Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae 31 (2020) 714–727

indagationes mathematicae

www.elsevier.com/locate/indag

# Optimal error estimate of the finite element approximation of second order semilinear non-autonomous parabolic PDEs

Antoine Tambue[a,b,c,*], Jean Daniel Mukam[d]

[a] Department of Computer science, Electrical engineering and Mathematical sciences, Western Norway University of Applied Sciences, Inndalsveien 28, 5063 Bergen, Norway

[b] Center for Research in Computational and Applied Mechanics (CERECAM), and Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, South Africa

[c] The African Institute for Mathematical Sciences (AIMS) of South Africa, 6–8 Melrose Road, Muizenberg 7945, South Africa

[d] Fakultät für Mathematik, Technische Universität Chemnitz, 09126 Chemnitz, Germany

Received 27 January 2020; received in revised form 11 June 2020; accepted 13 June 2020

Communicated by J.B. van den Berg

**Abstract**

In this work, numerical approximation of the second order non-autonomous semilinear parabolic partial differential equations (PDEs) is investigated using the classical finite element method. To the best of our knowledge, only the linear case is investigated in the literature. Using an approach based on evolution operator depending on two parameters, we obtain the error estimate of the semi-discrete scheme based on finite element method toward the mild solution of semilinear non-autonomous PDEs under polynomial growth and one-sided Lipschitz conditions of the nonlinear term. Our convergence rate is obtained with general non-smooth initial data, and is similar to that of the autonomous case. Such convergence result is very important in numerical analysis. For instance, it is one step forward for numerical approximation of non-autonomous stochastic partial differential equations with the finite element method.



*Keywords:* Non-autonomous parabolic partial differential equations; Finite element method; Error estimate; Two parameters evolution operator; one-sided Lipschitz condition; Polynomial growth condition

* Corresponding author at: Department of Computer science, Electrical engineering and Mathematical sciences, Western Norway University of Applied Sciences, Inndalsveien 28, 5063 Bergen, Norway.

*E-mail addresses:* antonio@aims.ac.za (A. Tambue), jean.d.mukam@aims-senegal.org (J.D. Mukam).

https://doi.org/10.1016/j.indag.2020.06.008

## 1. Introduction

Nonlinear partial differential equations are powerful tools in modelling real-world phenomena in many fields such as in geo-engineering. For instance oil and gas recovery from hydrocarbon reservoirs and mining heat from geothermal reservoirs can be modelled by nonlinear equations with possibly degeneracy appearing in the diffusion and transport terms. Since explicit solutions of many PDEs are rarely known, numerical approximations are powerful tools to provide realistic approximations. Approximations are usually done at two levels, namely space and time approximations. Only time approximations received so far some attentions (see for example [9] and reference therein), while the space approximation has been lacked. Our goal is to fill that gap by focusing on spatial approximation of the following advection–diffusion equation with a nonlinear reaction term using the finite element method

$$\frac{\partial u}{\partial t} = \mathcal{A}(t)u + F(t,u), \quad u(0)=u_0, \quad t\in(0,T], \quad T>0. \tag{1}$$

The mild solution is sought in the Hilbert space $H = L^2(\Lambda)$, where $\Lambda$ is assumed to have smooth boundary or is a convex polygon of $\mathbb{R}^d$ ($d = 1, 2, 3$). The second order differential operator $\mathcal{A}(t)$ is given by

$$\mathcal{A}(t)u = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(q_{i,j}(t,x)\frac{\partial u}{\partial x_j}\right) - \sum_{j=1}^{d} q_j(t,x)\frac{\partial u}{\partial x_j} + q_0(t,x)u, \tag{2}$$

where $q_{i,j}$, $q_j$ and $q_0$ are smooth coefficients on the spatial variable, i.e. $q_{i,j}, q_j, q_0 \in C^1(\Lambda)$, $i, j = 1,\dots,d$. We also assume $q_{i,j}, q_j, q_0$ to be time differentiable (i.e. for all $x \in \Lambda$ $q_{i,j}(;,x), q_j(.,x), q_0(.,x) \in C^1((0,T),\mathbb{R})$) and there exist $c_1 \geq 0$, $0 < \gamma \leq 1$ such that

$$|q_{i,j}(t,x) - q_{i,j}(s,x)| \leq c_1|t-s|^{\gamma}, \quad x\in\Lambda,\ t,s\in[0,T],\ i,j\in\{1,\dots,d\}.$$

Moreover, $q_{i,j}$ is assumed to satisfy the following ellipticity condition

$$\sum_{i,j=1}^{d} q_{i,j}(t,x)\xi_i\xi_j \geq c|\xi|^2, \quad (t,x)\in[0,T]\times\overline{\Lambda}, \tag{3}$$

where $c > 0$ is a constant. The finite element approximation of (1) with constant linear operator $\mathcal{A}(t) = \mathcal{A}$ are widely investigated in the scientific literature, see example [1,4,8,13] and the references therein. The finite volume method for $\mathcal{A}(t) = \mathcal{A}$ was recently investigated in [11]. If we turn our attention to the non-autonomous case, the list of references becomes remarkably short. For the linear homogeneous case ($F(t,u) = 0$), the finite element approximation has been investigated in [6], [1, Chapter III, Section 14.2]. The linear inhomogeneous version of (1) ($F(t,u) = f(t)$) was investigated in [5–7], [1, Chapter III, Section 12] and the references therein. To the best of our knowledge, the nonlinear case is not yet investigated in the scientific literature. This paper fills that gap by investigating the error estimate of the finite element method of (1) with a nonlinear term $F(t,u)$. The challenge increases when the nonlinear function satisfies the polynomial growth and the one-sided Lipschitz conditions than satisfying the global Lipschitz condition, which is restrictive and excludes many problems such as Allen–Cahn equation, for which $F(u) = u - u^3$. Our strategy is based on the introduction of an auxiliary problem, namely (65) lying on two parameters evolution operator and exploits carefully its smooth regularity properties. Our key intermediate result in Lemma 3.1 generalizes [13, Theorem 3.5] to semilinear problems with time dependent and not necessary self-adjoint operators. Furthermore, it also generalizes [13, Theorem 4.2], the results in [1, Chapter III, Section 12] and in [5–7] to smooth and non-smooth initial data. Note that Lemma 3.1 for non-smooth initial data is of great importance in numerical analysis. Our result is very useful while studying the convergence of the finite element method for many nonlinear problems, including stochastic partial differential equations(SPDEs), see for example [2,3,14] and references therein for time independent SPDEs. In fact, in the case of SPDEs, due to the Itô-isometry or the Burkhölder Davis–Gundy inequality, the non-smooth version of Lemma 3.1 cannot be applied since it brings degenerated integrals, which cause difficulties in the error estimates or reduce considerably the order of convergence. Hence, our result is more general than the existing results (even for linear problems) and has many applications. To sum up, our main contribution and main difficulties lie on

- Establishing Lemma 3.1, which is our key ingredient. Such results for autonomous problems with self-adjoint linear operator were thoroughly investigated in [13], revealing the importance of such estimate in numerical analysis. Here, the non-autonomous and not necessarily self-adjoint version is investigated.
- Estimating the semi-discrete error for semilinear problem with nonglobal Lipschitz condition (Theorem 3.2), which is more challenging and includes more realistic problems such as Allen–Cahn equation.

