^{-\frac{q}{2}(|az+b|^2 - |z|^2)}}\Big) dA(z) \\ &= |u(0)|^q e^{\frac{q}{2}|b|^2} \|f\|_q^q \end{aligned} \tag{1.7}$$

from which the sufficiency of the condition follows. Furthermore, $u(0) \neq 0$. If not, u reduces to the zero function. Now, the equality in (1.4) is immediate since for each $f \in \mathcal{F}_p$, the function $f_{ab}$ defined by

$$f_{ab}(z) := \frac{e^{-|b|^2}}{u(0)} f((z-b)/a)K_b(z),$$

is analytic and belongs to $\mathcal{F}_p$ with

$$\|f_{ab}\|_p^p = \frac{p|u(0)|^{-p}}{2\pi e^{p|b|^2}} \int\limits_{\mathbb{C}} |f((z-b)/a)|^p |K_b(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) = \frac{|u(0)|^{-p}}{e^{p|b|^2}} \|f\|_p^p.$$

Next, we proceed to show that $p < q$ is not admissible under the closed range assumption when $|a| = 1$. To this end, using the sequence $g_n(z) = z^n \in \mathcal{F}_p$ and applying (1.7)

$$\|W_{(u,\psi)} g_n\|_q = |u(0)| e^{\frac{1}{2}|b|^2} \|g_n\|_q.$$

Then, we argue as in the proof Theorem 1.1 to wards contradiction.
*Case 3*: It remains to show $W_{(u,\psi)}$ has no closed range whenever $0 < |a| < 1$. If $W_{(u,\psi)}$ is compact, then it is known that it can have closed range if and only if its range is finite dimensional. On the other hand, $W_{(u,\psi)}$ is injective and $\mathcal{F}_p$ is infinite dimensional. Consequently, the range of $W_{(u,\psi)}$ cannot be of finite dimensional and hence not closed. Note that for $p > q$, boundedness is also equivalent to compactness. Therefore, the same conclusion holds. If $u$ is a constant, then $W_{(u,\psi)}$ is a non-zero constant multiple of the composition operator $C_\psi$. Thus, the conclusion follows from Theorem 1.1 again.

Next, assume $0 < |a| < 1$, $p \le q$, and the operator is not compact. Suppose on the contrary that $W_{(u,\psi)}$ has a closed range. By defining inner product with usual dual pairing, we may apply [5, Proposition 6.4, p. 208] to deduce that zero is not in the approximate spectrum $\sigma_{ap}(W_{(u,\psi)})$. Consequently, by [5, Proposition 4.4, p. 359], zero is not in the left essential spectrum $\sigma_{el}(W_{(u,\psi)})$, and hence $W_{(u,\psi)}$ is a left semi-Fredholm operator. In particular, the composition operator $C_\psi$ is left semi-Fredholm and hence $0 \notin \sigma_l(C_\psi)$. Furthermore, since $W_{(u,\psi)}$ is injective and bounded from below, an application of the Open Mapping Theorem gives that $0 \notin \sigma_p(C_\psi)$ as well and hence $C_\psi$ has a closed range. This contradicts Theorem 1.1, and completes the proof.

## 2. Dynamical sampling with $W_{(u,\psi)}$ and $W^*_{(u,\psi)}$

Having completely identified the closed range $W_{(u,\psi)}$ on Fock spaces, we now turn to some application of the result on dynamical sampling from frame perspectives. Dynamical sampling has become an active area of research that connects frame theory to operator theory. It deals with representations of a given frames $\{f_n\}_{n=0}^\infty$ to the form $\{T^n f\}_{n=0}^\infty$ for some linear operator $T$ defined on a given Hilbert space $\mathcal{H}$ where

$$\{T^n f\}_{n=0}^\infty = \{f,\ Tf,\ T^2 f,\ T^3 f,\ ...\}$$

is the orbit of a function $f \in \mathcal{H}$ under $T$. A family $(f_j), j \in I$ of vectors in a Hilbert space $\mathcal{H}$ is a frame if there exist positive constants $A$ and $B$ such that for any $f \in \mathcal{H}$

$$A\|f\|_{\mathcal{H}}^2 \le \sum_{j\in I} |\langle f, f_j\rangle_{\mathcal{H}}|^2 \le B\|f\|_{\mathcal{H}}^2. \tag{2.1}$$

