FORMS OF EPISTEMIC FEEDBACK

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Feedback in a digital learning system has two functions, a pragmatic and an epistemic one. In previous papers, we were concerned with the pragmatic function, showing how digital feedback initiates acting with the system in groups of students. This paper addresses the epistemic function of this feedback by describing how feedback supports understanding algebraic principles. We report on an investigation of learning algebraic rules conducted with a virtual manipulative learning system, distinguishing, identifying and describing different forms of epistemic feedback, that is, how feedback provided by the learning system fosters disclosing and using algebraic principles.

Keywords: Algebra, epistemic feedback, integer arithmetic, representations, virtual manipulatives.

INTRODUCTION

Manipulatives have long been used in mathematics education to provide concrete representations of mathematical concepts. Symbols, however, are the primary representations used in mathematics, and students must become familiar with their use. We have been researching how to support this transition using digital feedback in a multimodal algebra learning system. In previous papers, we have focussed on the pragmatic function of feedback (Bikner-Ahsbahs et al., 2020, 2021), showing that digital feedback evokes students' actions by a feedback loop constituted in the learning groups. This paper reports on a complementary study in which we focus on the epistemic function of feedback in the same system. That is, we are investigating if our conceptually designed forms of feedback actually support the understanding of algebraic principles. Our report illustrates several kinds of epistemic feedback in the system, focussing on a virtual manipulative environment for integer addition. In this case study, we show particularly how this feedback allowed two boys to connect actions on the manipulatives to symbolic representations, in the process identifying and using several algebraic principles. We answer the research question of how the different kinds of feedback may foster disclosing and using algebraic principles.

THEORETICAL FRAME

As Duval (2006) notes, "mathematical processing always involves substituting some semiotic representation for another" (p. 107). Here we are interested in the substitution of symbolic representations for action on (virtual) physical objects. We make use of the ideas from research on representations by Kaput and Duval.

Kaput (1998) notes that already twenty years ago, "bi-directional links between pairs of the traditional … (numerical, graphical and character-string) notations have dominated the attention of educators and researchers" (p. 272) and that this focus informed software design. One example is the SimCalc project (see, e.g., Roschelle et al., 2000), in which actions on graphs affected other graphs and motions of cartoon figures. A danger of such environments, Kaput notes, is that, if the representations only refer to one another and not to the students' experiences in the world, any fluency in translation between them is empty.

Duval (2006) uses the concept of registers to clarify the kinds of transformations of representations that occur. When two different representations are linked, for example, symbols and manipulatives, two kinds of transformations can occur. Duval calls transformations that occur within a single register, that is, transformations of the manipulatives or of the symbols, 'treatments'. Transformations that occur between registers, he calls 'conversions'.

These transformations are central to learning using models. As Yopp (2018) puts it:

> The concept is conceived by individual learners through:
>
> • Working with objects that fall under the concept
>
> • Working with representations of those objects, and
>
> • Working with relationships between these objects and their representations. (p. 45)

THE MAL-SYSTEM

We are involved in a research project, the Multimodal Algebra Learning (MAL) project, with the goal of developing an "intelligent" system of algebra tiles with the ability to give feedback to its users (see, e.g., Janßen et al. 2017, 2019, 2020; Janßen & Döring, 2017; Janßen, 2017; Reinschlüssel et al. 2018). We developed an app with virtual algebra tiles for testing various features. The app provides the same visual and symbolic feedback as is planned for the physical tiles. It is based on a balance model, which associates the physical act of placing or removing objects on each side of the balance with the mathematical operations of adding to and subtracting from each side of an equation. Like a physical balance, the app provides feedback if the quantities on the two sides differ, but unlike a physical balance, the MAL-system can handle negative and unknown quantities.