The convergence rate achieved for semilinear problem is in agreement with many results in the literature for autonomous problems and for non-autonomous linear problems. More precisely, we achieve convergence order $\mathcal{O}\left(h^2t^{-1+\beta/2}+h^2\left(1+\ln(t/h^2)\right)\right)$ or $\mathcal{O}(h^\beta)$, where $\beta$ is a regularity parameter defined in Assumption 2.1. Furthermore, under optimal regularity of the nonlinear function $F$ or under a linear growth assumption on $F$, we achieve optimal convergence order $\mathcal{O}(h^2t^{-1+\beta/2})$. Following [11] and using the similar approach based on the two parameters evolution operator, this work can be extended to the finite volume method. The rest of this paper is structured as follows. In Section 2, the well-posedness result is provided along with the finite element approximation. The error estimate is analysed in Section 3 for both global Lipschitz nonlinear term and polynomial growth nonlinear term.

## 2. Mathematical setting and numerical approximation

### 2.1. Notations, settings and well-posedness problem

We denote by $\|\cdot\|$ the norm associated to the inner product $\langle\cdot,\cdot\rangle_H$ in the Hilbert space $H=L^2(\Lambda)$. We denote by $\mathcal{L}(H)$ the set of bounded linear operators in $H$. Let $\mathcal{C}:=\mathcal{C}(\overline{\Lambda},\mathbb{R})$ be the set of continuous functions equipped with the norm $\|u\|_{\mathcal{C}}:=\sup_{x\in\overline{\Lambda}}|u(x)|$, $u\in\mathcal{C}$. To ensure the well posedness of (1), we make the following assumptions.

**Assumption 2.1.** The initial data $u_0$ belongs to $\mathcal{D}\left((-A(0))^{\frac{\beta}{2}}\right)$, $0\leq\beta\leq 2$.

**Assumption 2.2.** The nonlinear function $F:[0,T]\times H\longrightarrow H$ is Lipschitz continuous, i.e. there exists a constant $K$ such that

$$\|F(t,v)-F(s,w)\|\leq K(|t-s|+\|v-w\|),\quad s,t\in[0,T],\quad v,w\in H. \tag{4}$$

We introduce two spaces $\mathbb{H}$ and $V$, such that $\mathbb{H}\subset V$, depending on the boundary conditions of $-\mathcal{A}(t)$. For Dirichlet boundary conditions, we take $V=\mathbb{H}=H_0^1(\Lambda)$. For Robin boundary condition, we take $V=H^1(\Lambda)$ and

$$\mathbb{H}=\{v\in H^2(\Lambda):\partial v/\partial v_{\mathcal{A}}+\alpha_0 v=0,\quad \text{on}\quad \partial\Lambda\},\quad \alpha_0\in\mathbb{R}, \tag{5}$$

where $\partial v/\partial v_{\mathcal{A}}$ stands for the differentiation along the outer conormal vector $v_{\mathcal{A}}$. One can easily check (see [1, Chapter III, (11.14′)]) that the bilinear operator $a(t)$, associated to $-\mathcal{A}(t)$ defined by $a(t)(u, v) = \langle -\mathcal{A}(t)u, v\rangle_H$, $u \in \mathcal{D}(\mathcal{A}(t))$, $v \in V$ satisfies

$$a(t)(v, v) \geq \lambda_0 \|v\|_1^2 - c_0\|v\|^2, \quad v \in V. \tag{6}$$

with $\lambda_0 > 0$ and $c_0 \in \mathbb{R}$. By adding and subtracting $c_0 u$ on the right hand side of (1), we have a new linear operator that we still call $\mathcal{A}(t)$ corresponding to the new bilinear form that we still call $a(t)$ such that the following coercivity property holds

$$a(t)(v, v) \geq \lambda_0 \|v\|_1^2, \quad v \in V, \quad t \in [0, T], \tag{7}$$

where $\lambda_0$ is a positive constant, independent of $t$. Note that $a(t)(\cdot, \cdot)$ is bounded in $V \times V$ ([1, Chapter III, (11.13)]), so the following operator $A(t) : V \to V^*$ is well defined

$$a(t)(u, v) = \langle -A(t)u, v\rangle \quad u, v \in V, \quad t \in [0, T],$$

where $V^*$ is the dual space of V and $\langle \cdot, \cdot\rangle$ the duality pairing between $V^*$ and $V$. Identifying $H$ to its adjoint space $H^*$, we have the following continuous and dense inclusions

$$V \subset H \subset V^*, \quad \text{and therefore} \quad \langle u, v\rangle_H = \langle u, v\rangle, \quad u \in H, \quad v \in V.$$

So if we want to replace $\langle \cdot, \cdot\rangle$ by the scalar product of $\langle \cdot, \cdot\rangle_H$ on $H$, we therefore need to have $A(t)u \in H$, for $u \in V$. So the domain of $-A(t)$ is defined as

$$D := \mathcal{D}(-A(t)) = \mathcal{D}(A(t)) = \{u \in V,\ A(t)u \in H\}.$$

It is well known (see [1, Chapter III, (11.11) & (11.11′)]) that in the case of Dirichlet boundary conditions $D = H_0^1(\Lambda) \cap H^2(\Lambda)$ and in the case of Robin boundary conditions $D = \mathbb{H}$ in (5). We write the restriction of $A(t) : V \longrightarrow V^*$ to $\mathcal{D}(A(t))$ again $A(t)$ which is therefore regarded as an operator of $H$ (more precisely the $H$ realization of $\mathcal{A}(t)$).

The coercivity property (7) implies that $-A(t)$ is a positive operator and its fractional powers are well defined [1,4]. The following equivalence of norms holds [1,4]

$$\|v\|_{H^\alpha(\Lambda)} \equiv \|((-A(t))^{\frac{\alpha}{2}})v\| := \|v\|_\alpha,\ v \in \mathcal{D}((-A(t))^{\frac{\alpha}{2}}) \cap H^\alpha(\Lambda), \quad \alpha \in [0, 2]. \tag{8}$$

It is well known that the family of operators $\{A(t)\}_{0\leq t\leq T}$ generates a family of two parameters operators $\{U(t, s)\}_{0\leq s\leq t\leq T}$ (see for example [10] or [1, Page 832]). The evolution equation (1) can be written as follows

$$\frac{du(t)}{dt} = A(t)u(t) + F(t, u(t)), \quad u(0) = u_0, \quad t \in (0, T]. \tag{9}$$

The following theorem provides the well-posedness of problem (1) (or (9)).