The constants $A$ and $B$ are called the lower and upper bounds of the frame respectively. It is called a tight frame when $A = B$ and a normalized tight frame whenever $A = B = 1$. Ever since introduced by Duffin and Schaeffer [6] in 1952 as a tool to study some problems in nonharmonic Fourier series, the theory of frame has found numerous applications in engineering, mathematics, and signal processing and data compression. Frames are generalizations of bases and their main advantage stems from the fact that a frame can be designed to be redundant while still providing a reconstruction formula for each vector in the space. Thus, identifying methods that generate new frames has been an interesting problem in the development of frame theory. A special type of frame is Riesz basis. A family $(f_j), j \in I$ of vectors in a Hilbert space $\mathcal{H}$ is a Riesz basis if it is complete and there exist constants $0 < A \leq B < \infty$ such that for any $c_j \in \ell^2(I)$

$$A\sum_{j\in I}|c_j|^2 \leq \Big\|\sum_{j\in I} c_j f_j\Big\|_{\mathcal{H}}^2 \leq B\sum_{j\in I}|c_j|^2.$$

The following lemma connects closed range operators and dynamical sampling.

**Lemma 2.1.** *Let $\mathcal{H}$ be a Hilbert space and $T$ be a bounded linear operator on $\mathcal{H}$. If $\{T^n f\}_{n=0}^{\infty}$ is a frame for some $f \in \mathcal{H}$, then*

(i) *$T$ is surjective.*
(ii) *$\|(T^*)^n g\|_{\mathcal{H}} \to 0$ as $n \to \infty$ for all $g \in \mathcal{H}$.*

Part (i) follows from a simple argument namely that if the orbit of $f$ is a frame, then for each $h \in \mathcal{H}$ there exists sequence $(c_n)$ such that

$$h = \sum_{n=1} c_n T^n f = T\Big(\sum_{n=1} c_n T^{n-1} f\Big).$$

As a consequence, the operator has a closed range which interestingly links us with Theorem 2.2 for the case of weighted composition operator on $\mathcal{F}_2$. The proof of part (ii) is available in [2].
For a better insight, we may first consider the problem with a bounded composition operator $C_\psi$ and verify that $\{C_\psi^n f\}_{n=0}^{\infty}$ cannot be a frame for any choice of $f$ in $\mathcal{F}_2$. To observe this, note that if $0 < |a| < 1$, then $C_\psi$ is compact and hence the conclusion follows once from [4]. On the other hand, if $|a| = 1$ and hence $b = 0$, then

$$\lim_{n\to\infty} \|C_\psi^{*n} K_z\|_2 = \lim_{n\to\infty} \|K_{\psi^n(z)}\|_2 = \lim_{n\to\infty} e^{\frac{1}{2}|a^n z|^2} = e^{\frac{1}{2}|z|^2} > 0$$

from which the assertion follows by Lemma 2.1. Thus, there exists no function $f$ in $\mathcal{F}_2$ for which its orbit under the composition operator represents a frame family. The orbit of any vector $f$ under $W_{(u,\psi)}$ has elements of the form

$$W_{(u,\psi)}^n f = f(\psi^n) u_n, \quad u_n = \prod_{j=0}^{n-1} u(\psi^j) \tag{2.2}$$

for all nonnegative integers $n$ and $\psi^0$ is the identity map. This shows that the product of weighted composition operators is another weighted composition operator with symbol $(u_n, \psi^n)$. The formula in (2.2) further displays a kind of interplay between the functions $\psi$ and $u$ and generates interest to ask whether the interplay results in dynamical sampling property for the weighted composition operators in contrast to the unweighed case. Disappointing enough, this is not the case either as seen in the next Theorem. We recall that the adjoint of a weighted composition operator on $\mathcal{F}_2$ is not necessarily a weighted composition operator [16]. Thus, an operator and its adjoint can have quite different dynamical structures but not in this case.