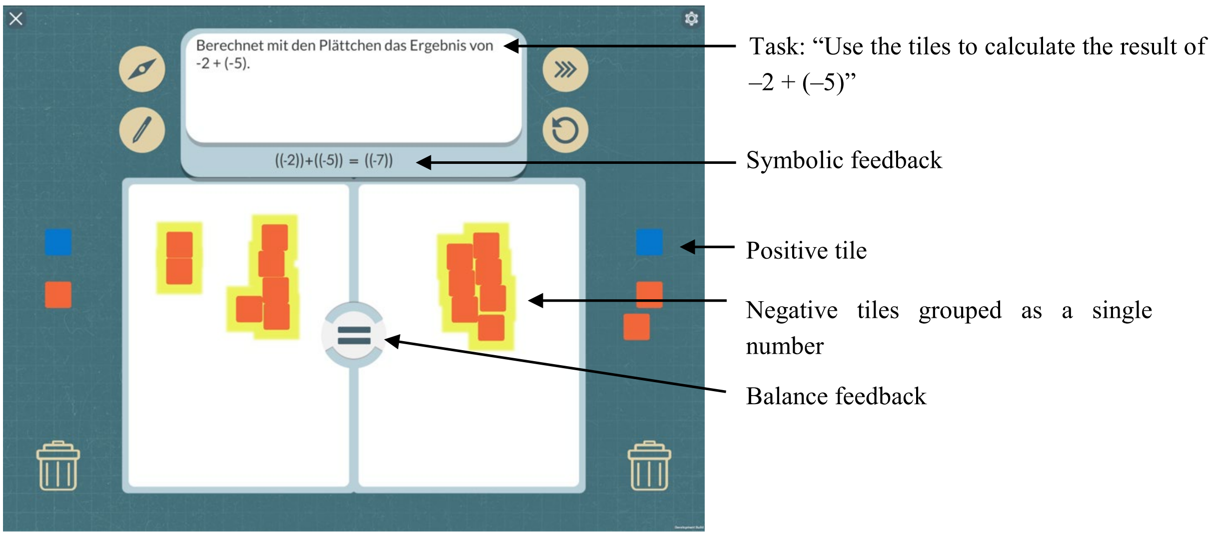

Figure 1. Some key features of the MAL-system

Some key features of the MAL-system are visible in Figure 1. Red square tiles, representing negative numbers, have been placed on both sides of the 'mat'. On the left side, they are grouped into a group of 5 tiles representing (–5) and a group of two tiles representing (–2). Because they are both on the same side, they are considered to be added, but because they are in two groups, the addends are represented, not the sum. On the right side, the tiles represent the sum (–7). The "balance feedback", the equal sign between the two sides, provides feedback that the two sides are equal. If they are not, it becomes a not-equal sign. Above the mat is the statement of the task and the symbolic feedback: "((–2))+((–5))=((–7))". The double parentheses are due to a bug in the programming. This symbolic feedback updates as tiles are regrouped, added or removed, but it cannot be changed directly. Both the balance feedback and the symbolic feedback are always visible.

To represent an equation like 3 – (3) = 0, a special zone is used, the subtraction zone (see Figure 2). Tiles in this zone are considered to be subtracted from the tiles outside it, on that side. In the symbolic feedback, the symbolic form of the tiles in a subtraction zone are always enclosed in parentheses to indicate that the subtraction sign is not a negative sign. The zone outside the subtraction zone is referred to as the 'addition zone'. Red tiles represent negative numbers that are added, while blue tiles in a subtraction zone represent positive numbers that are subtracted, making the distinction between these two situations visually evident.

In Duval's (2006) terms, the MAL-system operates in two registers, the tiles register and the symbolic register. The symbolic feedback provides a 'conversion' from the tiles register to the symbolic register. Manipulations of the tiles involve 'treatments' in the tiles register, and the balance feedback operates in that register as an indication that the manipulations are legitimate ones in that register.