**Theorem 2.1** (*[10]*). *Let Assumption 2.2 be fulfilled. If $u_0 \in H$, then the initial value problem (1) has a unique mild solution $u(t)$ given by*

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, s)F(s, u(s))ds, \quad t \in (0, T]. \tag{10}$$

*Moreover, if Assumption 2.1 is fulfilled, then the following space regularity holds*[1]

$$\|(-A(s))^{\frac{\beta}{2}}u(t)\| + \|F(u(t))\| \leq C\left(1 + \|(-A(s))^{\frac{\beta}{2}}u_0\|\right), \quad \beta \in [0, 2), \quad s, t \in [0, T]. \tag{11}$$

[1] This estimate also holds when $u$ is replaced by its semi-discrete version $u^h$ defined in Section 2.2.

### 2.2. Finite element discretization

Let $\mathcal{T}_h$ be a triangulation of $\Lambda$ with maximal length $h$. Let $V_h \subset V$ denote the space of continuous and piecewise linear functions over the triangulation $\mathcal{T}_h$. We define the projection $P_h$ from $H = L^2(\Lambda)$ to $V_h$ by

$$\langle P_h u, \chi\rangle_H = \langle u, \chi\rangle_H, \quad \chi \in V_h,\ u \in H. \tag{12}$$

For any $t \in [0, T]$, the discrete operator $A_h(t) : V_h \longrightarrow V_h$ is defined by

$$\langle A_h(t)\phi, \chi\rangle_H = -\left\langle(-A(t))^{1/2}\phi, (-A(t))^{*1/2}\chi\right\rangle_H = -a(t)(\phi, \chi), \quad \phi, \chi \in V_h. \tag{13}$$

Note that $(-A(t))^{*1/2}$ stands for the adjoint of $(-A(t))^{1/2}$.

The space semi-discrete version of problem (9) consists of finding $u^h(t) \in V_h$ such that

$$\frac{du^h(t)}{dt} = A_h(t)u^h(t) + P_h F(t, u^h(t)), \quad u^h(0) = P_h u_0, \quad t \in (0, T]. \tag{14}$$

For $t \in [0, T]$, we introduce the Ritz projection $R_h(t) : V \longrightarrow V_h$ defined by

$$\langle -A(t)R_h(t)v, \chi\rangle_H = a(t)(v, \chi), \quad v \in V, \quad \chi \in V_h. \tag{15}$$

It is well known (see e.g. [6, (3.2)] or [1]) that the following error estimate holds

$$\|R_h(t)v - v\| + h\|R_h(t)v - v\|_{H^1(\Lambda)} \leq Ch^r\|v\|_{H^r(\Lambda)}, \quad v \in V \cap H^r(\Lambda), \quad r \in [1, 2]. \tag{16}$$

The following error estimate also holds (see for example [6, (3.3)] or [1])

$$\|D_t\left(R_h(t)v - v\right)\| + h\|D_t\left(R_h(t)v - v\right)\|_{H^1(\Lambda)} \leq Ch^r\left(\|v\|_{H^r(\Lambda)} + \|D_t v\|_{H^r(\Lambda)}\right), \tag{17}$$

for any $r \in [1, 2]$ and $v \in V \cap H^r(\Lambda)$, where $D_t := \frac{\partial}{\partial t}$ and $D_t R_h(t) = R'_h(t)$ is the time derivative of $R_h$. According to the generation theory, $A_h(t)$ generates a two parameters evolution operator $\{U_h(t, s)\}_{0\leq s\leq t\leq T}$ (see for example [1, Page 839]). Therefore the mild solution of (14) can be written as follows

$$u^h(t) = U_h(t, 0)P_h u_0 + \int_0^t U_h(t, s)P_h F(s, u^h(s))ds, \quad t \in [0, T]. \tag{18}$$

In the rest of this paper, $C \geq 0$ stands for a constant independent of $h$, that may change from one place to another. It is well known (see example. [1, Chapter III, (12.3) & (12.4)]) that for any $0 \leq \gamma \leq \alpha \leq 1$ and $0 \leq s < t \leq T$, the following estimates hold[2]

$$\|(-A_h(t))^{\alpha}U_h(t, s)\|_{\mathcal{L}(H)} \leq C(t - s)^{-\alpha}, \quad \|U_h(t, s)(-A_h(s))^{\alpha}\|_{\mathcal{L}(H)} \leq C(t - s)^{-\alpha}. \tag{19}$$

## 3. Main result

### 3.1. Preliminaries result

We consider the following linear homogeneous problem: find $w \in D \subset V$ such that

$$w' = A(t)w, \quad w(\tau) = v, \quad t \in (\tau, T], \quad \text{with} \quad 0 \leq \tau \leq T. \tag{20}$$

The corresponding semi-discrete problem in space is: find $w_h \in V_h$ such that

$$w'_h(t) = A_h(t)w_h, \quad w_h(\tau) = P_h v, \quad t \in (\tau, T], \quad \text{with} \quad 0 \leq \tau \leq T. \tag{21}$$

The following lemma will be useful in our convergence analysis.

[2] These estimates remain true if $A_h(t)$ and $U_h(t, s)$ are replaced by $A(t)$ and $U(t, s)$ respectively.

**Lemma 3.1.** *Let $r \in [0,2]$ and $\gamma \leq r$. Let Assumption 2.2 be fulfilled. Then the following error estimate holds for the semi-discrete approximation (21)*

$$\begin{aligned}\|w(t)-w_h(t)\| &= \|[U(t,\tau)-U_h(t,\tau)P_h]v\|\\ &\leq Ch^r(t-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma,\ v\in\mathcal{D}\left((-A(0))^{\frac{\gamma}{2}}\right).\end{aligned}$$

**Proof.** We split the desired error as follows

$$w_h(t)-w(t) = (w_h(t)-R_h(t)w(t)) + (R_h(t)w(t)-w(t)) \equiv \theta(t)+\rho(t). \tag{22}$$