**Theorem 2.2.** *Let $W_{(u,\psi)}$ be bounded on $\mathcal{F}_2$. Then neither $\{W_{(u,\psi)}^n f\}_{n=0}^{\infty}$ nor $\{(W_{(u,\psi)}^*)^n f\}_{n=0}^{\infty}$ can be a frame for any choice of $f$ in $\mathcal{F}_2$.*

**Proof.** Let $\psi(z) = az + b,\ |a| \leq 1$ and suppose there exists an $f$ in $\mathcal{F}_2$ such that $\{W^n_{(u,\psi)} f\}_{n=0}^{\infty}$ is a frame. Then by Lemma 2.1, the operator has a closed range. An application of Theorem 1.2 ensures that $|a| = 1$ and $u = u(o)K_{-\overline{a}b}$ with $u(0) \neq 0$. Using the adjoint property $W^*_{(u,\psi)} K_w = \overline{u(w)} K_{\psi(w)}$ and a repeated iteration gives

$$W^{*n}_{(u,\psi)} K_w = \overline{u(z)} K_{\psi^n(w)}.$$

By (1.2), for all $n \in \mathbb{N}$

$$\begin{aligned}\|W^{*n}_{(u,\psi)} K_w\|_2 &= |u(w)| \|K_{\psi^n(w)}\|_2 = |u(w)| e^{\frac{1}{2}\left(|w|^2 + |\frac{b(1-a^n)}{1-a}|^2 + 2\Re(a^n w \overline{\frac{b(1-a^n)}{1-a}})\right)} \\ &\geq |u(w)| e^{\frac{1}{2}|w|^2} e^{\frac{|b|^2}{|1-a|^2} - 2\frac{|w||b|}{|1-a|}} = |u(0)| e^{-a\overline{b}w + \frac{|b|^2}{|1-a|^2} - 2\frac{|w||b|}{|1-a|}} > 0. \end{aligned} \tag{2.3}$$

Similarly for $a = 1$, since $u(0) \neq 0$

$$\|W^{*n}_{(u,\psi)} K_w\|_2 = |u(w)| \|K_{\psi^n(w)}\|_2 = |u(0)| e^{-\overline{b}w + \frac{1}{2}(|w|^2 + |nb|^2 + 2n\Re(w\overline{b}))} \to \infty \tag{2.4}$$

as $n \to \infty$. Now, by (2.3), (2.4) and part (ii) of Lemma 2.1, we arrive at a contradiction. Therefore, the assertion in the theorem is valid for $W_{(u,\psi)}$.

The proof for the adjoint operator is similar but need a computation with iterates of $W_{(u,\psi)}$ applied in a suitably selected sequence of functions. First note that by the Closed Range Theorem and Lemma 2.1, if $\{(W^*_{(u,\psi)})^n f\}_{n=0}^{\infty}$ is a frame, then $|a| = 1$ and $u(0) \neq 0$. We begin with the case $a = 1$ and hence $\psi^j(z) = z + jb$. Using (2.2), $W^n_{(u,\psi)} f = u_n f(\psi^n)$ where

$$u_n(z) = u(0)^n \prod_{j=0}^{n-1} K_{-b}(z + jb) = u(0)^n e^{-\overline{b}\sum_{j=0}^{n-1}(z+jb)} = u(0)^n e^{-\overline{b}nz - \frac{|b|^2}{2} n(n-1)}.$$

It follows that

$$u_n = u(0)^n e^{-\frac{|b|^2}{2} n(n-1)} K_{-nb}.$$

We may now consider a sequence $h_m(z) = \frac{e^{\frac{-m|b|^2}{2}}}{u(0)^m}$ and compute

$$\|W^n_{(u,\psi)} h_m\|_2 = \left|u(0) e^{\frac{|b|^2}{2}}\right|^{n-m} e^{-\frac{|b|^2}{2} n^2} \|K_{-nb}\|_2 = \left|u(0) e^{\frac{|b|^2}{2}}\right|^{n-m}$$

for all $n, m \in \mathbb{N}$. In particular when $n = m$

$$\|W^n_{(u,\psi)} f_n\|_2 = 1 \tag{2.5}$$

for all $n \in \mathbb{N}$. On the other hand, if $a \neq 1$ and $|a| = 1$, then $\psi^j(z) = a^j z + b\frac{1-a^j}{1-a}$. By using (2.2) again $u_n(z) = u(0)^n e^{h_n(z)}$ where

$$h_n(z) := -a\overline{b} \sum_{j=0}^{n-1} \left(a^j z + b\frac{1-a^j}{1-a}\right) = -a\overline{b}z\frac{1-a^n}{1-a} - \frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}.$$