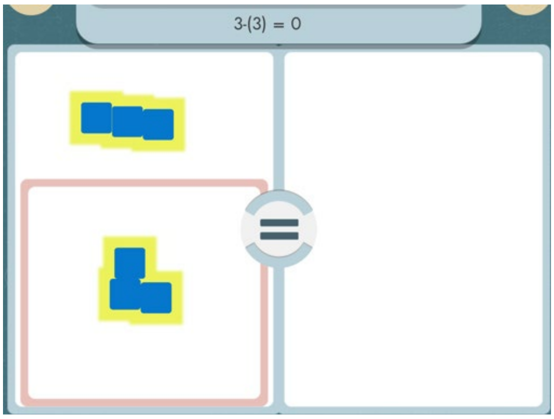

Figure 2. A Subtraction Zone used to represent 3 – 3 = 0

METHODS

The data analysed here come from a larger study exploring students' reactions to features in the MAL-system. Four pairs of Grade 5 students (aged about ten years old) from a German Gymnasium (upper stream secondary school) were given tasks about addition and subtractions of integers in a clinical interview setting outside the classroom (Hunting, 1997). The pairs were videotaped. We focus here on two boys, Simon and Timo, who were more vocal than the other pairs.

RESULTS AND ANALYSIS

In this section, we will describe results related to each kind of feedback and implications of the boys' reactions to that feedback. The boys' comments quoted below have been translated from the original German, and their names are pseudonyms.

The Tiles as Objects

One form of feedback that is not immediately obvious is the identification of the virtual tiles as behaving like physical objects that can be moved, and that have important properties like being countable and preserving quantity when rearranged. In the first task, only blue tiles were available, and the boys were asked to place a single tile and observe what happened. The boys placed three tiles, predicting that the symbolic feedback would show "3", and were surprised when it showed "1+1+1". We interpreted their surprise as an indication that they expected the symbolic feedback to count the tiles in the same way they would, which in turn indicates that they expected the tiles to be countable in the same way that physical tiles are. As the boys placed more tiles and rearranged them, the virtual tiles continued to behave as if they were physical objects. This meant that the boys could *make use of their prior experiences with physical objects* when working with the MAL-system.

Grouping Feedback

As they rearranged the tiles in Task 1, the boys accidentally brought two of them close enough that the yellow outline marking a group was activated, and the symbolic feedback changed. They then moved all the tiles together. Simon commented, "When you move them together, then they are counted together as one number", indicating an understanding of how grouping works in the app. This suggests that the grouping feedback, in conjunction with the symbolic feedback, had allowed the *grouping convention to become transparent* (Meira, 1998) to the boys.

Balance Feedback

The first part of Task 2 states: "Represent the number 5 on the left side of the mat." As the boys did so, Timo commented, "At the beginning, there was an equal sign in the middle, that means it's equal." We believe he had observed that when no tiles are present, the balance feedback is "=" and that as soon as they started adding tiles to the left side, it changed to "≠". The next part of the task reads: "So far, you see an unequal sign in the middle. Move tiles from the stock to the right side and observe what happens. What do you have to do to make the unequal sign an equal sign?" Timo conjectured that the sign would again be an equal sign "when there is equally many in there" and confirmed his conjecture by moving two tiles from the left side to the right side and adding one tile to the right side from the stock, making three tiles on each side. Here we see the linking of the balance feedback to the state of the tiles on the two sides. Here Timo seems to understand the use of *the equal sign to represent the equality of two quantities*, which is fundamental to algebraic thinking. This understanding of the equal sign was not the only one they used, however. Later, in Tasks 4 and 5, Simon seemed to interpret the equal sign as an "instruction to calculate" (Behr et al., 1980).

In Task 3, which asked the boys to create a subtraction zone and explore how it worked, Timo again referred to the balance feedback. They had placed a tile on one side and made a subtraction zone. Timo placed a tile in it and said, "and when I put that in here, then it's equal. It's exactly equal and it shows an equal sign." The interviewer asked him why, and he explained, "Because one minus one is zero and zero is the same as zero, and there, there's zero." It is interesting that he does not simply say "Because one minus one is zero", which uses the equal sign to separate a calculation and its result, but continues "zero is the same as zero" *using the equal sign to mark equality,* which according to Knuth et al. (2008) is a prerequisite for further algebraic understanding. Here Timo's treatment in the tiles register supported his interpretation of the balance feedback within that register; no conversion to the symbolic register was involved.