Using the definition of $R_h(t)$ and $P_h$ ((12)–(13)), we can prove exactly as in [4] that

$$A_h(t)R_h(t) = P_hA(t), \quad t\in[0,T]. \tag{23}$$

One can easily compute the following derivatives

$$D_t\theta = A_h(t)w_h(t) - D_tR_h(t)w(t) - R_h(t)A(t)w(t), \tag{24}$$

$$D_t\rho = D_tR_h(t)w(t) + R_h(t)A(t)w(t) - A(t)w(t). \tag{25}$$

Endowing $V$ and the linear subspace $V_h$ with the norm $\|.\|_{H^1(\Lambda)}$, it follows from (16) that $R_h(t)\in\mathcal{L}(V,V_h)$, $t\in[0,T]$. By the definition of the differential operator, it follows that $D_tR_h(t)\in\mathcal{L}(V,V_h)$ for all $t\in[0,T]$. Hence $P_hD_tR_h(t) = D_tR_h(t)$ for all $t\in[0,T]$ and it follows from (25) that

$$P_hD_t\rho = D_tR_h(t)w(t) + R_h(t)A(t)w(t) - P_hA(t)w(t). \tag{26}$$

Adding and subtracting $P_hA(t)w(t)$ in (24) and using (23), it follows that

$$D_t\theta = A_h(t)\theta - P_hD_t\rho, \quad t\in(\tau,T], \tag{27}$$

From (24), the mild solution of $\theta$ is given by

$$\theta(t) = U_h(t,\tau)\theta(\tau) - \int_\tau^t U_h(t,s)P_hD_s\rho(s)ds. \tag{28}$$

Splitting the integral part of (28) in two and integrating by parts the first one yields

$$\begin{aligned}\theta(t) &= U_h(t,\tau)\theta(\tau) + U_h(t,\tau)P_h\rho(\tau) - U_h\left(t,(t+\tau)/2\right)P_h\rho\left((t+\tau)/2\right)\\ &\quad+\int_\tau^{(t+\tau)/2}\frac{\partial}{\partial s}\left(U_h(t,s)\right)P_h\rho(s)ds - \int_{(t+\tau)/2}^t U_h(t,s)P_hD_s\rho(s)ds.\end{aligned} \tag{29}$$

Using the expression of $\theta(\tau)$, $\rho(\tau)$ (see (22)) and the fact that $w_h(\tau) = P_hv$, it holds that $\theta(\tau)+P_h\rho(\tau) = 0$. Hence (29) reduces to

$$\begin{aligned}\theta(t) &= -U_h(t,s)P_h\rho((t+\tau)/2) + \int_\tau^{\frac{(t+\tau)}{2}}\frac{\partial}{\partial s}\left(U_h(t,s)\right)P_h\rho(s)ds\\ &\quad-\int_{\frac{(t+\tau)}{2}}^t U_h(t,s)P_hD_s\rho(s)ds.\end{aligned} \tag{30}$$

Taking the norms on both sides of (30) and using (19) yields

$$\begin{aligned}\|\theta(t)\| &\leq C\left\|\rho\left((t+\tau)/2\right)\right\| + \int_\tau^{\frac{(t+\tau)}{2}}\|U_h(t,s)A_h(s)\|_{\mathcal{L}(H)}\|\rho(s)\|ds + \int_{\frac{(t+\tau)}{2}}^t\|D_s\rho(s)\|ds\\ &\leq C\left\|\rho\left((t+\tau)/2\right)\right\| + \int_\tau^{\frac{(t+\tau)}{2}}(t-s)^{-1}\|\rho(s)\|ds + \int_{\frac{(t+\tau)}{2}}^t\|D_s\rho(s)\|ds.\end{aligned} \tag{31}$$

Using (16) and (17), it holds that

$$\|\rho(s)\| \leq Ch^r\|w(s)\|_r, \quad \|D_s\rho(s)\| \leq Ch^r\left(\|w(s)\|_r + \|D_s w(s)\|_r\right). \tag{32}$$

Note that the solution of (20) can be represented as follows.

$$w(s) = U(s,\tau)v, \quad s \geq \tau. \tag{33}$$

Pre-multiplying both sides of (33) by $(-A(s))^{\frac{r}{2}}$ and using (19) yields

$$\begin{aligned}\left\|(-A(s))^{\frac{r}{2}}w(s)\right\| &\leq \left\|(-A(s))^{\frac{r}{2}}U(s,\tau)(-A(\tau))^{-\frac{\gamma}{2}}\right\|_{\mathcal{L}(H)}\left\|(-A(\tau))^{\frac{\gamma}{2}}v\right\| \\ &\leq C(s-\tau)^{-\frac{(r-\gamma)}{2}}\left\|(-A(\tau))^{\frac{\gamma}{2}}v\right\| \leq C(s-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma. \end{aligned}\tag{34}$$

Therefore it holds that

$$\|w(s)\|_r \leq C(s-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma, \quad 0 \leq \gamma \leq r \leq 2, \quad \tau < s. \tag{35}$$

Substituting (35) in (32) yields

$$\|\rho(s)\|_r \leq Ch^r(s-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma. \tag{36}$$

Taking the derivative with respect to $s$ on both sides of (33) yields

$$D_s w(s) = -A(s)U(s,\tau)v. \tag{37}$$

As for (34), pre-multiplying both sides of (37) by $(-A(s))^{\frac{r}{2}}$ and using (19) yields

$$\|D_s w(s)\|_r \leq C(s-\tau)^{-1-\frac{(r-\gamma)}{2}}\|v\|_\gamma. \tag{38}$$

Substituting (35) and (38) in the second estimate of (32) yields

$$\begin{aligned}\|D_s\rho(s)\| &\leq Ch^r\left((s-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma + (s-\tau)^{-1-\frac{(r-\gamma)}{2}}\|v\|_\gamma\right) \\ &\leq Ch^r(s-\tau)^{-1-\frac{(r-\gamma)}{2}}\|v\|_\gamma. \end{aligned}\tag{39}$$

Substituting the first estimate of (32) and (39) in (31) and using (36) yields

$$\begin{aligned}\|\theta(t)\| &\leq Ch^r(t-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma + Ch^r\int_\tau^{\frac{t+\tau}{2}}(t-s)^{-1}(s-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma ds \\ &+ Ch^r\int_{\frac{t+\tau}{2}}^t(s-\tau)^{-1-\frac{(r-\gamma)}{2}}\|v\|_\gamma ds. \end{aligned}\tag{40}$$

Using the estimate

$$\int_\tau^{\frac{t+\tau}{2}}(t-s)^{-1}(s-\tau)^{-\frac{(r-\gamma)}{2}}ds + \int_{\frac{t+\tau}{2}}^t(s-\tau)^{-1-\frac{(r-\gamma)}{2}}ds \leq C(t-\tau)^{-\frac{(r-\gamma)}{2}},$$

it follows from (40) that

$$\|\theta(t)\| \leq Ch^r(t-\tau)^{-\frac{(r-\gamma)}{2}}\|v\|_\gamma. \tag{41}$$

Substituting (36) and (41) in (22) completes the proof of Lemma 3.1. ■

### 3.2. Error estimate of the semilinear problem under global lipschitz condition

**Theorem 3.1.** *Let Assumptions 2.1 and 2.2 be fulfilled. Let $u(t)$ and $u^h(t)$ be defined by (10) and (18) respectively. Then the following error estimate holds*

$$\|u(t) - u^h(t)\| \leq Ch^2t^{-1+\beta/2} + Ch^2(1+\ln(t/h^2)), \quad 0 < t \leq T. \tag{42}$$