Thus, we have

$$u_n = u(0)^n e^{-\frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}} K_{-\overline{a}b\frac{1-\overline{a}^n}{1-\overline{a}}}.$$

Now, considering a constant sequence of functions $g_m(z) = \left(\frac{e^{\frac{a|b|^2}{1-a}}}{u(0)}\right)^m$, we have

$$\|W_{(u,\psi)}^n g_m\|_2 \gtrsim \left|u(0)e^{-\frac{a|b|^2}{1-a}}\right|^{n-m} \left\|K_{-\overline{a}b\frac{1-\overline{a}^n}{1-\overline{a}}}\right\|_2 \gtrsim \left|u(0)e^{-\frac{a|b|^2}{1-a}}\right|^{n-m}$$

for all $n, m \in \mathbb{N}$. In particular when $n = m$

$$\|W_{(u,\psi)}^n g_n\|_2 \gtrsim 1 \tag{2.6}$$

for all $n \in \mathbb{N}$. Now by (2.5), (2.6) and part (ii) of Lemma 2.1, we have a contradiction again. Therefore, $\{(W_{(u,\psi)}^*)^n f\}_{n=0}^\infty$ can not be a frame for any $f \in \mathcal{F}_2$. $\square$

Theorem 2.2 can be alternatively proved using part (i) of Lemma 2.1 and some results from linear dynamics of weighted composition operators. We preferred to use the above approach since it in addition shows how part (ii) of Lemma 2.1 is a highly restrictive condition. It even fails tests with sequences of constant functions and reproducing kernel in the space.

The adjoint of a bounded composition operator on the Fock space is not necessarily a composition operator. As proved in [3, Lemma 2], for $\psi = az + b$, the adjoint of $C_\psi$ is rather a weighted composition operator where the weight function is a reproducing kernel, namely that $C_\psi^* = W_{(K_b,\phi)}$ where $\phi(z) = \overline{a}z$. From this and Theorem 1.2, we conclude the following.

**Corollary 2.3.** *Let $C_\psi$ be bounded on $\mathcal{F}_2$. Then neither $\{C_\psi^n f\}_{n=0}^\infty$ nor $\{(C_\psi^*)^n f\}_{n=0}^\infty$ can be a frame for any choice of $f$ in $\mathcal{F}_2$.*

### *2.1. Frame preserving $W_{(u,\psi)}$ and $W_{(u,\psi)}^*$*

The results above have revealed that there exists no function $f \in \mathcal{F}_2$ for which its orbits under the operator $W_{(u,\psi)}$ or its adjoint represents a frame for the space. A related question has been to identify conditions under which the operator preserves frame property. Recall that a bounded operator preserves frame when it maps frames into frames in the underlying space. In this section we answer the question and to prove the corresponding result need the following from [1,11,14].

**Proposition 2.4.** *Let $T$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then $T$ preserves*

(i) *frames on $\mathcal{H}$ if and only if $T^*$ is bounded below on $\mathcal{H}$, and the latter happens if and only if $T$ is surjective on $\mathcal{H}$.*
(ii) *tight frames if and only if there exists a positive constant $\lambda$ such that $\|T^*f\|_{\mathcal{H}} = \lambda\|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$.*
(iii) *normalized tight frames if and only if $T^*$ is an isometry in $\mathcal{H}$.*

Now we are able to prove the following result on frame preserving weighted composition operator and its adjoint.