Tile Colours for Representing the Sign

In the first five tasks, only blue tiles were available. In Task 6, red tiles were added to the store, and Timo immediately noticed them. The task asked them to find the result of 2 – 3. Timo remarked, "I think it could still work because it could be possible that the red tile is negative", and Simon agreed, "I think so, too". The boys followed the procedure they had used in previous tasks to find the positive difference between two numbers. On each side, they placed three blue tiles in a subtraction zone and two blue tiles outside it. This gave them a starting position in which the two sides were visually equal, and the balance feedback indicated “=”. They had learned that removing one tile from a subtraction zone and one from outside it preserves the equality. They did this twice on the right side, leaving one tile in the subtraction zone and none outside it.

The instruction in the task then told them to place a red tile and a blue tile together outside the subtraction zone. As he placed the tiles, Timo commented, “I think it stays equal,” adding, “yes!” when the balance feedback remained “=”. The balance feedback, operating in the symbolic register, provides evidence that a red-blue pair has a value of 0 in the tiles register. The boys made a direct connection between the tiles and integers. When the interviewer summarised “a red and a blue together always gives zero”, Timo explained, “because a red one is negative”, and Simon commented, “that [refers to a blue tile] is 1 and that [refers to a red tile] is, I think, minus 1.”

Zero Pairs Feedback

In Task 8, a new kind of feedback was introduced. When a red and a blue tile are grouped as a pair, the pair vanishes. They seemed to be aware of this new feedback and its mathematical meaning. Simon observed, “it disappears,” and Timo explained, “The result is 0 and therefore they are only additional tiles which are not needed.” The task then instructed them to place more tiles on the left side, so that the balance feedback stayed “=”. Timo immediate stated “simply zero pairs.” At first, they had difficulty placing zero pairs because the new feedback immediately removed them, but Timo realised that if they were not placed together, they would remain, thus identifying the principle of *adding a number and its additive inverse cancel each other***.**

Symbolic Feedback

The most important feedback in terms of making a connection between the boys’ actions on the manipulative and symbolic representations is the symbolic feedback, but this connection is made with the support of the other kinds of feedback we have already discussed.

The boys had noticed the symbolic feedback already in Task 1, when they saw that it showed a sum when tiles were placed apart on the mat, and a single number when they were grouped. However, in Task 2, in which they were to make as many representations of 5 as possible, the interviewer had to explain that “representations” referred to what was written in the symbolic feedback.

In Task 3, Timo was clearly attending to the symbolic feedback as he commented on two different symbolic expressions. They had placed the tiles as shown in Figure 2, when Timo remarked, “before there was a brackets calculation” and separated the group of tiles in the subtraction zone into a pair and a single tile. This changed the symbolic feedback to “3 – (2+1) = 0” and Timo noted, “with brackets.” We believe he had noticed the symbolic feedback earlier, when they were first placing the tiles in the subtraction zone and had not yet grouped them into a group of three.

The boys were also aware of the relationship between the symbolic feedback and the tiles when they began Task 4, which told them to begin by placing tiles to show “3 – 2” on both sides. They placed the correct tiles, but due to a bug in the software, the symbolic feedback did not show “3 – 2 = 3 – 2” as expected. They restarted the task and proceeded once the symbolic feedback was what they expected. They also referred directly to the symbolic feedback at the end of the task, when they were trying to rearrange tiles to change the symbolic feedback from “2 + (–2) – (1)” to a single number.

In Task 8, Timo explained why the symbolic feedback displays “3 + ((–3))” when they have a group of three blue tiles together with a group of three red tiles. He said, “So there appears a double bracket because, otherwise, it would calculate plus minus three, and that would not make sense, and therefore there appears the bracket because it is another calculation.” We believe he is saying that writing “plus minus three,” that is, “3 + –3” “would not make sense.” His exact meaning is not clear, but he is certainly relating the symbolic feedback to the tiles and to the arithmetic operations on natural numbers that he is familiar with. The brackets are doubled due to a bug in the software, but the boys did not remark on the doubling.