*In addition, if the nonlinear term $F$ satisfies the linear growth condition $\|F(t,v)\| \leq C\|v\|$ or if there exists $\delta > 0$ small enough such that $\|(-A(s))^{\delta}F(t,v)\| \leq Ct + C\|(-A(s))^{\delta}v\|$, $s,t \in [0,T]$, $v \in H$, then the following optimal error estimate holds*

$$\|u(t) - u^h(t)\| \leq Ch^2 t^{-1+\beta/2}, \quad 0 < t \leq T, \tag{43}$$

*where $\beta$ is defined in Assumption 2.1.*

**Remark 3.1.** Note that the hypothesis $\|F(t,v)\| \leq C\|v\|$ is not too restrictive. An example of class of nonlinearities for which such hypothesis is fulfilled is a class of functions satisfying $F(t,0) = 0$, $t \in [0,T]$. Concrete examples are operators of the form $F(t,v) = f(t)\frac{v}{1+|v|}$, with $f : [0,T] \longrightarrow \mathbb{R}$ continuous or bounded on $[0,T]$.

**Remark 3.2.** It is possible to obtain an error estimate without irregularities terms of the form $t^{-1+\beta/2}$. The drawback here is that the convergence rate will not be 2, but will depend on the regularity of the initial data. The proof follows the same lines as that of Theorem 3.1 using Lemma 3.1 and we have

$$\|u(t) - u^h(t)\| \leq Ch^{\beta}, \quad t \in [0,T].$$

**Proof.** of Theorem 3.1. We start with the proof of (42). Subtracting (18) from (10), taking the norm on both sides and using the triangle inequality yields

$$\begin{aligned}\|u(t) - u^h(t)\| &\leq \|U(t,0)u_0 - U_h(t,0)P_h u_0\| \\ &\quad + \left\| \int_0^t \left[U(t,s)F(s,u(s)) - U_h(t,s)P_h F\left(s,u^h(s)\right)\right] ds \right\| =: I_0 + I_1. \end{aligned} \tag{44}$$

Using Lemma 3.1 with $r = 2$ and $\gamma = \beta$ yields

$$I_0 \leq Ch^2 t^{-1+\beta/2}\|u_0\|_{\beta} \leq Ch^2 t^{-1+\beta/2}. \tag{45}$$

Using Assumption 2.2, (11) and (19) yields

$$\begin{aligned} I_1 &\leq \int_0^t \left\|U(t,s)\left[F(s,u(s)) - F\left(s,u^h(s)\right)\right]\right\| ds \\ &\quad + \int_0^t \left\|\left[U(t,s) - U_h(t,s)P_h\right] F\left(s,u^h(s)\right)\right\| ds \\ &\leq C\int_0^t \left\|u(s) - u^h(s)\right\| ds + C\int_0^t \left\|\left[U(t,s) - U_h(t,s)P_h\right] F\left(s,u^h(s)\right)\right\| ds. \end{aligned} \tag{46}$$

If $0 \leq t \leq h^2$, then using (19), Assumption 2.2 and (11) yields

$$\begin{aligned} I_1 &\leq C\int_0^t \|u(s) - u^h(s)\| ds + C\int_0^t \|U(t,s) - U_h(t,s)P_h\|_{\mathcal{L}(H)} ds \\ &\leq C\int_0^t \|u(s) - u^h(s)\| ds + C\int_0^t ds \leq C\int_0^t \|u(s) - u^h(s)\| ds \\ &\leq C\int_0^t \|u(s) - u^h(s)\| ds + C\int_0^{h^2} ds \leq Ch^2 + C\int_0^t \|u(s) - u^h(s)\| ds. \end{aligned}$$

If $0 < h^2 \leq t$, using Lemma 3.1 (with $r = 2$ and $\gamma = 0$), and splitting the second integral in two parts yields

$$\begin{aligned} I_1 &\leq C\int_0^t \|u(s) - u^h(s)\|ds + Ch^2 \int_0^{t-h^2} (t-s)^{-1}ds + Ch^2 \int_{t-h^2}^t (t-s)^{-1}ds \\ &\leq C\int_0^t \|u(s) - u^h(s)\|ds + Ch^2(1 + \ln(t/h^2)). \end{aligned} \tag{47}$$

Substituting (45) and (47) in (44), and applying Gronwall's lemma proves (42). To prove (43), we only need to re-estimate the term $I_3 := \int_0^t \left\|[U(t,s) - U_h(t,s)P_h]\, F\left(s, u^h(s)\right)\right\| ds$. Note that under assumption $\|(-A(s))^\delta F(t,v)\| \leq Ct + C\|(-A(s))^\delta v\|$, using Lemma 3.1 (with $r = 2$ and $\gamma = \delta$) and (11), following the same lines as above, we obtain easily that $I_3 \leq Ch^2$. Let us now estimate $I_3$ under the hypothesis $\|F(t,v)\| \leq C\|v\|$. Using Assumption 2.2, (11), we the help of the mild solution (18), we have

$$\|F(t,u^h(t))\| \leq \|u^h(t)\| \leq C|t-s|^\epsilon s^{-\epsilon}, \quad \|F(s,u^h(s)) - F(t,u^h(t))\| \leq C|t-s|^\epsilon s^{-\epsilon}, \tag{48}$$

for some $\epsilon \in (0,1)$ and any $s, t \in [0,T]$. Using Lemma 3.1 (with $r = 2$ and $\gamma = 0$), triangle inequality and (48) yields

$$\begin{aligned} I_1 &\leq Ch^2 \int_0^t (t-s)^{-1} \left\|F\left(s, u^h(s)\right) - F(t, u^h(t))\right\| ds \\ &\quad + Ch^2 \int_0^t (t-s)^{-1} \|F(t, u^h(t))\| ds \\ &\leq Ch^2 \int_0^t (t-s)^{-1+\epsilon} s^{-\epsilon} ds \leq Ch^2. \end{aligned}$$

Hence the new estimate of $I_1$ is given by

$$I_1 \leq Ch^2 + C\int_0^t \|u(s) - u^h(s)\|ds. \tag{49}$$

Substituting (45) and (49) in (44) and applying Gronwall's lemma proves (43) and the proof of Theorem 3.1 is completed. ■

### *3.3. Error estimate for the semilinear problem under polynomial growth and one-sided lipschitz conditions*

In this section, we take $\beta \in \left(\frac{d}{2}, 2\right)$ and make the following assumptions on the nonlinearity.

