Energy storage and stabilization of floating wind turbines

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Preface

This bachelor thesis is written at the Department of Mechanical and Marine Engineering at the Western University of Applied Sciences (WNUAS), in which the Mechanical Engineering program is located. The thesis is written in cooperation with Professor Thomas J. Impelluso and Professor Jan Michael Simon Bartl.

Abstract

This project implements the Moving Frame Method (MFM) in dynamics to model how inertial disks placed inside the tower of an offshore floating wind turbine can stabilize the wind turbine and reduce strain on moorings. Furthermore, such inertial disks would be able to generate excess power captured from the motion of the waves. The MFM leverages Lie Group Theory, Cartan's moving frames, and a new notation from the geometrical physics. By utilizing the height of the wind turbine and its mass, the moment captured from the waves will be much greater. By being able to convert energy from both wind and waves, each wind turbine installation will generate more power in total, as well as increasing their reliability. This research studies the effect of two inertial disks; one counter induced pitch from the wind turbines preexisting rotor, and the other turned 90 degrees from the first to reduce/capture the yaw the waves will induce on the structure, such that the floating turbine can remain facing the wind. Utilizing Web Graphics Library to show the effect of the inertial disks on the turbine, creates a more visual understanding of their effect.

Sammendrag

Dette prosjektet bruker The Moving Frame Method (MFM) i dynamikk til å lage en modell på hvordan svinghjul plassert på innsiden av tårnet til en flytende vindturbin kan stabilisere den, og redusere slitasje på moringene. I tillegg vil de kunne generere energi fanget fra bevegelsen til bølgene. MFM utnytter Lie Gruppe Teoriene, Cartan sine bevegende koordinatsystem, og en ny metode basert på geometriske prinsipper. På grunn av vindturbinens høyde og masse vil momentet bølgene utgjør være betraktelig større, og en gyroskopisk bølgegenerator vil da kunne fange energien lettere. Ved å kunne fange energi fra både bølge og vind, vil hver vindturbin totalt produsere mer energi, samtidig som de vil produsere den mer pålitelig. Denne forskningsrapporten ser på effekten fra to svinghjul, en som roterer motsatt fra vindturbinens blader som reduserer pitch'en bladenes bevegelse påfører strukturen, og en snudd 90 grader fra den første for å redusere/fange yaw'en bølgene vil påføre strukturen, slik at bladene kan være rettet mot vinden. Vi bruker Web Graphics Library for å gi en visuell forståelse av effekten svinghjulene har på vindturbinen.

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Nomenclature

[B]: Transformation matrix to generalized coordinates

[C]: Transformation matrix for prescribed rates

[D]: Combined angular velocity matrix

e: Unit basis vector

E: Frame connection matrix

{F}: Force and moment list

{F*}: Generalized force and moment list

g: Gravitational acceleration

H: Angular momentum

I3: 3×3 Identity matrix

J: 3×3 Mass moment of inertia matrix

K: Kinetic energy

L: Linear momentum

L: Lagrangian

m: Mass

[M]: Mass Matrix

[M*]: Reduced mass matrix

[N*]: Reduced non-linear velocity matrix

q(t): Generalized coordinates

{$\dot{q}(t)$}: Generalized velocity list

{$\ddot{q}(t)$}: Generalized acceleration list

{$\dot{r}(t)$}: Generalized prescribed velocity list

δW: Virtual work

{$\delta\dot{X}$}: Virtual Cartesian velocities

{$\delta \vec{X}$}: Virtual Cartesian displacements

Ω: Time rate of the frame connection matrix

ω: Angular velocity components

$\vec{\omega}$: Skew-symmetric angular velocity matrix

R: 3×3 Rotation matrix

[T*]: Reduced velocity matrix for prescribed rates

W: Negative work function

1. Introduction

With the environmental challenges and changes in the climate the world now is facing, the focus on finding new sustainable energy sources have increased, and in 2015 Norway presented the 2030 Agenda. This agenda presents 17 Sustainable Development Goals (SDG's) decided by all UN member states as a universal guideline for national and international actions, emphasizing a holistic approach to achieving sustainable development for all [1]. The government of Norway places great importance in accomplishing these goals, one of which is a focus on sustainable energy.

Offshore wind energy has become increasingly popular in recent years. One reason is the resistance to land-based wind turbines, due to their impact on nature while being constructed and operated. Research states that offshore wind energy has the potential to sustain the entire world's energy consumption 18 times over [2]. With its long coastline and experience in offshore installations, Norway has great opportunities to invest and develop offshore energy.

There are many benefits and challenges when building wind turbines offshore, as opposed to on land. Offshore, a wind turbine is subject to much harsher elements than on land; stronger winds as well as waves. Off the Norwegian coast, the ocean is mostly too deep to have a fixed foundation. One solution is to have floating wind turbines, moored by chains or wires. However, the motion of the waves and the force of the wind will affect the floating wind turbine.

It is still desirable to build sustainable and long lasting offshore wind turbines due to several benefits; bottom piled wind turbines cannot easily be built at depths greater than around 60 meters, making the Norwegian coast mostly unviable, as their construction greatly impacts and damages marine life [3], and the size of the floating wind turbine is less limited by the supporting structures' capabilities.

The goal of this paper is to explore the possibility of using gyroscopes placed inside floating turbines as a stabilizing tool. The idea is that rotating disks/flywheels placed inside the tower of the turbine can stabilize the wind turbine, reduce strain on moorings, as well as store excess power captured from the motion of the waves.

Similar projects have been done at WUSAP in cooperation with Professor Thomas J. Impelluso. Two such papers are "Dual Gyroscope Wave Energy Converter" [4] and "Modeling Crane Induced Ship Motion Using the Moving Frame Method" [5]. This research is lightly based on work done in these papers.

The project implements the Moving Frame Method (MFM) in dynamics to model the motion of the floating turbine and the forces acting on it with and without the gyroscopes. The MFM leverages Lie Group Theory, Cartan's moving frames and a new notation from the principle of geometrical physics. This makes it possible to extract the equations of motion. The method solves the equations numerically using a relatively simple numerical integration scheme. Then, the Cayley-Hamilton theorem and Rodriguez's formula reconstructs the rotation matrix for the floating wind turbine and completes the analysis.

Furthermore, an online web page is created where the research results are presented qualitatively as a 3D simulation. This is done by using WebGL, Javascript and HTML. By displaying the motions in 3D one can more intuitively see and understand the effects the disks have on the structure, as one will be comparing to visual instances along with the data, instead of only the data.

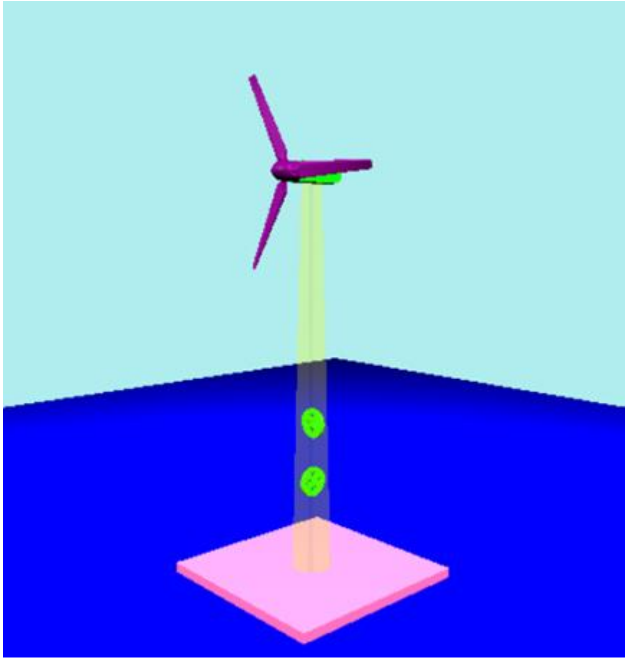

Figure 1: Model of floating wind turbine in WebGL.

2. Method

This section 2 is divided into sub-sections. First, the method is presented. This is followed by a deeper explanation of The Moving Frame Method, followed by the calculations. The code, The Cayley-Hamilton theorem, prescribed variables, and the 3D drawings of the floating wind turbine will be found in appendixes.

2.1 The moving frame method

This chapter introduces The Moving Frame Method and the advantages of using that method in this project.

Dynamics is a branch of mechanics that studies forces in action, a mathematical description of bodies in motion and the forces acting upon them.

Isaac Newton (1642 –1727) developed Dynamics, as a discipline of study. He formalized this discipline to explain the motion of the planets around the sun by developing the calculus. He was primarily concerned with particle motion and proposed three laws of motion, that are still used today to study the motion of particles.

Later, the Swiss mathematician Leonhard Euler (1707-1783) extended Newton's particle physics to rigid-body dynamics. Newton's equation of motion describes the motion of particles. Together, Newton's and Euler's equations of motion describe the motion of rigid bodies.

In the years after Newton and Euler, other mathematicians, Sophus Lie, Elie Cartan, Joseph-Louis Lagrange, and William Rowan Hamilton have contributed to Dynamics.

Dynamics is distinguished by two sub-disciplines: kinematics and kinetics. Kinematics concerns with the geometry of motion and its evolution with time without consideration of the forces and moments that induce such motion. Kinetics relies on kinematic variables and applies the Newton and Euler equations of motion to evaluate the effect of forces and moments on motion of rigid bodies.

Traditional rigid body dynamics was developed utilizing vector cross products. This is because in the 18th century, moving frames and rotation group theory were not available. Moving frames and the rotational group theory are what distinguish the new approach for the study of the dynamics of rigid bodies.

In the second half of the 19th century, Lie group theory was developed by a Norwegian mathematician, Sophus Lie (1842-1899). Subsequently, in the first half of the 20th century, a French mathematician, Élie Joseph Cartan (1869-1951) developed differential geometry using moving frames and advanced the Lie group theory. The latter includes the special orthogonal group, SO(3), which is also called the rotation group. Cartan's moving frame method was further advanced by Shiing-Shen Chern (1911-2004), who studied under Cartan and introduced the moving frame method to the U.S, [6]

The moving frame method is presented utilizing the compact notation, introduced by Theodore Frankel (1997). Specifically, Frankel's frame notation enables one to compute kinematic and kinetic quantities symbolically using the rotation group in a systematic and efficient manner, especially in three-dimensions. As a result, readers will see that vector cross products are replaced by matrix multiplications.

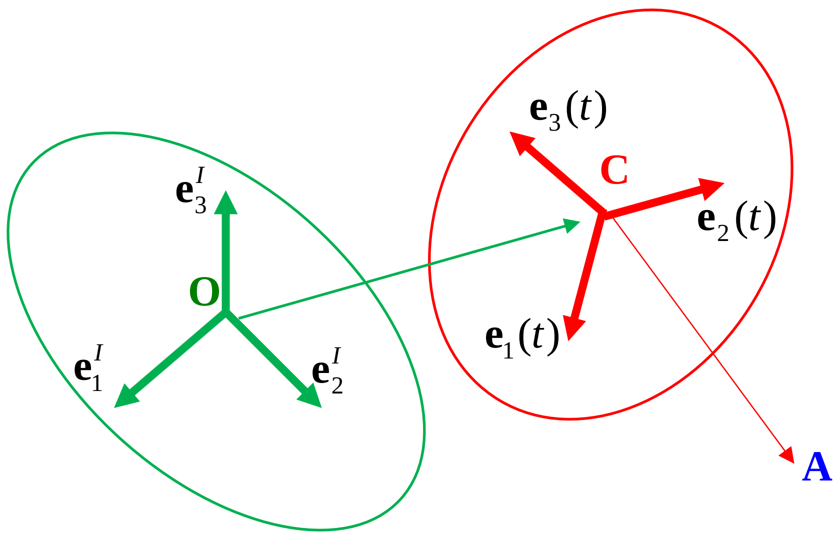

Figure 2: Observing a moving point A, from a moving frame ***e(t)****, which, in turn moves from an inertial body frame* $\mathbf{e}^I$*. [7]*

The reader may read more about The MFM in [8].

2.2 The Model

The floating wind turbine model is built using Creo Parametrics.

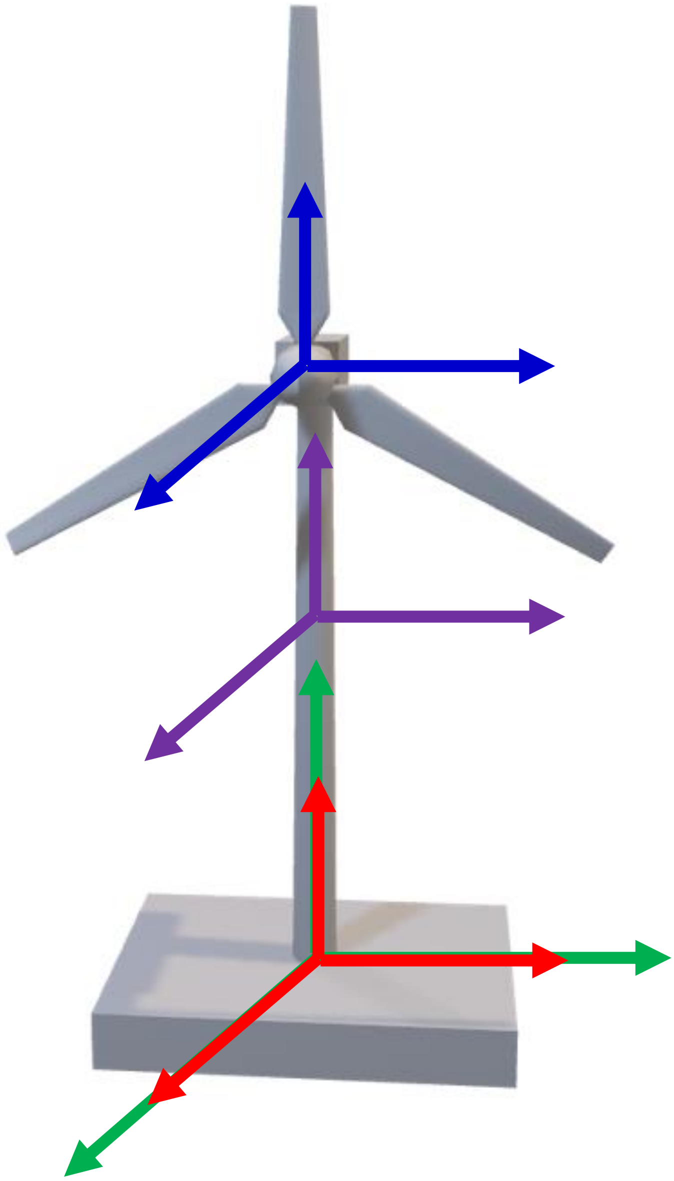

Figure 3: The model in Creo Parametric with unlabeled reference frames for the inertial, float, tower and blades.

The analysis commences with separating the model into parts. At the center of each body is a time-dependent moving frame. From the base of the model (the float), there is a systematic progression upwards. Each body is supplied with a numbered frame in ascending order. The multi-body system consists of five linked bodies. The float is body1, the tower is body 2, the blades are body 3 and the rotors are body 4 and 5. Each body get a moving Cartesian coordinate system:

$S_C^{(\alpha)}(t) = \left\{ S_1^{(\alpha)} S_2^{(\alpha)} S_3^{(\alpha)} \right\}^T$, where the subscript $\alpha = 1, 2, 3, 4, 5$

From this coordinate system, the frame bases are derived through directional derivatives in assumed Euclidean space.

2.3 Implementing the MFM

In this paper it is decided to split the float and the tower into separate parts. Usually they do not rotate or move relative to one another; they are fixed. However, as floating wind turbines are still in development, it could be useful for future work. Many floats are designed by a different company than the tower, and new designs may emerge where they are connected differently than today, and there is no loss in locking the two together, as is done later on. The work will now progress by defining the frame relations of each body, as if the tower and float moves independently. Later it is done calculations as if the float and the tower are fixed.

Construction of the float:

A frame is placed at the center of mass of the float as shown in figure 4 below:

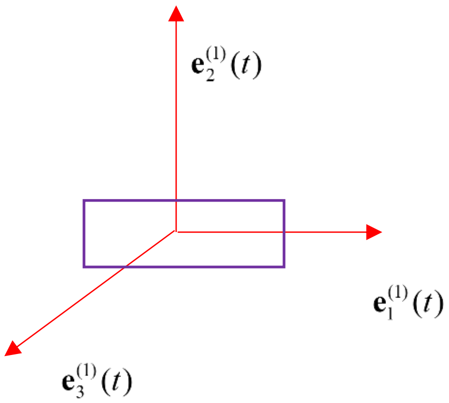

Figure 4 : The float with its frame.

- The 1-axis of float frame points to the right.
- The 2-axis of the float frame points inward, into the page.
- The 3-axis of the float frame points up.