**Theorem 2.5.** *Let $W_{(u,\psi)}$ be bounded on $\mathcal{F}_2$ and hence $\psi(z) = az + b$, $|a| \le 1$. Then the following statements are equivalent.*

(i) *$W_{(u,\psi)}$ preserves frame;*
(ii) *$|a| = 1$ and $u = u(0)K_{-\overline{a}b}$ with $u(0) \neq 0$;*
(iii) *$W_{(u,\psi)}^*$ preserves frame;*
(iv) *$W_{(u,\psi)}$ preserves Riesz bases;*

(v) $W^*_{(u,\psi)}$ *preserves Riesz bases.*

**Proof.** The equivalency of the statements in (i), (ii) and (iii) directly follows from Proposition 2.4, Theorem 1.2 and the Closed Range Theorem. Thus, we only need to prove $(i) \Leftrightarrow (iv)$ and $(ii) \Leftrightarrow (v)$. Now (iv) implies (i) follows trivially as all Riesz basis are frames and applying Proposition 2.4. Thus, we proceed to show the converse statement. In [9], it was proved that a frame $(f_j), j \in I$ is a Riesz bases if and only if it is $\omega$ independent. That is if

$$\sum_{j\in I} c_j f_j = 0$$

for some sequence of scalars $(c_j)$, then $c_j = 0$ for all $j \in I$. In view of this, suppose $(f_j), j \in I$ is a Riesz basis and

$$\sum_{j\in I} c_j W_{(u,\psi)} f_j = 0.$$

Note that since a Riesz basis is a frame, if $W_{(u,\psi)}$ preserves a Riesz basis, then by Theorem 1.2, $|a| = 1$, $u(0) \neq 0$ and $u = u(0)K_{-\overline{a}b}$. Taking these necessary conditions into account,

$$\sum_{j\in I} c_j W_{(u,\psi)} f_j = \sum_{j\in I} c_j u(0) K_{-\overline{a}b} f_j(\psi) = u(0)K_{-\overline{a}b} \sum_{j\in I} c_j f_j(\psi) = 0$$

if and only if

$$\sum_{j\in I} c_j f_j(\psi) = 0. \tag{2.7}$$

Furthermore, since $\psi(z) = az + b,\ a \neq 0$ interpolates all points in the complex plane, the relation in (2.7) holds only if

$$\sum_{j\in I} c_j f_j = 0$$

from which $c_j = 0$ for all $j$ since $(f_j)$ is a Riesz basis.

Next, we show that (ii) and (v) are equivalent. Assuming (ii), for two functions $f$ and $g$ in $\mathcal{F}_2$, the given condition and change of variable imply

$$\left\langle W^*_{(u,\psi)} f, g \right\rangle = \left\langle f, W_{(u,\psi)} g \right\rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{u(0)g(az)} e^{-|z|^2} dA(z) = \left\langle \overline{u(0)} C_\Phi f, g \right\rangle$$

where $\Phi(z) = z/a$. Therefore

$$W^*_{(u,\psi)} f = \overline{u(0)} C_\Phi f. \tag{2.8}$$

Suppose now that $(f_j), j \in I$ is a Riesz basis and

$$\sum_{j\in I} c_j W^*_{(u,\psi)} f_j = 0.$$

Using (2.8) and $u(0) \neq 0$, this holds if and only if

$$\sum_{j\in I} c_j f_j(\Phi) = 0. \tag{2.9}$$

Furthermore, since $\Phi$ is a nonconstant, (2.9) holds only if

$$\sum_{j\in I} c_j f_j = 0,$$

which implies $c_j = 0$ for all $j$ as asserted.
Conversely, suppose $W^*_{(u,\psi)}$ preserves Riesz bases. Since Riesz bases are frames again, by Proposition 2.4, $W^*_{(u,\psi)}$ is bounded from below and hence has closed range. So statement (ii) follows from Theorem 1.2 and Closed Range Theorem. □

Now we apply the preceding result and prove the following on tight frames.

**Theorem 2.6.** *Let $\psi$ and $u$ be nonzero entire functions on $\mathbb{C}$ and the operator $W_{(u,\psi)}$ is bounded on $\mathcal{F}_2$ and hence $\psi(z) = az + b,\ |a| \leq 1$. Then*

(i) *$W_{(u,\psi)}$ preserves*
  (a) *tight frames on $\mathcal{F}_2$ if and only if $|a| = 1$, $b = 0$ and $u = u(0) \neq 0$.*
  (b) *normalized tight frames on $\mathcal{F}_2$ if and only if $|a| = 1, b = 0$, $u = u(0)$ and $|u(0)| = 1$.*

(ii) *$W^*_{(u,\psi)}$ preserves tight frames on $\mathcal{F}_2$ if and only if $W_{(u,\psi)}$ does.*

(iii) *$W^*_{(u,\psi)}$ preserves normalized tight frames on $\mathcal{F}_2$ if and only if $W_{(u,\psi)}$ does.*

Theorem 2.6 and Theorem 2.5 provide interesting descriptions of the conditions required by $W_{(u,\psi)}$ or its adjoint to preserve various forms of frame properties. All of the conditions are different and depend on the special required property to be preserved.