**Assumption 3.1.** There exist four constants $L_0, L_1, c_1, c_2 \in [0, \infty)$ such that the nonlinear function $F$ satisfies the following conditions

$$\langle F(s,w) - F(s,v), w - v\rangle_H \leq L_0\|w - v\|^2, \quad w, v \in H, \tag{50}$$

$$\|F(s,w)\| \leq L_1 + L_1\|w\|\left(1 + \|w\|_C^{c_1}\right), \quad w \in H, \tag{51}$$

$$\|F(s,w) - F(s,v)\| \leq L_1\|w - v\|\left(1 + \|u\|_C^{c_1} + \|v\|_C^{c_1}\right), \quad w, v \in H, \tag{52}$$

for any $s \in [0,T]$.

Let us recall the following Sobolev embedding theorem.

$$\mathcal{D}\left((-A(0))^{\delta}\right) \hookrightarrow C(\Lambda, \mathbb{R}), \quad \text{for} \quad \delta > \frac{d}{2}, \quad d \in \{1,2,3\}. \tag{53}$$

It is a classical solution that under Assumption 3.1, (9) has a unique mild solution $u$ satisfying $u \in C([0,T], H)$ (see for example [10]).

**Lemma 3.2.** *The mild solution $u$ of (9) satisfies the following space regularity estimates*[3]

$$\left\|(-A(0))^{\frac{\beta}{2}}u(t)\right\| \leq C, \quad \|u(t)\|_{C} \leq C, \quad t \in [0,T]. \tag{54}$$

**Proof.** Having the first estimate of (54), the Sobolev embedding (53) allows to have

$$\|u(t)\|_{C} \leq C\left\|(-A(0))^{\frac{\beta}{2}}u(t)\right\| \leq C, \quad \|u^h(t)\|_{C} \leq C\left\|(-A(0))^{\frac{\beta}{2}}u^h(t)\right\| \leq C, \quad t \in [0,T]. \tag{55}$$

Hence, it remains to prove the first estimate of (54). Note that $u$ satisfies

$$\frac{d}{dt}u(t) - A(t)u(t) = F(t, u(t)), \quad u(0) = u_0, \quad t \in (0,T]. \tag{56}$$

Taking the inner product in (56), using Assumption 3.1 and Cauchy–Schwartz's inequality yields

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u(s)\|^2 - \langle A(s)u(s), u(s)\rangle_H &= \langle F(s,u(s)) - F(s,0), u(s)\rangle_H + \langle F(s,0), u(s)\rangle_H \\ &\leq C\|u(s)\|^2 + \frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|F(s,0)\|^2.\end{aligned} \tag{57}$$

Using the coercivity estimate (7), it holds that

$$\begin{aligned}\lambda_0\|u(s)\|_1^2 \leq a(s)(u(s),u(s)) &= -\frac{1}{2}\frac{d}{ds}\|u(s)\|^2 + \frac{1}{2}\frac{d}{ds}\|u(s)\|^2 - \langle A(s)u(s), u(s)\rangle_H \\ &\leq -\frac{1}{2}\frac{d}{ds}\|u(s)\|^2 + C\|u(s)\|^2 + \frac{1}{2}\|F(s,0)\|^2.\end{aligned} \tag{58}$$

Using Cauchy-Schwartz's inequality, [12, Lemma 3.1] and (58), it holds that

$$\begin{aligned}|\langle A(s)u(s), u(s)\rangle_H| &= \left|\left\langle(-A(s))^{\frac{1}{2}}u(s), (-A^*(s))^{\frac{1}{2}}u(s)\right\rangle_H\right| \\ &\leq \frac{1}{2}\left\|(-A(s))^{\frac{1}{2}}u(s)\right\|^2 + \frac{1}{2}\left\|(-A^*(s))^{\frac{1}{2}}u(s)\right\|^2 \\ &\leq \frac{C}{2}\|u(s)\|_1^2 + \frac{1}{2}\left\|(-A^*(s))^{\frac{1}{2}}(-A(s))^{-\frac{1}{2}}\right\|_{\mathcal{L}(H)}^2\left\|(-A(s))^{\frac{1}{2}}u(s)\right\|^2 \\ &\leq \frac{C}{2}\|u(s)\|_1^2 + \frac{C}{2}\left\|(-A(s))^{\frac{1}{2}}u(s)\right\|^2 \leq C\|u(s)\|_1^2 = \frac{C}{\lambda_0}.\lambda_0\|u(s)\|_1^2 \\ &\leq -\frac{C}{2\lambda_0}\frac{d}{ds}\|u(s)\|^2 + C\|u(s)\|^2 + C\|F(s,0)\|^2.\end{aligned} \tag{59}$$

[3] These estimates still hold if $u$ is replaced by its discrete version $u^h$.

Note that from (57) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u(s)\|^2 &\leq C\|u(s)\|^2 + \frac{1}{2}\|F(s,0)\|^2 + \langle A(s)u(s), u(s)\rangle_H \\ &\leq C\|u(s)\|^2 + \frac{1}{2}\|F(s,0)\|^2 + |\langle A(s)u(s), u(s)\rangle_H|. \end{aligned} \tag{60}$$

Substituting (59) in (60) yields

$$\left(\frac{1}{2} + \frac{C}{2\lambda_0}\right)\frac{d}{ds}\|u(s)\|^2 \leq C\|u(s)\|^2 + C\|F(s,0)\|^2. \tag{61}$$

Integrating both sides of (61) over $[0,t]$ yields

$$\|u(t)\|^2 \leq C\int_0^t \|u(s)\|^2 ds + C\int_0^t \|F(s,0)\|^2 ds \leq C\int_0^t \|u(s)\|^2 ds + C. \tag{62}$$

Applying Gronwall's lemma to (62) yields

$$\|u(t)\|^2 \leq C. \tag{63}$$

Now using Assumptions 2.1, 3.1, (63), the fact that $u \in C([0,T],H)$ and the stability properties of the two parameters semigroup yields

$$\begin{aligned}\|(-A(0))^{\frac{\beta}{2}}u(t)\| &\leq \|(-A(0))^{\frac{\beta}{2}}U(t,0)u_0\| + \int_0^t \|(-A(0))^{\frac{\beta}{2}}U(t,s)F(s,u(s))\|ds \\ &\leq \|(-A(0))^{\frac{\beta}{2}}U(t,0)(-A(0))^{-\frac{\beta}{2}}\|_{\mathcal{L}(H)}\|(-A(0))^{\frac{\beta}{2}}u_0\| \\ &+ \int_0^t \|(-A(0))^{\frac{\beta}{2}}U(t,s)\|_{\mathcal{L}(H)}\|F(s,u(s))\|ds \\ &\leq C + \int_0^t (t-s)^{-\frac{\beta}{2}}\|u(s)\|(1+\|u(s)\|_C^{c_1})ds \\ &\leq C + C\int_0^t (t-s)^{-\frac{\beta}{2}}ds \leq C. \end{aligned} \tag{64}$$