An inertial frame from the moving float frame is deposited at the start of the analysis:

$$\mathbf{e}^I \equiv \mathbf{e}^{(1)}(0) \tag{1}$$

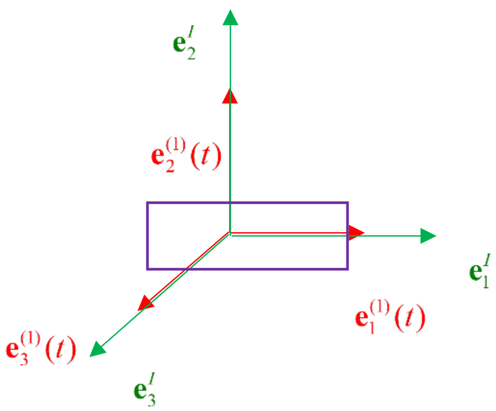

Figure 5: Float with inertial frame.

The frame relationship to the inertial frame is constructed through a rotation matrix: the elements of this rotation matrix will be pitch, yaw and roll.

$$\mathbf{e}^{(1)}(t) = \mathbf{e}^{I} R^{(1)}(t) \tag{2}$$

In itself $R^{(1)}(t)$ is an orthogonal rotation matrix wherein the inverse is its transpose. As such, it is a member of the Special Orthogonal Group, SO(3), and carries all properties of the group and its associated algebra.

Analysis of the displacement of the base frame from the inertial frame through a displacement constructed in the inertial frame is done. The form of this will be sway, heave and surge.

$$\mathbf{r}_c^{(1)}(t) = \mathbf{e}^{I} x_c^{(1)}(t) \tag{3}$$

To repeat, the rotations of the frame of the float are described by:

$$\mathbf{e}^{(1)}(t) = \mathbf{e}^{I} R^{(1)}(t) \tag{4}$$

Combining the rotation and translation information as follows, and designated it as a "frame connection" is then done:

$$\begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{I} & 0 \end{pmatrix} \begin{bmatrix} R^{(1)}(t) & x_c^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{5}$$

Expanding of the 0_3^T notation:

$$\begin{pmatrix} \mathbf{e}_1^{(1)}(t) & \mathbf{e}_2^{(1)}(t) & \mathbf{e}_3^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1^{I} & \mathbf{e}_2^{I} & \mathbf{e}_3^{I} & \mathbf{0} \end{pmatrix} \begin{bmatrix} R_{11}^{(1)}(t) & R_{12}^{(1)}(t) & R_{13}^{(1)}(t) & x_{1c}^{(1)}(t) \\ R_{21}^{(1)}(t) & R_{22}^{(1)}(t) & R_{23}^{(1)}(t) & x_{2c}^{(1)}(t) \\ R_{31}^{(1)}(t) & R_{32}^{(1)}(t) & R_{33}^{(1)}(t) & x_{3c}^{(1)}(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

The base frame was expanded, but not the position, due to the block-nature of the matrix. Thus, one can consolidate notation and define the "frame connection matrix":

$$\begin{pmatrix} E^{1}(t) \end{pmatrix} = \begin{bmatrix} R^{(1)}(t) & x_c^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{7}$$

Thus:

$$\begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{I} & 0 \end{pmatrix} \begin{pmatrix} E^{1}(t) \end{pmatrix} \tag{8}$$

Equation (9) recapitulates Eqns. (3) and (4).

Some definitions:

First body's frame connection:

$$\begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} \tag{9}$$

Inertial frame connection:

$$\begin{pmatrix} \mathbf{e}^{I} & 0 \end{pmatrix} \tag{10}$$

First body's frame connection matrix:

$$\begin{bmatrix} R^{(1)}(t) & x_c^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{11}$$

Next, the time rate of the frame connection matrix:

$$\left(\dot{E}^1(t)\right) = \begin{bmatrix} \dot{R}^{(1)}(t) & \dot{x}_c^{(1)}(t) \\ 0 & 0 \end{bmatrix} \tag{12}$$

The inverse of the frame connection matrix for the first body is known analytically, as it is a member of SE(3):

$$\left(E^1(t)\right)^{-1} = \begin{bmatrix} \left(R^{(1)}(t)\right)^T & -\left(R^{(1)}(t)\right)^T x_c^{(1)}(t) \\ 0 & 1 \end{bmatrix} \tag{13}$$

Computing this product:

$$\Omega^{(1)}(t) \equiv \left(E^1(t)\right)^{-1} \dot{E}^1(t) \tag{14}$$

By differentiating equation (9) and using the inverse to recast the result in the moving frame connection, this is found:

$$\begin{pmatrix} \dot{\mathbf{e}}^{(1)}(t) & \dot{\mathbf{r}}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)} \end{pmatrix} \Omega^{(1)}(t) \tag{15}$$

In the spirit of Cartan, the rate of the frame connection is expressed in terms of the same frame connection.

$$\begin{pmatrix} \dot{\mathbf{e}}^{(1)}(t) & \dot{\mathbf{r}}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} \begin{bmatrix} \overleftrightarrow{\omega}^{(1)}(t) & \left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{16}$$

Equation (17) can be separated as:

$$\dot{\mathbf{e}}^{(1)}(t) = \mathbf{e}^{(1)}(t)\overleftrightarrow{\omega}^{(1)}(t) \tag{17}$$

$$\dot{\mathbf{r}}_c^{(1)}(t) = \mathbf{e}^{(1)}(t)\left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t) = \mathbf{e}^{(1)}(t)\dot{s}_c^{(1)}(t) \tag{18}$$

Here, “x” is used when the coordinates are expressed in the inertial frame, and “s” when they are expressed in a moving frame.

Overview of the Cartesian coordinate rates for the turbine:

$$\{\dot{X}(t)\} = \begin{Bmatrix} \dot{x}_C^{(1)}(t) \\ \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix}$$

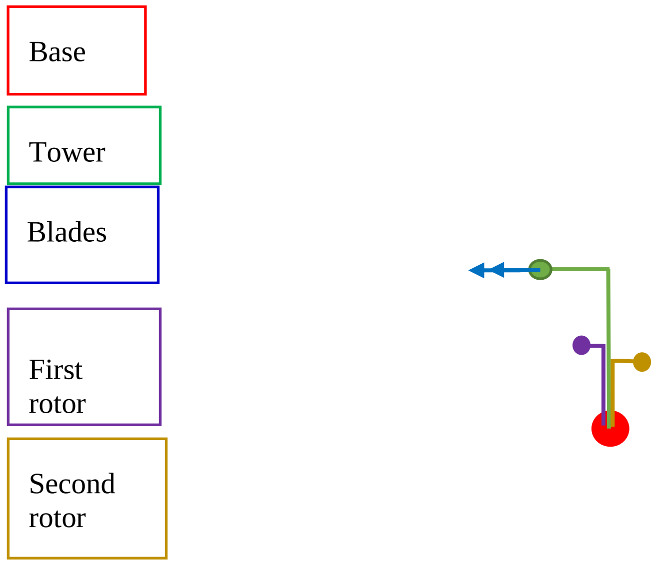

Figure 6: Cartesian coordinate rates with color coding and a figure showing their relations.

There are three branches off the main tree, as evinced by Figure 6c:

- “Trunk to arm”
- “Trunk to leg”
- “Trunk to head”

If this had been a continuous branch, the expressions would start to get very complicated. However, here, the last few bodies will be the same form, as they are primary branches.

In theory, for this wind turbine, there will be 30 rows. Five mass members and, for each mass: three rotations and three displacements, before the first two mass members are connected.

Recast and summarize the equations provided by Eqn. (17):

$$\dot{\mathbf{r}}_c^{(1)}(t) = \mathbf{e}^{(1)}(t)\left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t) \tag{19}$$

While

$$\mathbf{e}^{(1)} = \mathbf{e}^{I}\left(R^{(1)}(t)\right) \tag{20}$$

Through orthogonality, it can be stated as:

$$\mathbf{e}^{(1)}\left(R^{(1)}(t)\right)^{T} = \mathbf{e}^{I} \tag{21}$$

Thus:

$$\dot{\mathbf{r}}_{c}^{(1)}(t) = \mathbf{e}^{I}\dot{x}_{c}^{(1)}(t) \tag{22}$$

These parameters are unknown, and must be found:

Pitch, yaw and roll: $\overleftrightarrow{\omega^{(1)}(t)}$

Sway, heave and surge: $\dot{x}_{c}^{(1)}(t)$

Before continuing, and in the context of our current status, state the variables to be used:

- Cartesian velocities (displacement rates and angular velocities of each and every body)
- Minimal generalized (or essential) velocities (the minimum number of variables needed to describe the system)
- Prescribed velocities (those variables that will drive the system and which do not derive from applied forces or moments).

Formulate the expressions in terms of rates; not angles or positions, because Hamilton's Principle, and the extended Principle of Virtual Work, does this with rates.

Define all variables:

- The float will pitch, yaw and roll: $\omega^{(1)}(t)$, there are three of these for each direction.
  - Cartesian and minimal generalized.
- The float will translate: $\dot{x}_{C}^{(1)}(t)$, the heave, surge and sway.
  - Cartesian and minimal generalized
- The tower will rotate about the vertical: Essential $\dot{\theta}^{(2)}$
  - Minimal Generalized, this issue is addressed in the development.
- The blades will spin $\dot{\xi}^{(3)}$
  - Prescribed
- The first correcting rotor: $\dot{\phi}^{(4)}$
  - Prescribed
- The second correcting rotor: $\dot{\psi}^{(5)}$
  - Prescribed

$$\left\{\dot{X}(t)\right\}=\begin{Bmatrix}\dot{x}_C^{(1)}(t)\\ \omega^{(1)}(t)\\ \dot{x}_C^{(2)}(t)\\ \omega^{(2)}(t)\\ \dot{x}_C^{(3)}(t)\\ \omega^{(3)}(t)\\ \dot{x}_C^{(4)}(t)\\ \omega^{(4)}(t)\\ \dot{x}_C^{(5)}(t)\\ \omega^{(5)}(t)\end{Bmatrix}\qquad \left\{\dot{q}(t)\right\}=\begin{Bmatrix}\dot{x}_c^{(1)}(t)\\ \omega^{(1)}(t)\\ \dot{\theta}^{(2)}(t)\end{Bmatrix}\qquad \left\{\dot{r}(t)\right\}=\begin{Bmatrix}\dot{\xi}^{(3)}\\ \dot{\phi}^{(4)}\\ \dot{\psi}^{(5)}\end{Bmatrix} \tag{23}$$

- Above are the Cartesian velocities.
- In the middle, are the essential velocities:
  - This is needed to compute by numerical integration
- On the far right are the prescribed rotations, they play a role in the motion, but they are not unknown; they must be worked in.

The spin rates of the blades are prescribed, in the assumption that the calculation starts when the blades are already sped up by the wind. This will also allow a more stable code for the numerical scheme.

The goal is to construct B and C, a linear relationship, by working through each link.

$$\left\{\dot{X}(t)\right\}=\left[B(t)\right]\left\{\dot{q}(t)\right\}+\left[C(t)\right]\left\{\dot{r}(t)\right\} \tag{24}$$

The first body is finished. it is identically correct and goes in as the identity. Most is blank, and will be developed further on, as signified by the questionmark.

$$\left\{\dot{X}(t)\right\} = \begin{Bmatrix} \dot{x}_C^{(1)}(t) \\ \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1^{(1)}(t) \\ \dot{x}_2^{(1)}(t) \\ \dot{x}_3^{(1)}(t) \\ \omega_1^{(1)}(t) \\ \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) \\ \dot{x}_1^{(2)}(t) \\ \dot{x}_2^{(2)}(t) \\ \dot{x}_3^{(2)}(t) \\ \omega_1^{(2)}(t) \\ \omega_2^{(2)}(t) \\ \omega_3^{(2)}(t) \\ \dot{x}_1^{(3)}(t) \\ \dot{x}_2^{(3)}(t) \\ \dot{x}_3^{(3)}(t) \\ \omega_1^{(3)}(t) \\ \omega_2^{(3)}(t) \\ \omega_3^{(3)}(t) \\ \dot{x}_1^{(4)}(t) \\ \dot{x}_2^{(4)}(t) \\ \dot{x}_3^{(4)}(t) \\ \omega_1^{(4)}(t) \\ \omega_2^{(4)}(t) \\ \omega_3^{(4)}(t) \\ \dot{x}_1^{(5)}(t) \\ \dot{x}_2^{(5)}(t) \\ \dot{x}_3^{(5)}(t) \\ \omega_1^{(5)}(t) \\ \omega_2^{(5)}(t) \\ \omega_3^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & & & & & & \\ & & & ? & & & \\ & & & & & & \end{bmatrix} \begin{pmatrix} \dot{x}_1^{(1)}(t) \\ \dot{x}_2^{(1)}(t) \\ \dot{x}_3^{(1)}(t) \\ \omega_1^{(1)}(t) \\ \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) \\ \dot{\theta}^{(2)}(t) \end{pmatrix} + \left[C(t)\right]\left\{\dot{r}(t)\right\} \qquad (25)$$

Construction of the tower:

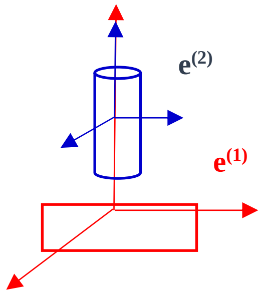

Figure 7: Float and tower with their frames

The center of mass of the tower must be found, which will not be at the geometric center, as the nacelle is accounted into the tower. This is found in appendix 1.

Place the coordinate system over the center of mass of the block to reduce the number of coordinates, but displace it upwards a distance *h*.

$$s_c^{(2/1)} = \begin{pmatrix} 0 \\ h_f/2 + h_t/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} \qquad \text{The geometric center of the tower.} \tag{26}$$

This will be treated as constant, to not allow the tower to move from the base.

The turbine will rotate about the vertical axis as can be seen by the following:

$$R^{(2/1)}(t) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{27}$$

Later, the following will be asserted to prevent relative rotation of the turbine from the base.

$$R^{(2/1)}(t) = I \tag{28}$$

Building the Frame Connection matrix:

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) = \left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_c^{(1)}(t)\right)\begin{bmatrix} R^{(2/1)}(t) & 0 \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} I & s_c^{(2/1)} \\ 0 & 1 \end{bmatrix} = \left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_c^{(1)}(t)\right)\begin{bmatrix} R^{(2/1)}(t) & R^{(2/1)}(t)s_c^{(2/1)} \\ 0_3^T & 1 \end{bmatrix} \tag{29}$$

For a rotation about the axis of the translation to the CM, this is true:

$$R^{(2/1)}(t)s_c^{(2/1)} = s_c^{(2/1)} \tag{30}$$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} * \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} \tag{31}$$

The following can be affirmed:

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) = \left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_c^{(1)}(t)\right)\begin{bmatrix} R^{(2/1)}(t) & s_c^{(2/1)} \\ 0_3^T & 1 \end{bmatrix} \tag{32}$$

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) = \left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_c^{(1)}(t)\right)E^{(2/1)}(t) \tag{33}$$

The above is the relative frame connection of the turbine, from the float.

The frame relation from the float to the inertial:

$$\left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_c^{(1)}(t)\right) = \left(\mathbf{e}^{I} \quad 0\right)E^{(1)}(t) \tag{34}$$

Thus, by the closure property of the SE(3) group:

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) = \left(\mathbf{e}^{I} \quad 0\right)E^{1}(t)E^{(2/1)}(t) \tag{35}$$

Then the multiplications:

$$E^{(2)}(t) = E^{(1)}(t)E^{(2/1)}(t) \tag{36}$$

$$E^{(2)}(t) = \begin{bmatrix} R^{(1)}(t) & x_c^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} R^{(2/1)}(t) & s_c^{(2/1)} \\ 0_3^T & 1 \end{bmatrix} \tag{37}$$

More simply stated as:

$$E^{(2)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t) & R^{(1)}(t)s_c^{(2/1)} + x_c^{(1)}(t) \\ 0 & 1 \end{bmatrix} \tag{38}$$

The rate and the invers of Eq. (39) is needed.