USE OF FEEDBACK IN ESTABLISHING NEW KNOWLEDGE

In Task 9, the boys were to represent the number –5 on the left side and then on the right side, find different possibilities for representing the same number. Simon began placing five red tiles on the left side to represent (–5) while at the same time, Timo created a subtraction zone on the right side and started placing blue tiles in it to represent 0 – 5. They next placed another blue tile in the subtraction zone and a matching tile in the addition zone on the right side to represent 1 – 6. Timo then wondered, "What happens then if I …" and he placed a red tile in the addition zone. The balance feedback showed that the two sides were no longer equal. Timo removed one blue tile from the subtraction zone, which restored the equality. He said, "That works, too. Because they are negative, they don't need to be in the subtraction zone". Simon agreed, "That would also work," and he removed another blue tile and added another red one to produce (–5) = (–2) – (3). Timo explained, "Yes, that would work too because it results in zero." This is an interesting remark. Unlike a zero pair, which gives the feedback that it is equal to zero by disappearing, the zero here is indicated more abstractly, by the fact that the balance feedback remains equal when a blue tile in the subtraction zone is exchanged for a red tile in the addition zone. Here the boys used the interaction of feedback from the tiles, the symbol feedback and the balance feedback to conclude that, as Simon put it, "We always place one more red tile in the addition zone, then we take one blue away. That stays always the same." He also noted, "The same works the other way around" and removed a red tile, placing a blue tile in the subtraction zone, observing that the balance feedback continued to indicate equality. The boys had discovered an important principle in integer arithmetic, that *subtraction of a number and addition of the opposite number are equivalent.*

CONCLUSIONS

The different kinds of feedback offered in the MAL-system allowed the boys to make links between their experiences of physical objects, the virtual tiles and symbolic expressions.

Conversions from the symbolic register to the tiles register and back again occurred throughout. Because the tasks presented symbolic representations, the boys converted to the tiles register when setting up the tasks. They were guided in this by their connecting the virtual tiles to real objects. The symbolic feedback provided a check on this process, converting the arrangements of tiles the boys produced back into the symbolic register. This feedback was combined with the grouping feedback to establish what representations in the tiles register correspond to numbers, and how addition is represented in the tiles register.

The subtraction zone does not correspond to a real-world object, and so establishing its meaning involved comparison of the symbolic feedback and the tiles representation. In Task 4, the boys said that tiles representing 3 – 2 on one side of the mat are equal (according to the balance feedback) to a single tile on the other side. This fits with the boys' real-world experience of subtraction as removing two objects from a set of three. The balance feedback in the tiles register gave them immediate confirmation that the subtraction zone representation corresponded to subtraction as taking away.

The balance feedback was also important, in Task 6, in establishing that a red tile is the opposite of a blue tile. The boys saw that a red-blue pair is equal to an empty side representing zero, and hence that the red tile represents a negative number. When zero pairs feedback was introduced in Task 8, this identification was reinforced.

In addition to using the feedback to establish how to convert between representations, there were also occasions where the boys used the feedback to reach conclusions. One example of this is in Task 9. They could explain why exchanging subtraction of a positive number corresponds to the addition of a negative number, but they also relied on the balance feedback to confirm that the action in the tiles register of exchanging a blue tile in the subtraction zone for a red tile in the addition zone, produces an equivalent symbolic expression.

The feedback in the MAL-system offers support for mathematics learning in a number of ways. It helps to establish the meanings of representations (through conversions), it supports the definition of new objects and operations (through conversions and treatments), and it helps to confirm hypotheses, primarily through conversions; hence, the feedback functions in various epistemic directions.

Our study shows that the combination of manipulative material, and feedback offered by software, makes it possible to overcome the danger pointed to by Kaput, of conversions between registers becoming a meaningless game. We also believe that the feedback offered by the MAL-system allowed Simon and Timo to discover for themselves an important principle in integer arithmetic, that *subtracting a number is equivalent to adding its opposite*. In contrast, using physical manipulatives alone require interventions by a teacher to say how representations should be interpreted. That is, physical manipulatives provide pragmatic feedback, while the digital feedback in the MAL-system provides both pragmatic and epistemic feedback.

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