**Proof.** (i) a) By Theorem 2.5, we have $|a| = 1$ and hence $u(z) = u(0)K_{-\overline{a}b}$ and $|u(0)| > 0$. By Proposition 2.4, the operator preserves tight frame if and only if there exists a positive number $\lambda$ such that $\|W^*_{(u,\psi)}f\|_2^2 = \lambda\|f\|_2^2$ for all functions $f \in \mathcal{F}_2$. In particular for $f = K_w$

$$\|W^*_{(u,\psi)}K_w\|_2^2 = |u(0)|^2\|K_{\psi(w)}\|_2^2 = \lambda\|K_w\|_2^2$$

and therefore,

$$|u(0)|^2 e^{|aw+b|^2-|w|^2} = |u(0)|^2 e^{2\Re(aw\overline{b})} = \lambda$$

for all $w \in \mathbb{C}$. This holds only if $b = 0$ and which further gives $\lambda = |u(0)|$. Therefore, $u(z) = u(0)$ is a constant.

Conversely, using (2.8)

$$\|W^*_{(u,\psi)}f\|_2^2 = \frac{1}{\pi}\int_C |u(0)|^2|f(z/a)|^2 e^{-|z|^2}dA(z) = |u(0)|^2\|f\|_2^2. \tag{2.10}$$

Therefore, by Proposition 2.4, the operator preserves tight frame with $\lambda = |u(0)|$.

(b) By Proposition 2.4, the operator preserves normalized tight frame if and only if $\|W^*_{(u,\psi)}f\|_2^2 = \|f\|_2^2$ for all functions $f \in \mathcal{F}_2$. From this and part (i), we observe that the additional necessity $\lambda = |u(0)| = 1$ holds. Setting $|u(0)| = 1$ in (2.10), the sufficiency follows.

To prove the statements in (ii)and (iii), first note that if $W^*_{u,\psi}$ preserves tight frames, then by Proposition 2.4 and Theorem 2.5

$$\|W_{(u,\psi)}K_w\|_2 = |u(0)|\|K_{w-\overline{a}b}\|_2 = \lambda\|K_w\| \tag{2.11}$$

for some $\lambda > 0$ and all $w \in \mathbb{C}$. Setting the norms of the corresponding kernel expressions, we observe that (2.11) holds for all $w \in \mathbb{C}$ only if $b = 0$ and $\lambda = |u(0)|$. The rest of the proof goes as in part (i) with little modifications. □

For the composition operator, our results can be simplified to the following series of equivalent statements.

**Corollary 2.7.** *Let $\psi$ be nonzero entire function on $\mathbb{C}$ and the composition operator $C_\psi$ is bounded on $\mathcal{F}_2$ and hence $\psi(z) = az + b$, $|a| \leq 1$ and $b = 0$ whenever $|a| = 1$. Then the following statements are equivalent.*

(i) *$C_\psi$ preserves frames on $\mathcal{F}_2$;*
(ii) *$C_\psi$ preserves tight frames on $\mathcal{F}_2$;*
(iii) *$C_\psi$ preserves normalized tight frames on $\mathcal{F}_2$;*
(iv) *$|a| = 1$;*
(v) *$C_\psi$ preserves Riesz basis on $\mathcal{F}_2$;*
(vi) *$C^*_\psi$ preserves frames on $\mathcal{F}_2$;*
(vii) *$C^*_\psi$ preserves tight frames on $\mathcal{F}_2$;*
(viii) *$C^*_\psi$ preserves normalized tight frames on $\mathcal{F}_2$;*
(ix) *$C^*_\psi$ preserves Riesz basis on $\mathcal{F}_2$.*

Unlike the conditions in Theorem 2.5, the corresponding conditions in Corollary 2.7 do not depend on the required specific frame properties to be preserved.

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