Since this is a member of SE(3), the inverse is:

- Transpose the term in the first row, first column; reverse the order and transpose each.
- Multiply that term by the term in the first row, second column and negate it.
- Leave the last row as is.

$$\left(E^{(2)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t)s_c^{(2/1)} + x_c^{(1)}(t)\right) \\ 0_3^T & 1 \end{bmatrix} \tag{39}$$

Take the rate of the frame connection matrix for the second body:

$$\dot{E}^{(2)}(t) = \begin{bmatrix} \dot{R}^{(1)}(t)R^{(2/1)}(t) + R^{(1)}(t)\dot{R}^{(2/1)}(t) & \dot{R}^{(1)}(t)s_c^{(2/1)} + \dot{x}_c^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{40}$$

Create $\Omega^{(2)}(t)$ by multiplying:

$$\Omega^{(2)}(t) = \left(E^{(2)}(t)\right)^{-1} \dot{E}^{(2)}(t) \tag{41}$$

$$\Omega^{(2)}(t) = \begin{bmatrix} \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t)s_c^{(2/1)} + x_c^{(1)}(t)\right) \\ 0_3^T & 1 \end{bmatrix} \begin{bmatrix} \dot{R}^{(1)}(t)R^{(2/1)}(t) + R^{(1)}(t)\dot{R}^{(2/1)}(t) & \dot{R}^{(1)}(t)s_c^{(2/1)} + \dot{x}_c^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{42}$$

$$\Omega^{(2)}(t) = \begin{bmatrix} \left(\left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t) + \overleftrightarrow{\omega^{(2/1)}(t)}\right) & \left(R^{(2/1)}(t)\right)^T \left(\overleftrightarrow{\omega^{(1)}(t)} s_c^{(2/1)} + \left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right) \\ 0_3^T & 0 \end{bmatrix} \tag{43}$$

The relation is now:

$$\begin{pmatrix} \dot{\mathbf{e}}^{(2)}(t) & \dot{\mathbf{r}}_c^{(2)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(2)}(t) & \mathbf{r}_c^{(2)}(t) \end{pmatrix} \Omega^{(2)}(t) \tag{44}$$

Or in matrix form:

$$\begin{pmatrix} \dot{\mathbf{e}}^{(2)}(t) & \dot{\mathbf{r}}_c^{(2)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(2)}(t) & \mathbf{r}_c^{(2)}(t) \end{pmatrix} \begin{bmatrix} \left(\left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t) + \overleftrightarrow{\omega^{(2/1)}(t)}\right) & \left(R^{(2/1)}(t)\right)^T \left(\overleftrightarrow{\omega^{(1)}(t)} s_c^{(2/1)} + \left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right) \\ 0_3^T & 0 \end{bmatrix} \tag{45}$$

Only two parts are needed:

$$\dot{\mathbf{e}}^{(2)}(t) = \mathbf{e}^{(2)}(t)\left(\left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t) + \overleftrightarrow{\omega^{(2/1)}(t)}\right) \tag{46}$$

$$\dot{\mathbf{r}}_c^{(2)}(t) = \mathbf{e}^{(2)}(t)\left(\left(R^{(2/1)}(t)\right)^T \left(\overleftrightarrow{\omega^{(1)}(t)} s_c^{(2/1)} + \left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right)\right) \tag{47}$$

The form in equations (46) and (47) gave the displacement in the moving frame (2). They are now moved to the inertial frame, and terms are moved to the right side, to allow the B-matrix to be built more easily.

$$\dot{\mathbf{r}}_c^{(2)}(t) = \mathbf{e}^{(1)}(t)\left(\overleftrightarrow{\omega^{(1)}(t)} s_c^{(2/1)} + \left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right) \tag{48}$$

Distribute the frame to the two terms:

$$\dot{\mathbf{r}}_c^{(2)}(t) = \mathbf{e}^{(1)}(t)\left(\overleftrightarrow{\omega^{(1)}(t)} s_c^{(2/1)}\right) + \mathbf{e}^{(1)}(t)\left(\left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right) \tag{49}$$

Vectorize the first term, and reverse the cross product:

$$\dot{\mathbf{r}}_c^{(2)}(t) = -\mathbf{s}_c^{(2/1)} \times \boldsymbol{\omega}^{(1)}(t) + \mathbf{e}^{(2)}(t)\left(\left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right) \tag{50}$$

Reapply the skew notation for the cross product and note the transpose of a skew as the negative of the transpose. Then pull out the frame:

$$\dot{\mathbf{r}}_c^{(2)}(t) = \mathbf{e}^{(1)}(t)\left(\left(\overleftrightarrow{s_c^{(2/1)}}^T \omega^{(1)}(t)\right) + \left(\left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t)\right)\right) \tag{51}$$

All needed variables are now on the right side.

The angular velocity term:

Use Eq. (48) to extract what is needed:

$$\overleftrightarrow{\omega^{(2)}(t)} = \left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t) + \overleftrightarrow{\omega^{(2/1)}(t)} \tag{52}$$

All omegas should be at the end of each term, to compute the linear relationships and create the B matrix. In 3D, the following expression is allowed [9]:

$$\left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t) = \overleftrightarrow{\left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t)} \tag{53}$$

Thus:

$$\overleftrightarrow{\omega^{(2)}(t)} = \left(\overleftrightarrow{R^{(2/1)}(t)}\right)^T \omega^{(1)}(t) + \overleftrightarrow{\omega^{(2/1)}(t)} \tag{54}$$

Removing the skew from all terms, converting the matrix expression to a column notation where omega is at the end:

$$\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + \omega^{(2/1)}(t) \tag{55}$$

A special format is used for $\omega^{(2/1)}(t)$, which is a relative angular velocity matrix and will be a standard form. There is only one rotation for the tower; a local rotation about one axis. In the following it is stated as unknown but will be disallowed towards the end.

Formulate an expression for equation (62).

$$\omega^{(2/1)}(t) = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} \tag{56}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{57}$$

$$\omega^{(2/1)}(t) = e_2\dot{\theta} \tag{58}$$

Thus:

$$\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + e_2\dot{\theta}^{(2)} \tag{59}$$

$$\dot{x}_c^{(2)}(t) = R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \omega^{(1)}(t) + \dot{x}_c^{(1)}(t) \tag{60}$$

Consider what happens if the float was tied to a support. Forces on the float translates it, but instead the cable stretches. There would be some deformation/stretch of the cable; in any scenario some translation, etc. Instead, assume there is no translation of the float and the attachment cable is rigid, then translational velocities are no longer needed. Now, restate the term:

1. $\omega^{(1)}(t) = \omega^{(1)}(t)$
2. $\dot{x}_c^{(2)}(t) = R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \omega^{(1)}(t)$
3. $\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + e_2\dot{\theta}^{(2)}$

The notation $\dot{\theta}^{(2)}$ will be reformed as $\dot{\theta}$ for simplicity, and finally set as $\theta = \dot{\theta} = 0$

$$\left\{\dot{X}(t)\right\}=\begin{Bmatrix}\omega^{(1)}(t)\\ \dot{x}_C^{(2)}(t)\\ \omega^{(2)}(t)\\ \dot{x}_C^{(3)}(t)\\ \omega^{(3)}(t)\\ \dot{x}_C^{(4)}(t)\\ \omega^{(4)}(t)\\ \dot{x}_C^{(5)}(t)\\ \omega^{(5)}(t)\end{Bmatrix}=\begin{bmatrix} I & 0\\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T & 0\\ \left(R^{(2/1)}(t)\right)^T & e_2\\ ? & ?\\ ? & ?\\ ? & ?\\ ? & ?\\ ? & ?\\ ? & ?\end{bmatrix}\begin{pmatrix}\omega^{(1)}(t)\\ \dot{\theta}(t)\end{pmatrix}+\begin{bmatrix} ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\end{bmatrix}\begin{Bmatrix}\dot{\xi}^{(3)}\\ \dot{\phi}^{(4)}\\ \dot{\psi}^{(5)}\end{Bmatrix} \tag{61}$$

Here, one problem must be anticipated: $R^{(1)}(t)$ is unknown. It does not derive from a standard rotation about one axis. This will cause a difficulty, because this rotation matrix is needed at every step in the numerical integration. However, $R^{(1)}(t)$ is obtained from the reconstruction with Caley Hamilton in Appendix 2.

The blades:

- The float will pitch, yaw and roll: $\omega^{(1)}(t)$, there are three of these for each direction.
- The tower will rotate about the vertical: essential $\dot{\theta}^{(2)}$
- The blades will spin $\dot{\xi}^{(3)}$
- The first correcting rotor will spin: $\dot{\phi}^{(4)}$
- The second correcting rotor will spin: $\dot{\psi}^{(5)}$

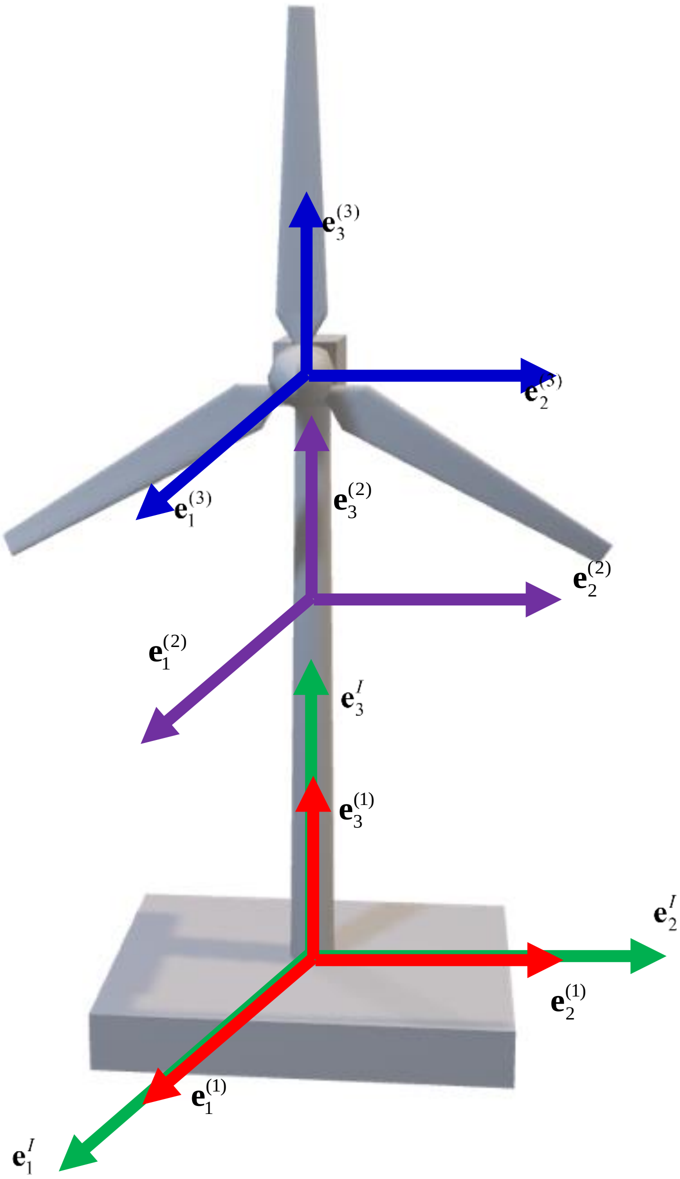

Figure 8: The tower with the frames.

Angular velocity of blades:

At the center of mass of the blades, place a moving frame $\mathbf{e}^{(3)}(t)$. To get to this frame move a distance "s" from the center of the tower up to the center of the blades and the depth "d" of the blades.

If the blades face out and rotate counterclockwise, then $\dot{\xi} < 0$.

$$R^{(3/2)}(t) = \begin{bmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{62}$$

$$\left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) \begin{bmatrix} I & \begin{pmatrix} 0 \\ -s \\ d \end{pmatrix} \\ 0_3^T & 1 \end{bmatrix} \begin{bmatrix} R^{(3/2)}(t) & 0 \\ 0 & 1 \end{bmatrix} \tag{63}$$

$$\left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) \begin{bmatrix} R^{(3/2)}(t) & \begin{pmatrix} 0 \\ -s \\ d \end{pmatrix} \\ 0 & 1 \end{bmatrix} \tag{64}$$

$$s_c^{(3/2)} = s_c^{(2/1)} + \begin{pmatrix} 0 \\ (height) \\ (depth) \end{pmatrix} \tag{65}$$

$$\left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) \begin{bmatrix} R^{(3/2)}(t) & s_c^{(3/2)} \\ 0 & 1 \end{bmatrix} \tag{66}$$

Thus:

$$E^{(3/2)}(t) = \begin{bmatrix} R^{(3/2)}(t) & s_c^{(3/2)} \\ 0_3^T & 1 \end{bmatrix} \tag{67}$$

The Blades, the first and second correcting rotor are each a branch from the tree. The analysis of each one will have the same form.

For the blades:

$$R^{(3/2)}(t) = \begin{bmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{68}$$

$$\left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right)\begin{bmatrix} R^{(3/2)}(t) & s_c^{(3/2)} \\ 0 & 1 \end{bmatrix} \tag{69}$$

$$E^{(3/2)} = \begin{bmatrix} R^{(3/2)} & s_c^{(3/2)} \\ 0 & 1 \end{bmatrix} \tag{70}$$

For the first correcting rotor:

$$R^{(4/2)}(t) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{71}$$

$$\left(\mathbf{e}^{(4)}(t) \quad \mathbf{r}_c^{(4)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right)\begin{bmatrix} R^{(4/2)} & s_c^{(4/2)} \\ 0 & 1 \end{bmatrix} \tag{72}$$

$$E^{(4/2)} = \begin{bmatrix} R^{(4/2)} & s_c^{(4/2)} \\ 0 & 1 \end{bmatrix} \tag{73}$$

For the second correcting rotor:

$$R^{(5/2)}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \tag{74}$$

$$\left(\mathbf{e}^{(5)}(t) \quad \mathbf{r}_c^{(5)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right)\begin{bmatrix} R^{(5/2)} & s_c^{(5/2)} \\ 0 & 1 \end{bmatrix} \tag{75}$$

$$E^{(5/2)} = \begin{bmatrix} R^{(5/2)} & s_c^{(5/2)} \\ 0 & 1 \end{bmatrix} \tag{76}$$

At this point, simplify, and ignore the translations of the float.

For the turbine

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_c^{(2)}(t)\right) = \left(\mathbf{e}^I \quad 0\right) E^{(2)}(t) \tag{77}$$

Restate the $E^{(2)}(t)$, Eq (38).

$$E^{(2)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t) & R^{(1)}(t)s_c^{(2/1)} + x_c^{(1)}(t) \\ 0 & 1 \end{bmatrix} \tag{78}$$

$$E^{(2)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t) & R^{(1)}(t)s_c^{(2/1)} \\ 0 & 1 \end{bmatrix} \tag{79}$$

$$\left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right) = \left(\mathbf{e}^I \quad 0\right) \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t) & R^{(1)}(t)s_c^{(2/1)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^{(3/2)} & s_c^{(3/2)} \\ 0 & 1 \end{bmatrix} \tag{80}$$

Define, as a result of the Group Properties of SE(3), the following

$$E^{(3)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t) & R^{(1)}(t)s_c^{(2/1)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^{(3/2)} & s_c^{(3/2)} \\ 0 & 1 \end{bmatrix} \tag{81}$$

Thus:

$$\left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right) = \left(\mathbf{e}^I(t) \quad 0\right) E^{(3)}(t) \tag{82}$$

Then:

$$E^{(3)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t)R^{(3/2)}(t) & R^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)s_c^{(2/1)} \\ 0 & 1 \end{bmatrix} \tag{83}$$

Now, form the inverse. It will be the inverse of a member of SE(3).

$$\left(E^{(3)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)s_c^{(2/1)}\right) \\ 0 & 1 \end{bmatrix} \tag{84}$$

Eliminating $R^{(1)}(t)$ in the top row, second column, using orthogonality:

$$\left(E^{(3)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(2/1)}(t) s_c^{(3/2)} + s_c^{(2/1)}\right) \\ 0 & 1 \end{bmatrix} \quad (85)$$

Using orthogonality again, in the top row, second column

$$\left(E^{(3)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(s_c^{(3/2)} + \left(R^{(2/1)}(t)\right)^T s_c^{(2/1)}\right) \\ 0 & 1 \end{bmatrix} \quad (86)$$

Now the time rate:

$$\dot{E}^{(3)}(t) = \begin{bmatrix} \dot{E}_{11}^{(3)}(t) & \dot{E}_{12}^{(3)}(t) \\ 0 & 0 \end{bmatrix} \quad (87)$$

$$\dot{E}_{11}^{(3)}(t) = \dot{R}^{(1)}(t) R^{(2/1)}(t) R^{(3/2)}(t) + R^{(1)}(t) \dot{R}^{(2/1)}(t) R^{(3/2)}(t) + R^{(1)}(t) R^{(2/1)}(t) \dot{R}^{(3/2)}(t) \quad (88)$$

$$\dot{E}_{12}^{(3)}(t) = \dot{R}^{(1)}(t) R^{(2/1)}(t) s_c^{(3/2)} + R^{(1)}(t) \dot{R}^{(2/1)}(t) s_c^{(3/2)} + \dot{R}^{(1)}(t) s_c^{(2/1)} \quad (89)$$