This completes the proof of the lemma. ■

**Theorem 3.2.** *Let $u(t)$ and $u^h(t)$ be solution of* (9) and (14) *respectively. Let* Assumption 3.1 *be fulfilled. If $u_0 \in \mathcal{D}((-A(0))^{\frac{\beta}{2}})$, then the following error estimate holds*

$$\|u(t) - u^h(t)\| \leq Ch^\beta, \quad t \in [0,T].$$

**Proof.** We introduce the following auxiliary equation

$$\frac{d\widetilde{u}^h(t)}{dt} = A_h(t)\widetilde{u}^h(t) + P_hF(t,u(t)). \tag{65}$$

Note that the mild solution of (65) is given by

$$\widetilde{u}^h(t) = U_h(t,0)P_hu_0 + \int_0^t U_h(t,s)P_hF(s,u(s))ds, \quad t \in [0,T]. \tag{66}$$

Using triangle inequality, it holds that

$$\|u(t) - u^h(t)\| \leq \|u(t) - \widetilde{u}^h(t)\| + \|\widetilde{u}^h(t) - u^h(t)\|. \tag{67}$$

Let us start by estimating the first term in (67). Subtracting (66) from (10) and using triangle inequality, it holds that

$$\begin{aligned}\|u(t)-\widetilde{u}^h(t)\| &\leq \|(U(t,0)-U_h(t,0)P_h)\,X_0\| \\ &\quad+\left\|\int_0^t (U(t,s)-U_h(t,s)P_h)\,F(s,u(s))ds\right\| \\ &=: II_1+II_2. \end{aligned} \tag{68}$$

Using Lemma 3.1 with $r=\alpha=\beta$ and Assumption 2.1 yields

$$II_1 \leq Ch^{\beta}\|(-A(0))^{\frac{\beta}{2}}X_0\| \leq Ch^{\beta}. \tag{69}$$

Using Lemma 3.1 (with $r=\beta$ and $\gamma=0$), Assumption 3.1 and Lemma 3.2 yields

$$\begin{aligned} II_{21} &\leq Ch^{\beta}\int_0^t (t-s)^{-\frac{\beta}{2}}\|F(s,u(s))\|ds \leq Ch^{\beta}\int_0^t (t-s)^{-\frac{\beta}{2}}\|u(s)\|\left(1+\|u(s)\|^{c_1}\right)ds \\ &\leq Ch^{\beta}\int_0^t (t-s)^{-\frac{\beta}{2}}ds \leq Ch^{\beta}. \end{aligned} \tag{70}$$

Substituting (69) and (70) in (68) yields

$$\|u(t)-\widetilde{u}^h(t)\| \leq Ch^{\beta}, \quad t\in[0,T]. \tag{71}$$

Let us introduce the following error representation $\widetilde{e}^h(t) := \widetilde{u}^h(t)-u^h(t)$. It is obvious to see that $\widetilde{e}^h(t)$ is differentiable with respect to the time and satisfies

$$\frac{d}{dt}\widetilde{e}^h(t)-A_h(t)\widetilde{e}^h(t)=P_h\left(F(t,u(t))-F(t,u^h(t))\right), \quad t\in(0,T], \quad \widetilde{e}^h(0)=0. \tag{72}$$

Taking the inner product on both sides of (72), using (50) and Cauchy-Schwartz's inequality yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2-\left\langle A_h(s)\widetilde{e}^h(s),\widetilde{e}^h(s)\right\rangle_H \\ &=\left\langle F(s,\widetilde{u}^h(s))-F(s,u^h(s)),\widetilde{e}^h(s)\right\rangle_H+\left\langle F(s,u(s))-F(s,\widetilde{u}^h(s)),\widetilde{e}^h(s)\right\rangle_H \\ &\leq C\|\widetilde{e}^h(s)\|^2+C\|F(s,u(s))-F(s,\widetilde{u}^h(s))\|\|\widetilde{e}^h(s)\|. \end{aligned} \tag{73}$$

Using Cauchy schwartz inequality, it follows from (73) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2-\left\langle A_h(s)\widetilde{e}^h(s),\widetilde{e}^h(s)\right\rangle_H \\ &\leq C\|\widetilde{e}^h(s)\|^2+C\|F(s,u(s))-F(s,\widetilde{u}^h(s))\|^2+C\|\widetilde{e}^h(s)\|^2, \end{aligned} \tag{74}$$

Using the coercivity estimate (7), (74) and the fact that $\widetilde{e}^h(s)\in V_h$ yields

$$\begin{aligned} \lambda_0\|\widetilde{e}^h(s)\|_1^2 &\leq a(s)\left(\widetilde{e}^h(s),\widetilde{e}^h(s)\right)=-\left\langle A_h(s)\widetilde{e}^h(s),\widetilde{e}^h(s)\right\rangle_H \\ &=-\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2+\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2-\left\langle A_h(s)\widetilde{e}^h(s),\widetilde{e}^h(s)\right\rangle_H \\ &\leq -\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2+C\|\widetilde{e}^h(s)\|^2 \\ &\quad+C\|F(s,u(s))-F(s,\widetilde{u}^h(s))\|^2+C\|\widetilde{e}^h(s)\|^2. \end{aligned} \tag{75}$$

Using (74), it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2 \\
&\leq C\|\widetilde{e}^h(s)\|^2 + C\|F(u(s)) - F(\widetilde{u}^h(s))\|^2 + C\|\widetilde{e}^h(s)\|_1^2 + \left\langle A_h(s)\widetilde{e}^h(s), \widetilde{e}^h(s)\right\rangle_H \\
&\leq C\|\widetilde{e}^h(s)\|^2 + C\|F(s,u(s)) - F(s,\widetilde{u}^h(s))\|^2 + C\|\widetilde{e}^h(s)\|^2 + \left|\left\langle A_h(s)\widetilde{e}^h(s), \widetilde{e}^h(s)\right\rangle_H\right|.
\end{aligned} \tag{76}$$

Since $\widetilde{e}^h(s) \in V_h$, using Cauchy-Schwartz's inequality, the equivalence of norms [4, (2.12)] and [12, Lemma 3.1], it holds that

$$\begin{aligned}
\left|\left\langle A_h(s)\widetilde{e}^h(s), \widetilde{e}^h(s)\right\rangle\right| &= \left|\left\langle(-A_h(s))^{\frac{1}{2}}\widetilde{e}^h(s), (-A_h(s)^*)^{\frac{1}{2}}\widetilde{e}^h(s)\right\rangle_H\right| \\
&\leq \frac{1}{2}\left\|(-A_h(s))^{\frac{1}{2}}\widetilde{e}^h(s)\right\|^2 + \frac{1}{2}\|(-A_h(s)^*)^{\frac{1}{2}}\widetilde{e}^h(s)\|^2 \\
&\leq \frac{C}{2}\|\widetilde{e}^h(s)\|_1^2 + \frac{1}{2}\|(A_h^*)^{\frac{1}{2}}(-A_h(s))^{-\frac{1}{2}}\|^2_{\mathcal{L}(H)}\|(-A_h(s))^{\frac{1}{2}}\widetilde{e}^h(s)\|^2 \\
&\leq \frac{C}{2}\|\widetilde{e}^h(s)\|_1^2 + \frac{C}{2}\|(-A_h(s))^{\frac{1}{2}}\widetilde{e}^h(s)\|^2 \leq C\|\widetilde{e}^h(s)\|_1^2.
\end{aligned} \tag{77}$$