Form omega using partial symbolic notation:

$$\Omega^{(3)}(t) = \left(E^{(3)}\right)^{-1} \dot{E}^{(3)} \quad (90)$$

$$\Omega^{(3)}(t) = \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(s_c^{(3/2)} + \left(R^{(2/1)}(t)\right)^T s_c^{(2/1)}\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{E}_{11}^{(3)}(t) & \dot{E}_{12}^{(3)}(t) \\ 0 & 0 \end{bmatrix} \quad (91)$$

Now, turn to this:

$$\left(\dot{\mathbf{e}}^{(3)}(t) \quad \dot{\mathbf{r}}_c^{(3)}(t)\right) = \left(\mathbf{e}^I \quad 0\right) \Omega^{(3)}(t) \quad (92)$$

Pull out and work the $\overleftrightarrow{\omega^{(3)}}$ first

$$\begin{aligned}&\overleftrightarrow{\omega^{(3)}} = \\ &\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(\dot{R}^{(1)}(t)R^{(2/1)}(t)R^{(3/2)}(t)\right)+ \\ &\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t)\dot{R}^{(2/1)}(t)R^{(3/2)}(t)\right)+ \\ &\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t)R^{(2/1)}(t)\dot{R}^{(3/2)}(t)\right)\end{aligned} \tag{93}$$

Collapse terms to identity matrices due to orthogonality:

$$\begin{aligned}&\overleftrightarrow{\omega^{(3)}} = \\ &\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \dot{R}^{(1)}(t)R^{(2/1)}(t)R^{(3/2)}(t)+ \\ &\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \dot{R}^{(2/1)}(t)R^{(3/2)}(t)+ \\ &\left(R^{(3/2)}(t)\right)^T \dot{R}^{(3/2)}(t)\end{aligned} \tag{94}$$

Transform the above with the shorthand contraction in 3D:

$$\overleftrightarrow{\omega^{(3)}(t)} = \overleftrightarrow{\left(R^{(3/1)}(t)\right)^T \omega^{(1)}(t)} + \overleftrightarrow{\left(R^{(3/2)}(t)\right)^T \omega^{(2/1)}(t)} + \overleftrightarrow{\omega^{(3/2)}(t)} \tag{95}$$

Lift all skews:

$$\omega^{(3)}(t) = \left(R^{(3/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(3/2)}(t)\right)^T \omega^{(2/1)}(t) + \omega^{(3/2)}(t) \tag{96}$$

Omega-3 is found, and all terms are moved to the right. One may now utilize a simpler form for that last $\omega^{(3/2)}(t)$:

$$\omega^{(3)}(t) = \left(R^{(3/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(3/2)}(t)\right)^T \dot{\theta}^{(2)}e_2 + \dot{\xi}^{(3)}e_3 \tag{97}$$

Drop the superscript on the single angle terms to simplify:

$$\omega^{(3)}(t) = \left(R^{(3/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(3/2)}(t)\right)^T \dot{\theta}e_2 + \dot{\xi}e_3 \tag{98}$$

Fill in the angular velocity of the blades:

$$\left\{\dot{X}(t)\right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I & 0 \\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T & 0 \\ \left(R^{(2/1)}(t)\right)^T & e_2 \\ ? & ? \\ \left(R^{(3/1)}(t)\right)^T & \left(R^{(3/2)}(t)\right)^T e_3 \\ ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} \omega^{(1)}(t) \\ \dot{\theta}(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_3 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{99}$$

Translational Velocity of Blades:

$$\left(E^{(3)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(s_c^{(3/2)} + \left(R^{(2/1)}(t)\right)^T s_c^{(2/1)}\right) \\ 0 & 1 \end{bmatrix} \tag{100}$$

Repeat some of what is found:

$$E^{(3)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(2/1)}(t)R^{(3/2)}(t) & R^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)s_c^{(2/1)} \\ 0 & 1 \end{bmatrix} \tag{101}$$

$$\left(E^{(3)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(s_c^{(3/2)} + \left(R^{(2/1)}(t)\right)^T s_c^{(2/1)}\right) \\ 0 & 1 \end{bmatrix} \tag{102}$$

$$\left(E^{(3)}(t)\right)^{-1} = \begin{bmatrix} \left(\left(E^{(3)}(t)\right)^{-1}\right)_{11} & \left(\left(E^{(3)}(t)\right)^{-1}\right)_{12} \\ 0 & 1 \end{bmatrix} \tag{103}$$

$$\dot{E}^{(3)}(t) = \begin{bmatrix} \dot{E}_{11}^{(3)}(t) & \dot{E}_{12}^{(3)}(t) \\ 0 & 0 \end{bmatrix} \tag{104}$$

$$\dot{E}_{11}^{(3)}(t) = \dot{R}^{(1)}(t)R^{(2/1)}(t)R^{(3/2)}(t) + R^{(1)}(t)\dot{R}^{(2/1)}(t)R^{(3/2)}(t) + R^{(1)}(t)R^{(2/1)}(t)\dot{R}^{(3/2)}(t) \tag{105}$$

$$\dot{E}_{12}^{(3)}(t) = \frac{d}{dt}\left(R^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)s_c^{(2/1)}\right) \tag{106}$$

$$\dot{E}_{12}^{(3)}(t) = \dot{R}^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)\dot{R}^{(2/1)}(t)s_c^{(3/2)} + \dot{R}^{(1)}(t)s_c^{(2/1)} \tag{107}$$

$$\Omega^{(3)}(t) = \left(E^{(3)}(t)\right)^{-1}\dot{E}^{(3)}(t) \tag{108}$$

$$\Omega^{(3)}(t) = \begin{bmatrix} \left(\left(E^{(3)}(t)\right)^{-1}\right)_{11} & \left(\left(E^{(3)}(t)\right)^{-1}\right)_{12} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{E}_{11}^{(3)}(t) & \dot{E}_{12}^{(3)}(t) \\ 0 & 0 \end{bmatrix} \tag{109}$$

$$\Omega^{(3)}(t) = \begin{bmatrix} \left(\left(E^{(3)}(t)\right)^{-1}\right)_{11}\dot{E}_{11}^{(3)}(t) & \left(\left(E^{(3)}(t)\right)^{-1}\right)_{11}\dot{E}_{12}^{(3)}(t) \\ 0 & 0 \end{bmatrix} \tag{110}$$

$$\left(\dot{\mathbf{e}}^{(3)}(t) \quad \dot{\mathbf{r}}_c^{(3)}(t)\right) = \left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right)\Omega^{(3)}(t) \tag{111}$$

$$\left(\dot{\mathbf{e}}^{(3)}(t) \quad \dot{\mathbf{r}}_c^{(3)}(t)\right) = \left(\mathbf{e}^{(3)}(t) \quad \mathbf{r}_c^{(3)}(t)\right)\begin{bmatrix} \left(\left(E^{(3)}(t)\right)^{-1}\right)_{11}\dot{E}_{11}^{(3)}(t) & \left(\left(E^{(3)}(t)\right)^{-1}\right)_{11}\dot{E}_{12}^{(3)}(t) \\ 0 & 0 \end{bmatrix} \tag{112}$$

$$\dot{\mathbf{r}}_c^{(3)} = \mathbf{e}^{(3)}(t)\left(\left(E^{(3)}(t)\right)^{-1}\right)_{11}\dot{E}_{12}^{(3)}(t) \tag{113}$$

For the translational velocity; multiply this for the last term:

$$\dot{\mathbf{r}}_c^{(3)} = \mathbf{e}^{(3)}(t)\begin{pmatrix} \left(\left(R^{(3/2)}(t)\right)^T\left(R^{(2/1)}(t)\right)^T\left(R^{(1)}(t)\right)^T\right) \\ \dot{R}^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)\dot{R}^{(2/1)}(t)s_c^{(3/2)} + \dot{R}^{(1)}(t)s_c^{(2/1)} \end{pmatrix} \tag{114}$$

Isolate the equation after the frame:

$$\begin{pmatrix} \left(\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T\right) \\ \dot{R}^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)} + R^{(1)}(t)\dot{R}^{(2/1)}(t)s_c^{(3/2)} + \dot{R}^{(1)}(t)s_c^{(2/1)} \end{pmatrix} \tag{115}$$

Simplify it by multiplying the above to get the one below:

$$\begin{pmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(\dot{R}^{(1)}(t)R^{(2/1)}(t)s_c^{(3/2)}\right) + \\ \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t)\dot{R}^{(2/1)}(t)s_c^{(3/2)}\right) + \\ \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(\dot{R}^{(1)}(t)s_c^{(2/1)}\right) \end{pmatrix} \tag{116}$$

Simplify this term by moving its part "down" to their own frames by consolidating orthogonality, and using the term for angular velocity:

$$\dot{\mathbf{r}}_c^{(3)} = \mathbf{e}^{(2)}(t)\left(\left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t)s_c^{(3/2)} + \overleftrightarrow{\omega^{(2/1)}(t)} s_c^{(3/2)}\right) + \mathbf{e}^{(1)}(t)\left(\overleftrightarrow{\omega^{(1)}(t)} s_c^{(2/1)}\right) \tag{117}$$

By reversing the cross product, moving the terms to the first frame, and then compressing, it may be stated as:

$$\dot{\mathbf{r}}_c^{(3)} = \mathbf{e}^{(1)}(t)\left(R^{(2/1)}\right)^T \left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \omega^{(2)}(t) + \mathbf{e}^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \omega^{(1)}(t) \tag{118}$$

Move it to the inertial, consolidate and extract to get:

$$\dot{x}_c^{(3)} = \left(R^{(1)}R^{(2/1)}\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \omega^{(2)}(t) + R^{(1)}R^{(2/1)}\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \omega^{(1)}(t)\right) \tag{119}$$

Because the angular velocity terms are done first, the translational velocity terms become cleaner and easier to work with. Now, insert the previously found omegas:

Float

$$\omega^{(1)}(t) = \omega^{(1)}(t) \tag{120}$$

Tower

$$\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + e_2\dot{\theta}^{(2)} \tag{121}$$

Insert them into this:

$$\dot{x}_c^{(3)} = R^{(1)}R^{(2/1)}\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \omega^{(2)} + R^{(1)}R^{(2/1)}\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \omega^{(1)} \tag{122}$$

Then distribute and group some terms:

$$\dot{x}_c^{(3)} = \left(R^{(1)}R^{(2/1)}\left(\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \left(R^{(2/1)}(t)\right)^T + \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\right)\right)\omega^{(1)}(t) + R^{(1)}R^{(2/1)}\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \dot{\theta}^{(2)}e_2 \tag{123}$$

Define the B_{41}:

$$B_{41} = R^{(1)}R^{(2/1)}\left(\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \left(R^{(2/1)}(t)\right)^T + \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\right) \tag{124}$$

or

$$B_{42} = R^{(1)}R^{(2/1)}\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \dot{\theta}^{(2)}e_2 \tag{125}$$

With the exception of R(1), all of these are known expressions. They will be found from the code when it does the multiplication of matrices.

Summarized, this was found:

$$\dot{x}_c^{(3)} = B_{41}\omega^{(1)}(t) + B_{42}\dot{\theta}^{(2)} \tag{126}$$

These equations will be added to the Cartesian velocities $\left\{\dot{X}(t)\right\}$, as $\dot{x}_C^{(3)}(t)$ in equation (101).

First correcting rotor:

Angular velocity of first correcting rotor

Angular velocity for the blades was:

$$\omega^{(3)}(t) = \left(R^{(3/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(3/2)}(t)\right)^T \dot{\theta}e_2 + \dot{\xi}e_3 \tag{127}$$

Now the only change is that the “3” becomes a “4” and the name of the angles have changed

This was the rotational matrix for the blades:

$$R^{(3/2)}(t) = \begin{bmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{128}$$

This is for the outward facing rotor:

$$R^{(4/2)}(t) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{129}$$

The only other change is the last term. These are direct from the tower, and not from the blades, which is why it is noted as $R^{(4/2)}(t)$ instead of $R^{(4/3)}(t)$. It will be $\dot{\phi}e_3$, because it is the prescribed spin of the rotor, not the blades. However, they face in the same direction, only difference is in the distance up to the first rotor, which should be less than the distance to the top blades.

$$\omega^{(4)}(t) = \left(R^{(4/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(4/2)}(t)\right)^T \dot{\theta}e_2 + \dot{\phi}e_3 \tag{130}$$

The B-matrix so far:

$$\left\{\dot{X}(t)\right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I & 0 \\ R^{(1)}(t)\overleftrightarrow{s_c^{(2/1)}}^T & 0 \\ R^{(2/1)}(t)^T & e_2 \\ B_{41} & B_{42} \\ \left(R^{(3/1)}(t)\right)^T & \left(R^{(3/2)}(t)\right)^T e_3 \\ ? & ? \\ \left(R^{(4/1)}(t)\right)^T & \left(R^{(4/2)}(t)\right)^T e_3 \\ ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} \omega^{(1)}(t) \\ \dot{\theta}(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{41} & 0 & 0 \\ e_3 & 0 & 0 \\ ? & ? & ? \\ 0 & e_3 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{131}$$

Note:

$R^{(4/2)}(t)$ is direct: the second rotor spins from the turbine, not from the blades. However, with (4/1), there is a product:

$$R^{(4/1)}(t) = R^{(4/2)}(t)R^{(2/1)}(t) \tag{132}$$

Translational Velocity of first correcting rotor

These equations were stated earlier for the blades:

$$B_{41} = \left(R^{(1)} R^{(2/1)} \left(\left(\overleftrightarrow{s_c^{(3/2)}} \right)^T \left(R^{(2/1)}(t) \right)^T + \left(\overleftrightarrow{s_c^{(2/1)}} \right)^T \right) \right) \tag{133}$$

$$B_{42} = R^{(1)} R^{(2/1)} \left(\overleftrightarrow{s_c^{(3/2)}} \right)^T \dot{\theta}^{(2)} e_2 \tag{134}$$

$$\dot{x}_c^{(3)} = B_{41} \omega^{(1)}(t) + B_{42} \dot{\theta}^{(2)} \tag{135}$$

All 3's becomes 4's in the superscript notation. The outward facing rotor is identical in form to the blades and faces the same way. In the subscript notation on the left side above, it indicates the row of the B matrix, but the confusing part is that the top row for the translation of the body is deleted.

This "3" changes to a "4": $\left(\overleftrightarrow{s_c^{(3/2)}} \right)^T$ because the correcting rotor extends from the tower.

This does not change: $\left(\overleftrightarrow{s_c^{(2/1)}} \right)^T$ because this located the CM of the tower from the base.

Thus:

$$B_{61} = \left(R^{(1)} R^{(2/1)} \left(\left(\overleftrightarrow{s_c^{(4/2)}} \right)^T \left(R^{(2/1)}(t) \right)^T + \left(\overleftrightarrow{s_c^{(2/1)}} \right)^T \right) \right) \tag{136}$$

$$B_{62} = R^{(1)} R^{(2/1)} \left(\overleftrightarrow{s_c^{(4/2)}} \right)^T \dot{\theta}^{(2)} e_2 \tag{137}$$

$$\dot{x}_c^{(4)} = B_{61} \omega^{(1)}(t) + B_{62} \dot{\theta}^{(2)} \tag{138}$$

Fill in:

$$\left\{ \dot{X}(t) \right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I & 0 \\ R^{(1)}(t) \overleftrightarrow{s_c^{(2/1)}}^T & 0 \\ \left(R^{(2/1)}(t) \right)^T & e_2 \\ B_{41} & B_{42} \\ \left(R^{(3/1)}(t) \right)^T & \left(R^{(3/2)}(t) \right)^T e_3 \\ B_{61} & B_{62} \\ \left(R^{(4/1)}(t) \right)^T & \left(R^{(4/2)}(t) \right)^T e_3 \\ ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} \omega^{(1)}(t) \\ \dot{\theta}(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_3 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{139}$$

Second correcting rotor:

The angular velocity

The rotation matrix changes because now it faces in the 1-direction. The height changes. The variable changes. One may try to predict the angular velocity, from the term for the blades.

This was for the blades:

$$\omega^{(3)}(t) = \left(R^{(3/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(3/2)}(t)\right)^T \dot{\theta} e_2 + \dot{\xi} e_3 \tag{140}$$

This time, "3" becomes "5", but it is important to note; $R^{(5/2)}(t)$ will be a single direct rotation of the second rotor from the tower. It will be this:

$$R^{(5/2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \tag{141}$$

$R^{(5/1)}(t) = R^{(5/2)}(t) R^{(2/1)}(t)$ will rotate the second rotor from the tower and the tower from the float. There is one more change:

$$\omega^{(5)}(t) = \left(R^{(5/1)}(t)\right)^T \omega^{(1)}(t) + \left(R^{(5/2)}(t)\right)^T \dot{\theta} e_2 + \dot{\psi} e_1 \tag{142}$$

Translational velocity of second correcting rotor

$$B_{81} = \left(R^{(1)} R^{(2/1)} \left(\left(\overleftrightarrow{s_c^{(5/2)}} \right)^T \left(R^{(2/1)}(t) \right)^T + \left(\overleftrightarrow{s_c^{(2/1)}} \right)^T \right) \right) \tag{143}$$

$$B_{82} = R^{(1)} R^{(2/1)} \left(\overleftrightarrow{s_c^{(5/2)}} \right)^T \dot{\theta}^{(2)} e_2 \tag{144}$$

$$\dot{x}_c^{(4)} = B_{61} \omega^{(1)}(t) + B_{62} \dot{\theta}^{(2)} \tag{145}$$

$$\left\{ \dot{X}(t) \right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I & 0 \\ R^{(1)}(t) \overleftrightarrow{s_c^{(2/1)}}^T & 0 \\ \left(R^{(2/1)}(t)\right)^T & e_2 \\ B_{41} & B_{42} \\ \left(R^{(3/1)}(t)\right)^T & \left(R^{(3/2)}(t)\right)^T e_3 \\ B_{61} & B_{62} \\ \left(R^{(4/1)}(t)\right)^T & \left(R^{(4/2)}(t)\right)^T e_3 \\ B_{81} & B_{82} \\ \left(R^{(5/1)}(t)\right)^T & \left(R^{(5/2)}(t)\right)^T e_1 \end{bmatrix} \begin{pmatrix} \omega^{(1)}(t) \\ \dot{\theta}(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{146}$$

Every term is now defined, and may be stated more simply as:

$$\begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = B\begin{pmatrix} \omega^{(1)}(t) \\ \dot{\theta}(t) \end{pmatrix} + C\begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{147}$$

The B-Matrix:

$$\begin{bmatrix} I & 0 \\ R^{(1)}(t)\overleftrightarrow{s_c^{(2/1)}}^T & 0 \\ \left(R^{(2/1)}(t)\right)^T & e_2 \\ B_{41} & B_{42} \\ \left(R^{(3/1)}(t)\right)^T & \left(R^{(3/2)}(t)\right)^T e_3 \\ B_{61} & B_{62} \\ \left(R^{(4/1)}(t)\right)^T & \left(R^{(4/2)}(t)\right)^T e_3 \\ B_{81} & B_{82} \\ \left(R^{(5/1)}(t)\right)^T & \left(R^{(5/2)}(t)\right)^T e_1 \end{bmatrix} \tag{148}$$

It is 27 rows and 4 columns.

If the tower does not rotate, the following will be a correct simplification:

$$R^{(2/1)}(t) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{149}$$

Becomes:

$$R^{(2/1)}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{150}$$

As an identity matrix, it plays no role in the analysis.

Initially, this was equation 146:

$$\left\{\dot{X}(t)\right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I & 0 \\ R^{(1)}(t)\overleftrightarrow{s_c^{(2/1)}}^T & 0 \\ \left(R^{(2/1)}(t)\right)^T & e_2 \\ B_{41} & B_{42} \\ \left(R^{(3/1)}(t)\right)^T & \left(R^{(3/2)}(t)\right)^T e_3 \\ B_{61} & B_{62} \\ \left(R^{(4/1)}(t)\right)^T & \left(R^{(4/2)}(t)\right)^T e_3 \\ B_{81} & B_{82} \\ \left(R^{(5/1)}(t)\right)^T & \left(R^{(5/2)}(t)\right)^T e_1 \end{bmatrix} \begin{pmatrix} \omega^{(1)}(t) \\ \dot{\theta}(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{166}$$

It now becomes this:

$$\left\{\dot{X}(t)\right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I \\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \\ I \\ B_{41} \\ \left(R^{(3/1)}(t)\right)^T \\ B_{61} \\ \left(R^{(4/1)}(t)\right)^T \\ B_{81} \\ \left(R^{(5/1)}(t)\right)^T \end{bmatrix} \left(\omega^{(1)}(t)\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{151}$$

Now, go through each term and strike out the R that is now and identity matrix:

$$R^{(3/1)}(t) = R^{(3/2)}(t)R^{(2/1)}(t) \tag{152}$$

Which becomes this:

$$\left\{\dot{X}(t)\right\} = \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} = \begin{bmatrix} I \\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \\ I \\ B_{41} \\ \left(R^{(3/2)}(t)\right)^T \\ B_{61} \\ \left(R^{(4/2)}(t)\right)^T \\ B_{81} \\ \left(R^{(5/2)}(t)\right)^T \end{bmatrix} \left(\omega^{(1)}(t)\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{bmatrix} \begin{Bmatrix} \dot{\xi}^{(3)} \\ \dot{\phi}^{(4)} \\ \dot{\psi}^{(5)} \end{Bmatrix} \tag{153}$$

This leaves the B terms and the C terms.

B41

B42 is no longer needed:

$$B_{41} = R^{(1)}R^{(2/1)}\left(\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T \left(R^{(2/1)}(t)\right)^T + \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\right) \tag{154}$$

It now becomes:

$$B_{41} = R^{(1)}\left(\left(\overleftrightarrow{s_c^{(3/2)}}\right)^T + \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\right) \tag{155}$$

B61

B62 is no longer needed:

$$B_{61} = R^{(1)}R^{(2/1)}\left(\left(\overleftrightarrow{s_c^{(4/2)}}\right)^T \left(R^{(2/1)}(t)\right)^T + \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\right) \tag{156}$$

It now becomes:

$$B_{61} = R^{(1)}\left(\left(\overleftrightarrow{s_c^{(4/2)}}\right)^T + \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\right) \tag{157}$$

And if the first rotor is below the center of mass of the tower, you go back down.

B81

$$B_{81} = R^{(1)} R^{(2/1)} \left(\left(\overleftrightarrow{s_c^{(5/2)}} \right)^T \left(R^{(2/1)}(t) \right)^T + \left(\overleftrightarrow{s_c^{(2/1)}} \right)^T \right) \tag{158}$$

Which becomes:

$$B_{81} = R^{(1)} \left(\left(\overleftrightarrow{s_c^{(5/2)}} \right)^T + \left(\overleftrightarrow{s_c^{(2/1)}} \right)^T \right) \tag{159}$$

The following are the new forms:

$$R^{(1)} = \begin{bmatrix} R_{11}^{(1)}(t) & R_{12}^{(1)}(t) & R_{13}^{(1)}(t) \\ R_{21}^{(1)}(t) & R_{22}^{(1)}(t) & R_{23}^{(1)}(t) \\ R_{31}^{(1)}(t) & R_{32}^{(1)}(t) & R_{33}^{(1)}(t) \end{bmatrix} \tag{160}$$

$$R^{(3/2)}(t) = \begin{bmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{161}$$

$$R^{(4/2)}(t) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{162}$$

$$R^{(5/2)}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \tag{163}$$

$$B_{41} = R^{(1)} \left(\overleftrightarrow{s_c^{(3/2)} + s_c^{(2/1)}} \right)^T \tag{164}$$

$$B_{61} = R^{(1)} \left(\overleftrightarrow{s_c^{(4/2)} + s_c^{(2/1)}} \right)^T \tag{165}$$

$$B_{81} = R^{(1)} \left(\overleftrightarrow{s_c^{(5/2)} + s_c^{(2/1)}} \right)^T \tag{166}$$

$$s_c^{(2/1)} = \begin{pmatrix} 0 \\ h_f + h_t/2 + c_{nt2} \\ c_{nt3} \end{pmatrix} = \begin{pmatrix} 0 \\ h_f + h_t/2 + \left(h_t/2 + h_n/2\right) \\ d_n/2 - d_t/2 \end{pmatrix} = \begin{pmatrix} 0 \\ h_f + h_t + h_n/2 \\ d_n/2 - d_t/2 \end{pmatrix} \tag{167}$$

$$s_c^{(3/2)} = \begin{pmatrix} 0 \\ 0 \\ d_t/2 + d_n/2 \end{pmatrix} \tag{168}$$

$$s_c^{(3/1)} = s_c^{(3/2)} + s_c^{(2/1)} = \begin{pmatrix} 0 \\ 0 \\ d_t/2 + d_n/2 \end{pmatrix} + \begin{pmatrix} 0 \\ h_f + h_t + h_n/2 \\ d_n/2 - d_t/2 \end{pmatrix} = \begin{pmatrix} 0 \\ h_f + h_t + h_n/2 \\ d_n + d_t \end{pmatrix} \tag{169}$$

$$s_c^{(4/1)} = s_c^{(2/1)} + s_c^{(4/2)} = \begin{pmatrix} 0 \\ h_f + h_t + h_n/2 \\ d_n/2 - d_t/2 \end{pmatrix} + \begin{pmatrix} 0 \\ -h_t/2 - h_n/2 \\ d_t/X - \left(d_n/2 - d_t/2\right) \end{pmatrix} \tag{170}$$

$$s_c^{(5/1)} = s_c^{(2/1)} + s_c^{(5/2)} = \begin{pmatrix} 0 \\ h_f + h_t + h_n/2 \\ d_n/2 - d_t/2 \end{pmatrix} + \begin{pmatrix} w_t/X \\ -h_t/2 - h_n/2 \\ d_t/2 - \left(d_n/2 - d_t/2\right) \end{pmatrix} \tag{171}$$

Here the “X” is a factor to offset the disks from the center of the tower. They could be in the center, and one would put X=0, though usually there is an elevator here. It is therefore more beneficial to make this a variable.

$$\left\{\dot{X}(t)\right\}=\begin{Bmatrix}\omega^{(1)}(t)\\ \dot{x}_C^{(2)}(t)\\ \omega^{(2)}(t)\\ \dot{x}_C^{(3)}(t)\\ \omega^{(3)}(t)\\ \dot{x}_C^{(4)}(t)\\ \omega^{(4)}(t)\\ \dot{x}_C^{(5)}(t)\\ \omega^{(5)}(t)\end{Bmatrix}=\begin{bmatrix}I\\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\\ I\\ R^{(1)}\left(\overleftrightarrow{s_c^{(3/2)}+s_c^{(2/1)}}\right)^T\\ \left(R^{(3/2)}(t)\right)^T\\ R^{(1)}\left(\overleftrightarrow{s_c^{(4/2)}+s_c^{(2/1)}}\right)^T\\ \left(R^{(4/2)}(t)\right)^T\\ R^{(1)}\left(\overleftrightarrow{s_c^{(5/2)}+s_c^{(2/1)}}\right)^T\\ \left(R^{(5/2)}(t)\right)^T\end{bmatrix}\omega^{(1)}(t)+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\e_3&0&0\\0&0&0\\0&e_3&0\\0&0&0\\0&0&e_1\end{bmatrix}\begin{Bmatrix}\dot{\xi}^{(3)}\\ \dot{\phi}^{(4)}\\ \dot{\psi}^{(5)}\end{Bmatrix} \tag{172}$$

$$B=\begin{bmatrix}I\\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\\ I\\ R^{(1)}\left(\overleftrightarrow{s_c^{(3/2)}+s_c^{(2/1)}}\right)^T\\ \left(R^{(3/2)}(t)\right)^T\\ R^{(1)}\left(\overleftrightarrow{s_c^{(4/2)}+s_c^{(2/1)}}\right)^T\\ \left(R^{(4/2)}(t)\right)^T\\ R^{(1)}\left(\overleftrightarrow{s_c^{(5/2)}+s_c^{(2/1)}}\right)^T\\ \left(R^{(5/2)}(t)\right)^T\end{bmatrix}\qquad \dot{B}=\begin{bmatrix}I\\ \dot{R}^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T\\ I\\ \dot{R}^{(1)}\left(\overleftrightarrow{s_c^{(3/2)}+s_c^{(2/1)}}\right)^T\\ \left(\dot{R}^{(3/2)}(t)\right)^T\\ \dot{R}^{(1)}\left(\overleftrightarrow{s_c^{(4/2)}+s_c^{(2/1)}}\right)^T\\ \left(\dot{R}^{(4/2)}(t)\right)^T\\ \dot{R}^{(1)}\left(\overleftrightarrow{s_c^{(5/2)}+s_c^{(2/1)}}\right)^T\\ \left(\dot{R}^{(5/2)}(t)\right)^T\end{bmatrix} \tag{173}$$

Return to the linear relationship between the relevant variables:

$$\left\{\dot{X}(t)\right\} = \left[B(t)\right]\left\{\dot{q}(t)\right\} + \left[C(t)\right]\left\{\dot{r}(t)\right\} \tag{174}$$

Now, several terms are inspected.

The modified mass matrix:

$$\left[M^*(t)\right] \equiv \left[B(t)\right]^T \left[M\right]\left[B(t)\right] \tag{175}$$

B-transpose has 4 rows and 27 columns, M* will be 4 x 4.

And this:

$$\left[N^*(t)\right] \equiv \left[B(t)\right]^T \left(\left[M\right]\left[\dot{B}(t)\right] + \left[D(t)\right]\left[M\right]\left[B(t)\right]\right) \tag{176}$$

And the forces

$$\left\{F^*(t)\right\} = \left[B(t)\right]^T \left\{F(t)\right\} \tag{177}$$

B and C are found in the previous page:

$$\left[T^*(t)\right] \equiv \left[B(t)\right]^T \left[D(t)\right]\left[M\right]\left[C\right] \tag{178}$$

A variational approach then provides this summative equation of motion:

Equation of Motion

$$\left[M^*(t)\right]\left\{\ddot{q}(t)\right\} + \left[N^*(t)\right]\left\{\dot{q}(t)\right\} = \left\{F^*(t)\right\} - \left[T^*(t)\right]\left\{\dot{r}(t)\right\} \tag{179}$$

For ease, drop all brackets and time dependencies:

$$M^*\ddot{q} + N^*\dot{q} = F^* - T^*\dot{r} \tag{180}$$

State them all:

$$M^* \equiv B^T MB \tag{181}$$

$$N^* \equiv \left(B^T M\dot{B} + B^T DMB\right) \tag{182}$$

$$T^* \equiv B^T DMC \tag{183}$$

$$F^* = B^T F \tag{184}$$

$$M^*\ddot{q} + N^*\dot{q} = F^* - T^*\dot{r} \tag{185}$$

The only terms above that are not time dependent is the M and $\dot{r}$

Note that $\dot{r}$ is given: the speed of the rotors and the blades will be an input on the web page. In future work, it will not be prescribed, they will all accelerate with motors.

Now, begin the easy way to integrate. It will not be stable for long, but is enough for now.

Numerical Integration

Set:

$$p = \{\dot{q}(t)\} = \omega^{(1)}(t) = \begin{pmatrix} \omega_1^{(1)}(t) \\ \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) \end{pmatrix} \tag{186}$$

Thus:

$$M^* \dot{p} + N^* p = F^* - T^* \dot{r} \tag{187}$$

With forward approcimation, the solution is: $\dot{p}_n = \dfrac{p_{n+1} - p_n}{\Delta t}$

Thus:

$$M^* \dot{p} + N^* p = F^* - T^* \dot{r} \tag{188}$$

It becomes:

$$M_n^* \dot{p}_n + N_n^* p_n = F_n^* - T_n^* \dot{r} \tag{189}$$

And that becomes:

$$M_n^* \left(\frac{p_{n+1} - p_n}{\Delta t} \right) + N_n^* p_n = F_n^* - T_n^* \dot{r} \tag{190}$$

Or:

$$M_n^* \left(p_{n+1} - p_n \right) = \Delta t \left(F_n^* - T_n^* \dot{r} - N_n^* p_n \right) \tag{191}$$

Or:

$$p_{n+1} = \left(M_n^* \right)^{-1} \left(M_n^* p_n + \Delta t \left(F_n^* - T_n^* \dot{r} - N_n^* p_n \right) \right) \tag{192}$$

Or:

$$p_{n+1} = p_n + \left(\Delta t \left(M_n^* \right)^{-1} \left(F_n^* - T_n^* \dot{r} - N_n^* p_n \right) \right) \tag{193}$$

Starting with:

$$p_1 = p_0 + \left(\Delta t\left(M_0^{\ *}\right)^{-1}\left(F_0^{\ *} - T_0^{\ *}\dot{r} - N_0^{\ *}p_0\right)\right) \tag{194}$$

Assume that at the start, $p_0 = 0$ for all four terms.