Substituting (77) in (76) and using (75) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2 &\leq C\|\widetilde{e}^h(s)\|^2 + C\|F(s,u(s)) - F(s,\widetilde{u}^h(s))\|^2 + \frac{C}{\lambda_0}\lambda_0\|\widetilde{e}^h(s)\|_1^2 \\
&\leq C\|\widetilde{e}^h(s)\|^2 + C\|F(s,u(s)) - F(s,\widetilde{u}^h(s))\|^2 - \frac{C}{2\lambda_0}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2.
\end{aligned} \tag{78}$$

Therefore, from (78) we have

$$\left(\frac{1}{2} + \frac{C}{2\lambda_0}\right)\frac{d}{ds}\|\widetilde{e}^h(s)\|^2 \leq C\|\widetilde{e}^h(s)\|^2 + C\|F(s,u(s)) - F(s,\widetilde{u}^h(s))\|^2. \tag{79}$$

Integrating both sides of (79) over $[0, t]$ yields

$$\|\widetilde{e}^h(t)\|^2 \leq C\int_0^t \|\widetilde{e}^h(s)\|^2 ds + C\int_0^t \|F(s,u(s)) - F(s,\widetilde{u}^h(s))\|^2 ds. \tag{80}$$

Using the Cauchy-Schwartz's inequality, Assumption 3.1, Lemma 3.2 and (71) yields

$$\begin{aligned}
\|\widetilde{e}^h(t)\|^2 &\leq C\int_0^t \|\widetilde{e}^h(s)\|^2 ds + C\int_0^t \|F(s,u(s)) - F(s,\widetilde{u}^h(s))\|^2 ds \\
&\leq C\int_0^t \|\widetilde{e}^h(s)\|^2 ds + C\int_0^t \|u(s) - \widetilde{u}^h(s)\|^2\left(1 + \|u(s)\|^{2c_1} + \|\widetilde{u}^h(s)\|^{2c_1}\right) ds \\
&\leq C\int_0^t \|\widetilde{e}^h(s)\|^2 ds + Ch^{2\beta}.
\end{aligned} \tag{81}$$

Applying the Gronwall's lemma to (81) yields

$$\|\widetilde{e}^h(t)\| \leq Ch^\beta, \quad t \in [0, T]. \tag{82}$$

This completes the proof of Theorem 3.2. ■

**Remark 3.3.** Along the same lines as in Theorem 3.2, one can obtain the error analysis in the following form

$$\|u(t) - u^h(t)\| \leq Ch^2 t^{-1+\beta/2} + Ch^2\left(1 + \ln(t/h^2)\right), \quad t \in [0, T]. \tag{83}$$

## Acknowledgments

The authors would like to thank the referees for their careful reading and comments that helped to improve this work.

## References

[1] H. Fujita, T. Suzuki, Evolutions problems (part1), in: P.G. Ciarlet, J.L. Lions (Eds.), Handb. Numer. Anal., II, North-Holland, 1991, pp. 789–928.
[2] M. Kovàcs, S. Larsson, F. Lindgren, Strong convergence of the finite element method with truncated noise for semilinear parabolic stochastic equations with additive noise, Numer. Algorithms 53 (2–3) (2010) 309–320, http://dx.doi.org/10.1007/s11075-009-9281-4.
[3] R. Kruse, Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise, IMA J. Numer. Anal. 34 (2014) 217–251, http://dx.doi.org/10.1093/imanum/drs055.
[4] S. Larsson, Nonsmooth Data Error Estimates with Applications to the Study of the Long-Time Behavior of the Finite Elements Solutions of Semilinear Parabolic Problems, Departement of Mathematics, Chalmers University of Technology, 1992, Preprint 6. http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.28.1250.
[5] D. Leykekhman, B. Vexler, Discrete maximal parabolic regularity for Galerkin finite element methods for non-autonomous parabolic problems, SIAM J. Numer. Anal. 56 (4) (2018) 2178–2202, http://dx.doi.org/10.1137/17M114100X.
[6] M. Luskin, R. Rannacher, On the smoothing property of the Galerkin method for parabolic equations, SIAM J. Numer. Anal. 19 (1) (1981) 1–21, http://dx.doi.org/10.1137/0719003.
[7] H. Mingyou. V. Thomee, Some convergence estimates for semidiscrete type schemes for time-dependent nonselfadjoint parabolic equations, Math. Comp. 37 (1981) 327–346, http://dx.doi.org/10.1137/0714015.
[8] J.D. Mukam, A. Tambue, A note on exponential rosenbrock-euler method for the finite element discretization of a semilinear parabolic partial differential equation, Comput. Math. Appl. 76 (2018) 1719–1738, http://dx.doi.org/10.1016/j.camwa.2018.07.025.
[9] A. Ostermann, M. Thalhammer, Convergence of runge–kutta methods for nonlinear parabolic equations, Appl. Numer. Math. 42 (2002) 367–380, http://dx.doi.org/10.1016/S0168-9274(01)00161-1.
[10] P.E. Sobolevskii, Equations of parabolic type in a Banach space, Trudy Moskov. Mat. Obs. 10 (1961) 297–350.
[11] A. Tambue, An exponential integrator for finite volume discretization of a reaction-advection-diffusion equation, Comput. Math. Appl. 71 (9) (2016) 1875–1897, http://dx.doi.org/10.1016/j.camwa.2016.03.001.
[12] A. Tambue, J.M.T. Ngnotchouye, Weak convergence for a stochastic exponential integrator and finite element discretization of stochastic partial differential equation with multiplicative & additive noise, Appl. Numer. Math. 108 (2016) 57–86, http://dx.doi.org/10.1016/j.apnum.2016.04.013.
[13] V. Thomée, Galerkin Finite Element Method for Parabolic Problems, second ed., in: Springer Series in Computational Mathematics, vol. 25, Springer, Berlin, 2006.
[14] X. Wang, Strong convergence rates of the linear implicit euler method for the finite element discretization of SPDEs with additive noise, IMA J. Numer. Anal. 37 (2) (2016) 965–984, http://dx.doi.org/10.1093/imanum/drw016.