Thus:

$$p_1 = \left(\Delta t\left(M_0^{\ *}\right)^{-1}\left(F_0^{\ *} - T_0^{\ *}\dot{r}\right)\right) \tag{195}$$

Getting the T(0) :

$$T^* \equiv B^T DMC \tag{196}$$

Or:

$$T_0^{\ *} \equiv B_0^{\ T} D_0 MC_0 \tag{197}$$

That "D" in the middle contains something like this:

$$[D] \equiv \begin{bmatrix} \overleftrightarrow{\omega}^{(1)} & \underset{3\times3}{0} & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \overleftrightarrow{\omega}^{(2)} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \vdots & \vdots & \vdots & \ddots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \overleftrightarrow{\omega}^{(5)} \end{bmatrix} \tag{198}$$

Assume quiescent initial conditions. Thus, it is zero to start.

$$p_1 = \Delta t\left(M_0^{\ *}\right)^{-1} F_0^{\ *} \tag{199}$$

Or:

$$p_1 = \Delta t\left(B_0^{\ T} MB_0\right)^{-1} F_0^{\ *} \tag{200}$$

$$F_0^{\ *} = B_0^{\ T} F \tag{201}$$

Assume a constant force:

$$p_1 = \Delta t\left(B_0{}^T M B_0\right)^{-1} B_0{}^T F \tag{202}$$

Since the translation term is removed from the top row, one may no longer include buoyancy or gravity. There are five bodies; the only forces will be a motor that might be used to turn the tower M and -M. There will be an equal and opposite reaction to the float. The only other force will be the wind force, W, on the blades.

$$\{F(t)\} \equiv \begin{Bmatrix} M_C^{(1)}(t) \\ F_C^{(2)I}(t) \\ M_C^{(2)}(t) \\ F_C^{(3)I}(t) \\ M_C^{(3)}(t) \\ F_C^{(4)I}(t) \\ M_C^{(4)}(t) \\ F_C^{(5)I}(t) \\ M_C^{(5)}(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ We_2 \\ 0 \end{Bmatrix} \tag{203}$$

F is 27 x 1

M is:

$$[M] \equiv \begin{bmatrix} J_C^{(1)} & \underset{3\times3}{0} & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & m^{(2)} I_3 & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & J_C^{(2)} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \vdots & \vdots & \vdots & \ddots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & m^{(5)} I_3 & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & J_C^{(5)} \end{bmatrix} \tag{204}$$

Which is (27 x 27) and B is (27 x 3). B-transpose is (3 x 27)

$$p_1 = \Delta t\left(B_0{}^T M B_0\right)^{-1} B_0{}^T F \tag{205}$$

The inverse of a 3 by 3 is known without having to do a gauss inversion:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = {}^{1}\!/_{\det(A)} * Adj(A) \tag{206}$$

$$Adj(A) = \begin{bmatrix} a_{33}a_{22} - a_{32}a_{23} & -\left(a_{33}a_{12} - a_{32}a_{13}\right) & a_{23}a_{12} - a_{22}a_{13} \\ -\left(a_{33}a_{21} - a_{31}a_{23}\right) & a_{33}a_{11} - a_{31}a_{13} & -\left(a_{23}a_{11} - a_{21}a_{13}\right) \\ a_{32}a_{21} - a_{31}a_{22} & -\left(a_{32}a_{11} - a_{31}a_{12}\right) & a_{22}a_{11} - a_{21}a_{12} \end{bmatrix} \tag{207}$$

$$\det(A) = \begin{bmatrix} a_{11}\left(a_{33}a_{22} - a_{32}a_{23}\right) \\ -\,a_{21}\left(a_{33}a_{12} - a_{32}a_{13}\right) \\ +\,a_{31}\left(a_{23}a_{12} - a_{22}a_{13}\right) \end{bmatrix} \tag{208}$$

The B0 is still needed.

Here is B:

$$\begin{bmatrix} I \\ R^{(1)}(t)\left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \\ I \\ R^{(1)}\left(\overleftrightarrow{s_c^{(3/2)} + s_c^{(2/1)}}\right)^T \\ \left(R^{(3/2)}(t)\right)^T \\ R^{(1)}\left(\overleftrightarrow{s_c^{(4/2)} + s_c^{(2/1)}}\right)^T \\ \left(R^{(4/2)}(t)\right)^T \\ R^{(1)}\left(\overleftrightarrow{s_c^{(5/2)} + s_c^{(2/1)}}\right)^T \\ \left(R^{(5/2)}(t)\right)^T \end{bmatrix} \tag{209}$$

Assume, at first, that all rotational matrixes are identity matrixes.

The B(0):

$$\begin{bmatrix} I \\ \left(\overleftrightarrow{s_c^{(2/1)}}\right)^T \\ I \\ \left(\overleftrightarrow{s_c^{(3/2)} + s_c^{(2/1)}}\right)^T \\ I \\ \left(\overleftrightarrow{s_c^{(4/2)} + s_c^{(2/1)}}\right)^T \\ I \\ \left(\overleftrightarrow{s_c^{(5/2)} + s_c^{(2/1)}}\right)^T \\ I \end{bmatrix} \tag{210}$$

One may then solve this:

$$p_1 = \Delta t\left(B_0^{\ T} M B_0\right)^{-1} F_0^{\ *} \tag{211}$$

Which will give the omega for the first time-step:

$$p = \left\{\dot{q}(t)\right\} = \omega^{(1)}(t) = \begin{pmatrix} \omega_1^{(1)}(t) \\ \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) \end{pmatrix} \tag{212}$$

Then, turn to Cayley Hamilton and get R-updated from appendix 2.

Note which will be the updated R1, minus the Identity (the original R1).

Each term is devided by delta time-step, which gives R(1) and R(1)-dot at the first time step.

The iteration begins.

$$p_{n+1} = p_n + \left(\Delta t\left(M_n^{\ *}\right)^{-1}\left(F_n^{\ *} - T_n^{\ *}\dot{r} - N_n^{\ *} p_n\right)\right) \tag{213}$$

3. Results

The goal of this paper is a qualitative and visualized result. The Caley-Hamilton theorem is used to reconstruct the unknown rotation matrix of the floating wind turbine, though this is only valid for constant angular velocities. In this first pass, simplifications have been made to accommodate this. In order to solve the equations, numerical methods are applied. The next step is the coding, which can be viewed in Appendix 4. There, one can find the code for the numerical methods, the reader may view the codes directly at:

https://home.hvl.no/prosjekter/dynamics/2020/windmill/index.html

By utilizing WebGL (Web Graphics Library), alongside JavaScript and HTML, the goal is to show the effect of the inertial disks in the turbine. An online webpage was created where the research results are presented qualitatively as a 3D simulation. One can more intuitively see and understand the effects the disks have on the structure, as one is comparing to visual instances along with the data, instead of only the data.

The following are pictures of the website, where three simulations have been run:

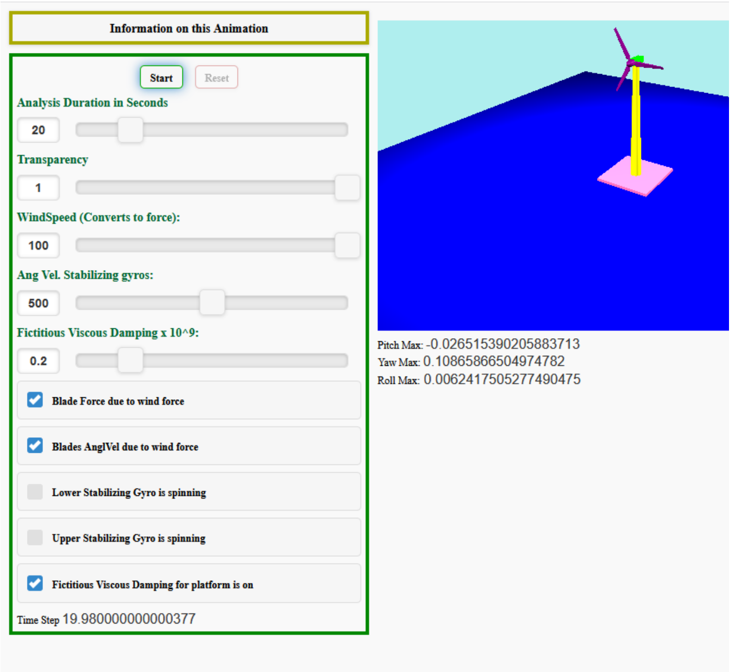

Figure 9: The wind turbine after a simulation without the effects of the stabilizing.

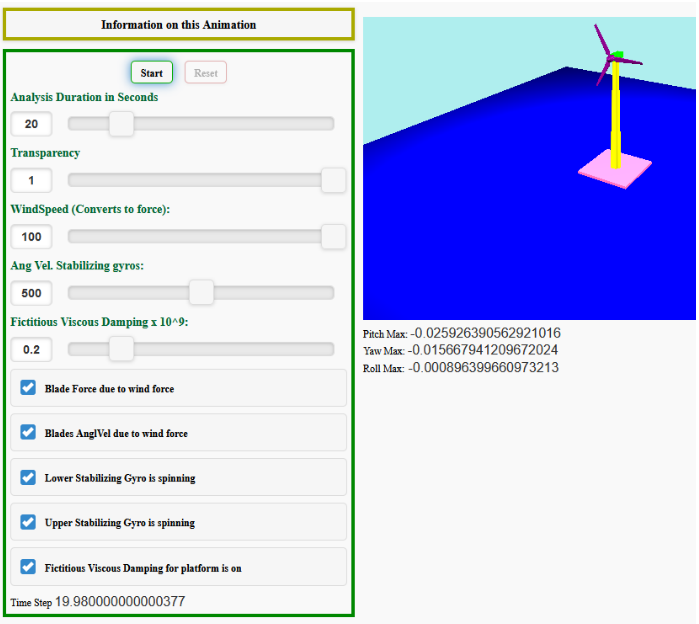

Figure 10: The wind turbine after a simulation with the effects of the stabilizing.

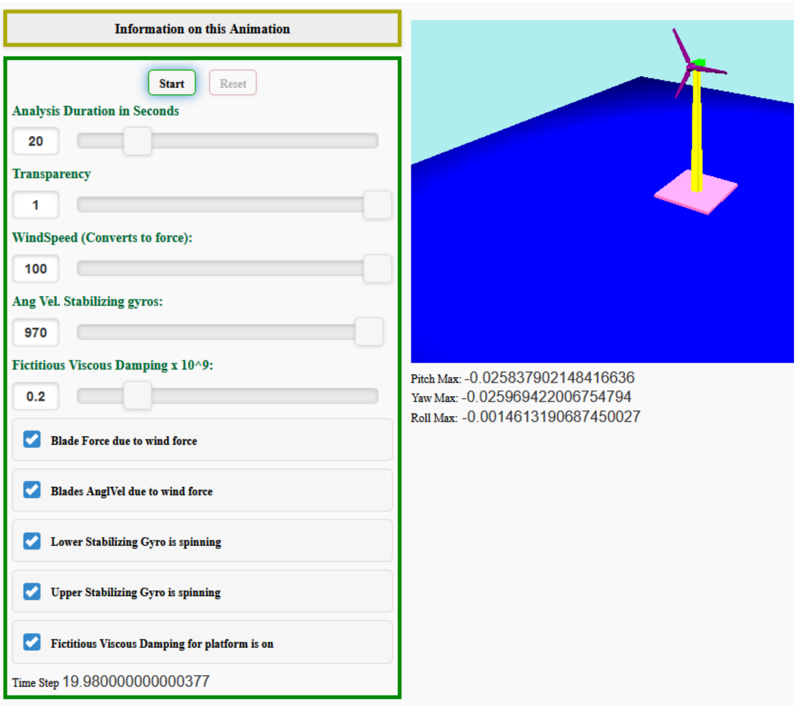

Figure 11: The wind turbine after a simulation, this time with the angular velocity of the disks to max.

Comparing the figures number 9, 10 and 11, one may note the effects of the stabilizing gyros on the number “Yaw Max”. It has decreased from figure 9 to figure 10, and in figure 11 the angular velocity is turned up to so high that it is negative. The roll goes from positive to negative as well. This indicates that the spinning disks influence the wind turbines movement.

4. Discussion

Initially, the tower could rotate from the base. While this was done in order to allow a freedom of design, it might not have been necessary. It might have been more desirable to create the simplest model possible, to give an easier approach, even though this paper showed a simplified version later. Still, a separate float allows it to have different material properties from the rest of the wind turbine, which is more realistic.

The scientific focus does, on the other hand, allow great freedom for those who are familiar with the MFM and may in that regard be a great tool. If anything, it shows the flexibility of the method, and how one may open for future work without hindering accurate results. It generates results without compromise to its flexibility and complexity.

The separation of the float opens for more options, such as having more than one wind turbine on the same float, or a mounted crane, for example. In other words; a base for calculating larger and more complex structures. It would be a good base for more complex gyroscopic wave energy converters placed inside floating wind turbines, or floating structures with more than one wind turbine, crane or similar.

One could have included a moving nacelle, instead of fixing it to the tower structure. This was done because it more closely resembled the real-life function of a wind turbine; it will turn to face the wind, and then the nacelle will lock in place. On the one hand, it could be allowed to move, to open for similar freedom of design as done with the float and include this motion as well.

On the other, this seemed redundant, as the nacelle and tower structure are often designed and constructed by the same manufacturer, while the float is often not. This would also be a quite complex function in comparison to the others shown in this paper. Due to time-restrictions, the paper had to be limited in this regard.

5. Conclusion and future work

This project creates a foundation to expand how to calculate the movement and momentum affecting a wind turbine, while exploring the possibility of using the gyroscopic effect to reduce yaw and roll created by the wind turbines' blades and wind. It shows the effect of spinning disks placed inside the wind turbine to reduce motion, as illustrated in the 3D simulation. These same spinning disks may also capture the moment induced by the motion of the waves as an additional energy source in future work. It may also be expanded to include more spinning disks or similar rotating elements.

The next step would be to introduce waves into the code to show the spinning disks effects on this motion. Further along, one could write a more complex code, including the forces like buoyancy and gravitation, which were removed for simplification in this project. One may also include freedom of rotation for the nacelle, changing one rotational matrix and then adding one more frame relation.

It could also be prudent to create a standardized model for floating elements and wind turbines explicitly for offshore use, as the one from DTU [10] used as reference in this paper, which is for land-based wind turbines. It should encompass basic geometric shapes, sizes and mass. Such a model could easily be implemented into this project and be a base for other, similar projects. This would allow further work on this paper, in order to create a simulation closer to a real-life scenario.

One could also build a small-scale model of a floating wind turbine with spinning disks and subject it to waves in a wave-pool. This could possibly show the potential of these spinning disks, and the dimensions needed for them to function.

6. References

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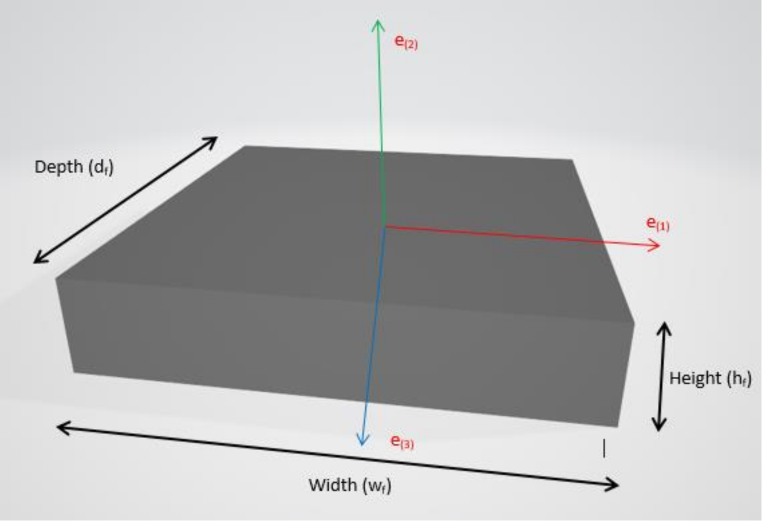
List of figures

Appendix 1: Physical Parameters

The following are the physical shapes used in the code to get the results. They are simplified, as this allows the numerical integrations in the code to be more stable.

Float:

Height: $h_f = 5m$
Width: $w_f = 20m$
Depth: $d_f = 20m$
Mass: $m_f = 300ton$



A.1 - Figure 1: The simple float model. They come in many various shapes.

Mass Moment of Inertia:

J-matrix:

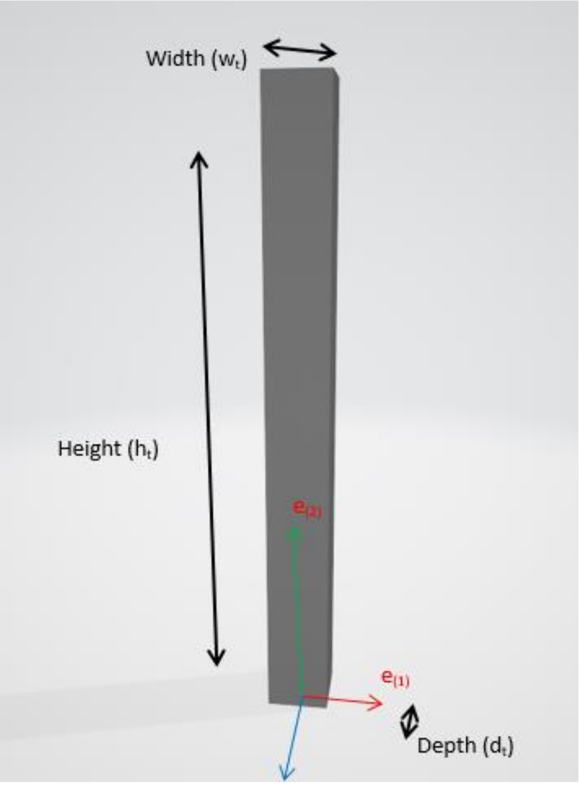
$$J_f = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12}m(h_f^2 + d_f^2) & 0 & 0 \\ 0 & \frac{1}{12}m(w_f^2 + d_f^2) & 0 \\ 0 & 0 & \frac{1}{12}m(w_f^2 + h_f^2) \end{bmatrix} \quad (A.1)$$

Center of mass:

The frame may be placed wherever, initially. The simplest thing is then to place it in the center of mass.

$$CM_f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (A.2)$$

Tower:



A.1 - Figure 2: The tower fuselage.

Height: $h_t = 100$m
Width: $w_t = 5$m
Depth: $d_t = 5$m
Mass: $m_t = 600$ ton

Mass Moment of Inertia:

J-matrix:

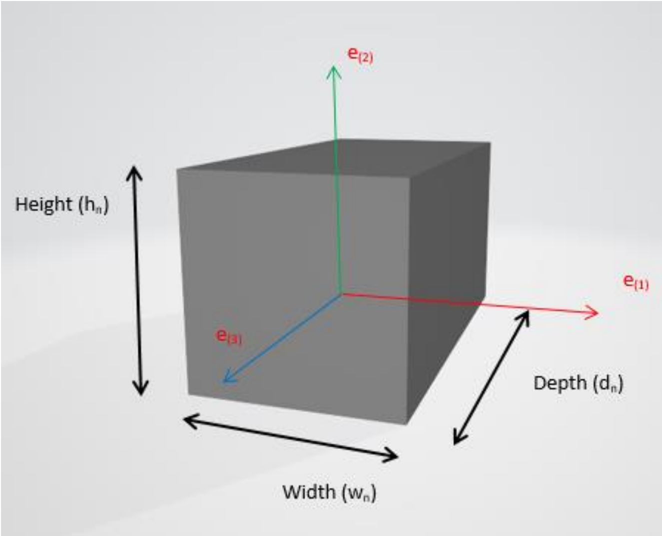
$$J_t = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12}m(h_t^2 + d_t^2) & 0 & 0 \\ 0 & \frac{1}{12}m(w_t^2 + d_t^2) & 0 \\ 0 & 0 & \frac{1}{12}m(w_t^2 + h_t^2) \end{bmatrix} \tag{A.3}$$

Center of mass:
From the center of the float to the center of the tower

$$CM_t = \begin{pmatrix} 0 \\ 0 \\ h_f/2 + h_t/2 \end{pmatrix} \tag{A.4}$$

Nacelle:

Height: $h_n = 6m$
Width: $w_n = 5m$
Depth: $d_n = 10m$
Mass: $m_n = 400$ ton



A.1 - Figure 3: The Nacelle.

Mass Moment of Inertia:

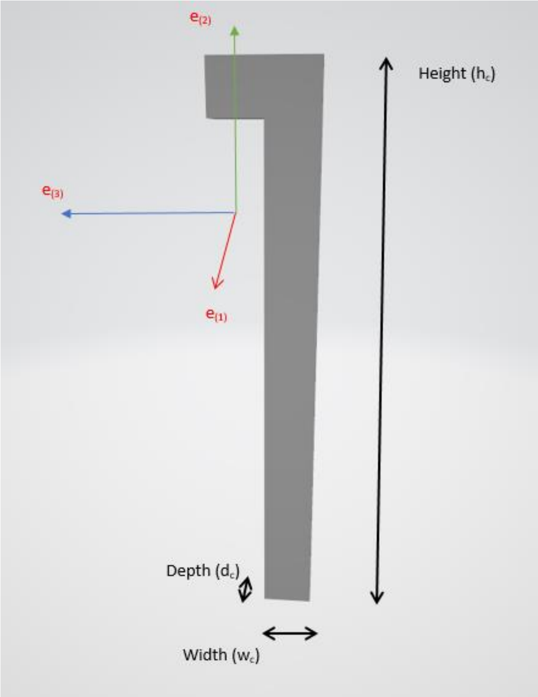
J-matrix:

$$J_n = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12}m(h_n^2 + d_n^2) & 0 & 0 \\ 0 & \frac{1}{12}m(w_n^2 + dn^2) & 0 \\ 0 & 0 & \frac{1}{12}m(w_n^2 + h_n^2) \end{bmatrix} \quad (A.5)$$

Center of mass:

$$CM_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (A.6)$$

Combined Tower and Nacelle (nt)



A.1 - Figure 4: The Tower and Nacelle combined.

The parallel axis theorem:

$$J_O = m\,\overrightarrow{s_C}\,\overrightarrow{s_C}^T + J_C = m\begin{bmatrix} s_{2C}^2 + s_{3C}^2 & -s_{1C}s_{2C} & -s_{1C}s_{3C} \\ -s_{1C}s_{2C} & s_{1C}^2 + s_{3C}^2 & -s_{2C}s_{3C} \\ -s_{1C}s_{3C} & -s_{2C}s_{3C} & s_{1C}^2 + s_{2C}^2 \end{bmatrix} + J_C \qquad (A.7)$$

First, find the CM of the new system of the combined tower and nacelle. Establish a temporary frame and put that frame at the CM of the tower. From that frame, locate the new CM:

Tower CM:

$$CM_t = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (A.8)$$

Nacelle CM:

$$CM_n = \begin{pmatrix} 0 \\ \dfrac{h_t + h_n}{2} \\ \dfrac{d_n - d_t}{2} \end{pmatrix} \qquad (A.9)$$

This is from the CM of the tower, then up and out along the third axis to the center of the nacelle.

Weigh both with the mass, add, and divide by total mass. Lowercase "c" is used for the combined structure.

$$c_{cm} = \frac{m_n}{m_t + m_n}\begin{pmatrix} 0 \\ \frac{h_t + h_n}{2} \\ \frac{d_n - d_t}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{(A.10)}$$

$$\overleftrightarrow{c_{cm}} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & 0 \\ -c_2 & 0 & 0 \end{bmatrix} \quad \text{(A.11)}$$

Now, put a frame in the new CM.

Moment of inertia of combined about the new CM

$$J_O = m\overleftrightarrow{s_C}\,\overleftrightarrow{s_C}^T + J_C = m\begin{bmatrix} s_{2C}^2 + s_{3C}^2 & -s_{1C}s_{2C} & -s_{1C}s_{3C} \\ -s_{1C}s_{2C} & s_{1C}^2 + s_{3C}^2 & -s_{2C}s_{3C} \\ -s_{1C}s_{3C} & -s_{2C}s_{3C} & s_{1C}^2 + s_{2C}^2 \end{bmatrix} + J_C \quad \text{(A.12)}$$

Moment of inertia of tower about new CM - O

$$J_{t*} = m_t\begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & 0 \\ -c_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{12}m_t(h_t^2 + d_t^2) & 0 & 0 \\ 0 & \frac{1}{12}m_t(w_t^2 + d_t^2) & 0 \\ 0 & 0 & \frac{1}{12}m_t(w_t^2 + h_t^2) \end{bmatrix} \quad \text{(A.13)}$$

Moment of inertia of Nacelle about the new CM

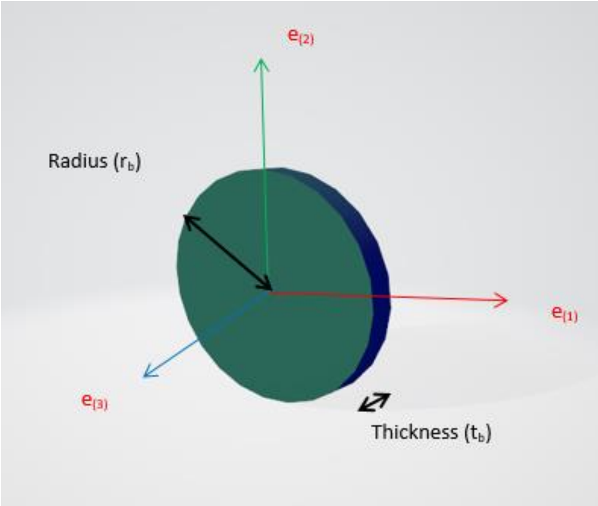
$$J_{n*} = m_n\begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & 0 \\ -c_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{12}m_n(h_n^2 + d_n^2) & 0 & 0 \\ 0 & \frac{1}{12}m(w_n^2 + d_n^2) & 0 \\ 0 & 0 & \frac{1}{12}m(w_n^2 + h_n^2) \end{bmatrix} \quad \text{(A.14)}$$

Now, add them to get the moment of inertia about the combined structure.

$$J_c = J_{t*} + J_{n*} \qquad (A.15)$$

Blade rotor

Radius: $r_b = 170$m
Thickness: t_b (negligible)
Mass: $m_b = 230$ ton



A.1 - Figure 5: Since the force of the wind is prescribed, the blades can be simplified to a disk.

Mass Moment of Inertia:

J-matrix:

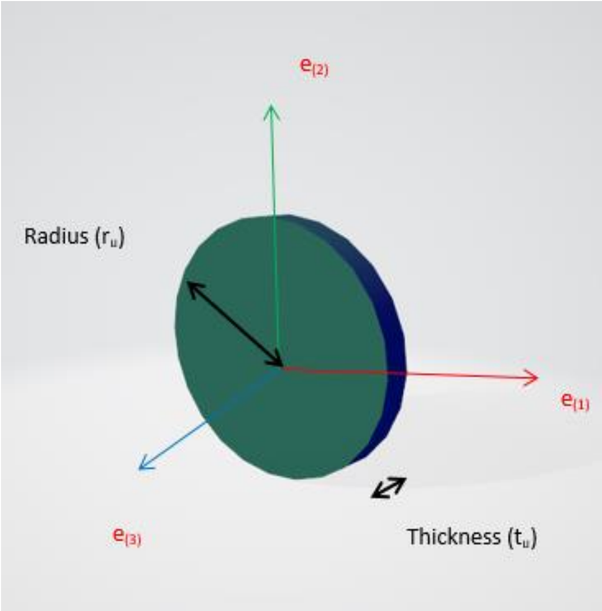
$$J_b = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} m_b h_b^2 & 0 & 0 \\ 0 & \frac{1}{12} m r_b^2 & 0 \\ 0 & 0 & \frac{1}{12} m_b h_b^2 \end{bmatrix} \qquad (A.16)$$

Center of mass:

$$CM_b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (A.17)$$

Upper Disk

Radius: r_u
Thickness: t_u (negligible)
Mass: m_u



A.1 - Figure 6: The upper rotating disk.

The thickness is negligible, and it becomes clear why, when one takes into account what dimensions of the disk has more impact of its moment of inertia.

Mass Moment of Inertia:

J-matrix:

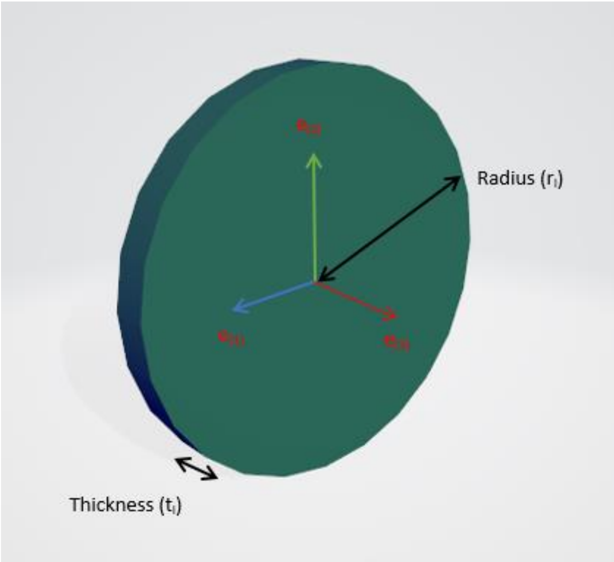
$$J_u = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} m_u h_u^2 & 0 & 0 \\ 0 & \frac{1}{12} m_u r_u^2 & 0 \\ 0 & 0 & \frac{1}{12} m_u h_u^2 \end{bmatrix} \tag{A.18}$$

Centre of mass:

$$CM_u = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.19}$$

Lower Disk

Radius: r_l
Thickness: t_l (negligible)
Mass: m_l



A.1 - Figure 7: The lower rotating disk.

Mass Moment of Inertia:

J-matrix:

$$J_l = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} m_l r_l^2 & 0 & 0 \\ 0 & \frac{1}{12} m_l h_l^2 & 0 \\ 0 & 0 & \frac{1}{12} m_l h_l^2 \end{bmatrix} \tag{A.20}$$

Centre of mass:

$$CM_l = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.21}$$

Appendix 2: Caley Hamilton Theorem

Reconstruction of R1 using Caley Hamilton:

The MFM introduces a restriction on the variation of the angular velocity for Hamilton's Principle. The numerical integration method is not precise enough and gives numerical noise over time.

The generalized coordinates include the omegas for the float.

However, one still need the R for the float: the rotation matrix.

Note:

SO(3) is the group of rotations

so(3) is the algebra of angular velocities

The rotation matrix for the foundation is needed for two reasons:

To rotate the float in the WebGL visualization.

Because the terms are used in the B-matrix.

$$R(t) = I + \overleftrightarrow{\omega(0)}\,t + \frac{1}{2!}t^2\,\overleftrightarrow{\omega(0)}\,\overleftrightarrow{\omega(0)} + \frac{1}{3!}t^3\,\overleftrightarrow{\omega(0)}\,\overleftrightarrow{\omega(0)}\,\overleftrightarrow{\omega(0)} + \frac{1}{4!}t^4\,\overleftrightarrow{\omega(0)}\,\overleftrightarrow{\omega(0)}\,\overleftrightarrow{\omega(0)}\,\overleftrightarrow{\omega(0)} + \ldots \quad \text{(A2.1)}$$

That is the solution; to multiply the skew matrix several times in a series.

It may also be written like this:

$$R(t) = I + \left(\overleftrightarrow{\omega(0)}\,t\right) + \frac{1}{2!}\left(\overleftrightarrow{\omega(0)}\,t\right)^2 + \frac{1}{3!}\left(\overleftrightarrow{\omega(0)}\,t\right)^3 + \frac{1}{4!}\left(\overleftrightarrow{\omega(0)}\,t\right)^4 + \ldots \quad \text{(A2.2)}$$

Appendix 3: Prescribed Rates

Kinetics of the wind turbine

The Lagrangian is defined as the difference between the kinetic and potential energy:

$$L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)=K^{(\alpha)}\left(q(t),\dot{q}(t),t\right)-U^{(\alpha)}\left(q(t),t\right) \quad \text{(A3.1)}$$

The Action is defined as the integral of the Lagrangian:

$$A=\int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt \quad \text{(A3.2)}$$

Hamilton's principle states that "the motion of a system occurs in such a way that the definite integral A, becomes a minimum for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed" [7]

Action equal to zero to get the equations of motion:

$$A=\int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt=0 \quad \text{(A3.3)}$$

Introduce then principle of Virtual Work:

$$\int_{t_0}^{t_1}\{\delta K(t)+\delta W(t)\}\,dt=0 \quad \text{(A3.4)}$$

$$\{\dot{X}(t)\}=[B(t)]\{\dot{q}(t)\}+[C(t)]\{\dot{r}(t)\} \quad \text{(A3.5)}$$

First, drop all brackets and dependencies.

$$\dot{X}=B\dot{q}+C\dot{r} \quad \text{(A3.6)}$$

One could here insert the kinetic energy and the work. It is, however, important to note that there are no moments working on the gyro; the spin is constant. There could be other forces and moments in other directions, but that would be another problem. Thus, F* has been contracted along with the q.

$$\int_{t_0}^{t_1}\left(\{\delta\dot{X}(t)\}^T[M]\{\dot{X}(t)\}+\{\delta q(t)\}^T\{F^*(t)\}\right)dt=0 \quad \text{(A3.7)}$$

Insert:

$$\{\delta\dot{X}\}=\frac{d}{dt}\{\delta\tilde{X}\}+[D]\{\delta\tilde{X}\} \quad \text{(A3.8)}$$

To get:

$$\int_{t_0}^{t_1}\left[\left(\frac{d}{dt}\{\delta\tilde{X}\}+[[D]\{\delta\tilde{X}\}]\right)^T[M]\{\dot{X}\}+\{\delta q\}^T\{F^*\}\right]dt=0 \quad \text{(A3.9)}$$

Perform the transpose:

$$\int_{t_0}^{t_1}\left[\left(\frac{d}{dt}\left\{\delta\tilde{X}\right\}^T + \left\{\delta\tilde{X}\right\}^T [D]^T\right)[M]\left\{\dot{X}\right\} + \{\delta q\}^T\left\{F^*\right\}\right]dt = 0 \tag{A3.10}$$

Skew symmetry of D:

$$\int_{t_0}^{t_1}\left[\left(\frac{d}{dt}\left\{\delta\tilde{X}\right\}^T - \left\{\delta\tilde{X}\right\}^T [D]\right)[M]\left\{\dot{X}\right\} + \{\delta q\}^T\left\{F^*\right\}\right]dt = 0 \tag{A3.11}$$

Distribute:

$$\int_{t_0}^{t_1}\left[\left(\frac{d}{dt}\left\{\delta\tilde{X}\right\}^T [M]\left\{\dot{X}\right\} - \left\{\delta\tilde{X}\right\}^T [D][M]\left\{\dot{X}\right\}\right) + \{\delta q\}^T\left\{F^*\right\}\right]dt = 0 \tag{A3.12}$$

Simplify:

$$\int_{t_0}^{t_1}\left[\left(\frac{d}{dt}\left\{\delta\tilde{X}\right\}^T M\dot{X} - \left\{\delta\tilde{X}\right\}^T DM\dot{X}\right) + \{\delta q\}^T F^*\right]dt = 0 \tag{A3.13}$$

Integration by parts:

$$\int_{t_0}^{t_1}\left[\left(-\left\{\delta\tilde{X}\right\}^T \frac{d}{dt}\left(M\dot{X}\right) - \left\{\delta\tilde{X}\right\}^T DM\dot{X}\right) + \{\delta q\}^T F^*\right]dt = 0 \tag{A3.14}$$

Adjust the sign:

$$\int_{t_0}^{t_1}\left[\left(\left\{\delta\tilde{X}\right\}^T \frac{d}{dt}\left(M\dot{X}\right) + \left\{\delta\tilde{X}\right\}^T DM\dot{X}\right) - \{\delta q\}^T F^*\right]dt = 0 \tag{A3.15}$$

Pull out a term:

$$\int_{t_0}^{t_1}\left[\left(\left\{\delta\tilde{X}\right\}^T \left(\frac{d}{dt}\left(M\dot{X}\right) + DM\dot{X}\right)\right) - \{\delta q\}^T F^*\right]dt = 0 \tag{A3.16}$$

Use this:

$$\left\{\delta\tilde{X}(t)\right\}^T = \{\delta q(t)\}^T [B(t)]^T \tag{A3.17}$$

Or this:

$$\int_{t_0}^{t_1}\left[\left(\{\delta q(t)\}^T [B(t)]^T \left(\frac{d}{dt}\left(M\dot{X}\right) + DM\dot{X}\right)\right) - \{\delta q\}^T F^*\right]dt = 0 \tag{A3.18}$$

Distribute the B and pull out the q:

$$\int_{t_0}^{t_1}\left[\left(\{\delta q(t)\}^T \left(B^T \frac{d}{dt}\left(M\dot{X}\right) + B^T DM\dot{X}\right) - F^*\right)\right]dt = 0 \tag{A3.19}$$

The variation is arbitrary; put it as such:

$$\left(B^T \frac{d}{dt}\left(M\dot{X}\right) + B^T DM\dot{X}\right) - F^* = 0 \tag{A3.20}$$

Now, insert the principle of Virtual Work: $\dot{X} = B\dot{q} + C\,\dot{r}$:

$$\left(B^T \frac{d}{dt}\left(M\left(B\dot{q} + C\,\dot{r}\right)\right) + B^T DM\left(B\dot{q} + C\,\dot{r}\right)\right) - F^* = 0 \tag{A3.21}$$

Distribute the time derivative:

$$\left(B^T\left(M\left(B\ddot{q} + \dot{B}\dot{q} + C\ddot{r} + \dot{C}\dot{r}\right)\right) + B^T DM\left(B\dot{q} + C\,\dot{r}\right)\right) - F^* = 0 \tag{A3.22}$$

Distribute all other terms:

$$\left(B^T MB\ddot{q} + B^T M\dot{B}\dot{q} + B^T MC\ddot{r} + B^T M\dot{C}\dot{r} + B^T DMB\dot{q} + B^T DMC\,\dot{r}\right) - F^* = 0 \tag{A3.23}$$

Consolidate:

$$B^T MB\ddot{q} + \left(B^T M\dot{B} + B^T DMB\right)\dot{q} + B^T MC\ddot{r} + \left(B^T M\dot{C} + B^T DMC\right)\dot{r} - F^* = 0 \tag{A3.24}$$

Bring terms to the right:

$$B^T MB\ddot{q} + \left(B^T M\dot{B} + B^T DMB\right)\dot{q} = F^* - B^T MC\ddot{r} - \left(B^T M\dot{C} + B^T DMC\right)\dot{r} \tag{A3.25}$$

Defined as:

$$\left[M^*(t)\right] \equiv \left[B(t)\right]^T \left[M\right]\left[B(t)\right] \tag{A3.26}$$

$$\left[N^*(t)\right] \equiv \left[B(t)\right]^T \left(\left[M\right]\left[\dot{B}(t)\right] + \left[D(t)\right]\left[M\right]\left[B(t)\right]\right) \tag{A3.27}$$

$$M^*\ddot{q} + N^*\dot{q} = F^* - B^T MC\ddot{r} - \left(B^T M\dot{C} + B^T DMC\right)\dot{r} \tag{A3.28}$$

If r-dot is constant, then r-double dot is zero.

$$M^*\ddot{q} + N^*\dot{q} = F^* - \left(B^T M\dot{C} + B^T DMC\right)\dot{r} \tag{A3.29}$$

Review all terms.

The D and M are the same.

$$D \equiv \begin{bmatrix} \ddot{\omega}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddot{\omega}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddot{\omega}^{(3)} \end{bmatrix} \tag{A3.30}$$

$$M = \begin{bmatrix} J^{(1)} & 0 & 0 & 0 & 0 \\ 0 & m^{(2)}I & 0 & 0 & 0 \\ 0 & 0 & J^{(2)} & 0 & 0 \\ 0 & 0 & 0 & m^{(3)}I & 0 \\ 0 & 0 & 0 & 0 & J^{(3)} \end{bmatrix} \tag{A3.31}$$

For the B matrix, one can eliminate the last column.

Showing the original B, when the full theory was partitioned:

$$B \equiv \begin{bmatrix} I_3 & \underset{3x1}{0} & \underset{3x1}{0} \\ -R^{(1)}\ddot{s}^{(2/1)} & \underset{3x1}{0} & \underset{3x1}{0} \\ R_3^{(2/1)T} & e_3 & \underset{3x1}{0} \\ -R^{(1)}\left(\ddot{s}^{(2/1)} + \overleftrightarrow{R_3^{(2/1)}\ddot{s}^{(3/2)}}\right) & R^{(1)}R_3^{(2/1)}\ddot{s}^{(3/2)}e_3 & \underset{3x1}{0} \\ \left(R_3^{(2/1)}R_1^{(3/2)}\right)^T & \left(R_1^{(3/2)}\right)^T e_3 & e_1 \end{bmatrix} \equiv \begin{bmatrix} I_3 & \underset{3x1}{0} & \underset{3x1}{0} \\ B_{21} & \underset{3x1}{0} & \underset{3x1}{0} \\ B_{31} & e_3 & \underset{3x1}{0} \\ B_{41} & B_{42} & \underset{3x1}{0} \\ B_{51} & B_{52} & e_1 \end{bmatrix} \tag{A3.32}$$

This will be the current B, drop the last column:

$$B \equiv \begin{bmatrix} I_3 & \underset{3x1}{0} \\ -R^{(1)}\ddot{s}^{(2/1)} & \underset{3x1}{0} \\ R_3^{(2/1)T} & e_3 \\ -R^{(1)}\left(\ddot{s}^{(2/1)} + \overleftrightarrow{R_3^{(2/1)}\ddot{s}^{(3/2)}}\right) & R^{(1)}R_3^{(2/1)}\ddot{s}^{(3/2)}e_3 \\ \left(R_3^{(2/1)}R_1^{(3/2)}\right)^T & \left(R_1^{(3/2)}\right)^T e_3 \end{bmatrix} \equiv \begin{bmatrix} I_3 & \underset{3x1}{0} \\ B_{21} & \underset{3x1}{0} \\ B_{31} & e_3 \\ B_{41} & B_{42} \\ B_{51} & B_{52} \end{bmatrix} \tag{A3.33}$$

$$B^T \equiv \begin{bmatrix} I_3 & (B_{21})^T & (B_{31})^T & (B_{41})^T & (B_{51})^T \\ \underset{1x3}{0} & \underset{1x3}{0} & (e_3)^T & (B_{42})^T & (B_{52})^T \end{bmatrix} \tag{A3.34}$$

C is needed. C will take the spin of the gyro into all Cartesian components:

$$\begin{Bmatrix} \omega_1^{(1)} \\ \omega_2^{(1)} \\ \omega_3^{(1)} \\ \dot{x}_1^{(2)} \\ \dot{x}_2^{(2)} \\ \dot{x}_3^{(2)} \\ \omega_1^{(2)} \\ \omega_2^{(2)} \\ \omega_3^{(2)} \\ \dot{x}_1^{(3)} \\ \dot{x}_2^{(3)} \\ \dot{x}_3^{(3)} \\ \omega_1^{(3)} \\ \omega_2^{(3)} \\ \omega_3^{(3)} \end{Bmatrix} = \underset{15x3}{\left[\ B(t) \right]} \begin{Bmatrix} \omega_1^{(1)} \\ \omega_2^{(1)} \\ \omega_3^{(1)} \\ \dot{\theta} \end{Bmatrix} + \underset{15x1}{\left[\ C(t) \right]} \left(\dot{\phi}\right) \tag{A3.35}$$

$$\begin{pmatrix} \omega^{(1)} \\ \dot{x}^{(2)} \\ \omega^{(2)} \\ \dot{x}^{(3)} \\ \omega^{(3)} \end{pmatrix} = \underset{5x2}{\left[\ B(t) \right]} \begin{pmatrix} \omega^{(1)} \\ \dot{\theta} \end{pmatrix} + \underset{5x1}{\begin{bmatrix} \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ e_1 \end{bmatrix}} \left(\dot{\phi}\right) \tag{A3.36}$$

Here, $\dot{r} = \dot{\phi}$

Thus, this is desired:

$$B^T DMC\dot{r} \tag{A3.37}$$

$$\begin{bmatrix} I_3 & (B_{21})^T & (B_{31})^T & (B_{41})^T & (B_{51})^T \\ \underset{1x3}{0} & \underset{1x3}{0} & (e_3)^T & (B_{42})^T & (B_{52})^T \end{bmatrix} \begin{bmatrix} \ddot{\omega}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddot{\omega}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddot{\omega}^{(3)} \end{bmatrix} \begin{bmatrix} J^{(1)} & 0 & 0 & 0 & 0 \\ 0 & m^{(2)}I & 0 & 0 & 0 \\ 0 & 0 & J^{(2)} & 0 & 0 \\ 0 & 0 & 0 & m^{(3)}I & 0 \\ 0 & 0 & 0 & 0 & J^{(3)} \end{bmatrix} \begin{bmatrix} \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ e_1 \end{bmatrix} \dot{\phi} \tag{A3.38}$$

$$\begin{bmatrix} I_3 & (B_{21})^T & (B_{31})^T & (B_{41})^T & (B_{51})^T \\ \underset{1x3}{0} & \underset{1x3}{0} & (e_3)^T & (B_{42})^T & (B_{52})^T \end{bmatrix} \begin{bmatrix} J^{(1)}\vec{\vec{\omega}}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J^{(2)}\vec{\vec{\omega}}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J^{(3)}\vec{\vec{\omega}}^{(3)} \end{bmatrix} \begin{bmatrix} \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ e_1 \end{bmatrix} \dot{\phi} \tag{A3.39}$$

$$\begin{bmatrix} I_3 & (B_{21})^T & (B_{31})^T & (B_{41})^T & (B_{51})^T \\ \underset{1x3}{0} & \underset{1x3}{0} & (e_3)^T & (B_{42})^T & (B_{52})^T \end{bmatrix} \begin{bmatrix} \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ \underset{3x1}{0} \\ \dot{\phi} J^{(3)}\vec{\vec{\omega}}^{(3)} e_1 \end{bmatrix} \tag{A3.40}$$

$$\begin{bmatrix} (B_{51})^T \dot{\phi} J^{(3)}\vec{\vec{\omega}}^{(3)} e_1 \\ (B_{52})^T \dot{\phi} J^{(3)}\vec{\vec{\omega}}^{(3)} e_1 \end{bmatrix} \tag{A3.41}$$

The Force on the blades:

A reference wind turbine made by DTU [10], is used to get the thrust coefficient, the rotors square area, rated windspeed and blade radius.

$$C_T = 0,85 \tag{A3.42}$$

Formula for wind force:

$$F_T = C_T \left(\frac{1}{2} \rho_{air} A_{rotor} u^2{}_{wind} \right) \tag{A3.43}$$

$$F_T = C_T \left(\rho_{air} R^2 \pi u^2{}_{wind} \right) \tag{A3.44}$$

Density of air is found from [11].

$$F_T = 0,85 \left(1,2 \, {}^{kg}\!/\!{}_{m^3} \cdot \left(\frac{170m}{2} \right)^2 \pi \cdot \left(11,5 \, {}^{m}\!/\!{}_{s} \right)^2 \right) \tag{A3.45}$$

Appendix 4: The Code

Various WebGL functions were written to carry out this work. The only one we choose to list here, in the Appendix, is the numerical integration itself. For all other associated modules, the reader may view the codes directly at:

https://home.hvl.no/prosjekter/dynamics/2020/windmill/index.html

```
function myIntegrate(document ) {

        var damp = myTime.damping
        var i, j
        var dt                  = myTime.dt;
        var dti                 = 1/dt
        var nstop               = myTime.nstop
        var presc               = zeroMatrix(3, 1)
        var pn                  = zeroMatrix(3, 1)
        var B                   = zeroMatrix(27, 3)
        var Bdot                = zeroMatrix(27,3)
        var BT                  = zeroMatrix(3, 27)

        var DM                  = zeroMatrix(27, 27)
        var BTDM                = zeroMatrix(3, 27)
        var MB                          = zeroMatrix(27, 3)
        var MBdot               = zeroMatrix(27, 3)
        var BTMB                = zeroMatrix(3, 3)
        var BTMBdot             = zeroMatrix(3, 3)

        var R11                 = zeroMatrix(3, 3)
        var R11d                = zeroMatrix(3, 3)
        var R11dot              = zeroMatrix(3, 3)

        var BTDMB               = zeroMatrix(3, 3)
        var BTDMC               = zeroMatrix(3, 3)
        var BTDMBdot            = zeroMatrix(3, 3
        var BTMBdotmBTDMB               = zeroMatrix(3, 3)
```

```
var Mx              = zeroMatrix(3, 3)
var Tx              = zeroMatrix(3, 3)
var Fx              = zeroMatrix(3, 1)
var Nx              = zeroMatrix(3, 3)

var MxI                     = zeroMatrix(3, 3)

var damping         = zeroMatrix(3, 1)
var pn              = zeroMatrix(3, 1)
var Nxpn            = zeroMatrix(3, 1)
var Txr             = zeroMatrix(3, 1)

var Viscous         = zeroMatrix(3, 1)
var FxmTxr          = zeroMatrix(3, 1)
var FxmTxrmNxp      = zeroMatrix(3, 1)

var MxIFxmTxrmNxp   = zeroMatrix(3, 1)

var pnp1                    = zeroMatrix(3, 1)
var ptemp           = zeroMatrix(3, 1)

var test1                   = zeroMatrix(3, 3)
var test2                   = zeroMatrix(3, 3)
var x1              = zeroMatrix(3, 1)
var x2              = zeroMatrix(3, 1)
var x3              = zeroMatrix(3, 1)
var of              = zeroMatrix(3, 3)
var oc              = zeroMatrix(3, 3)
var ob              = zeroMatrix(3, 3)
var ou              = zeroMatrix(3, 3)
var ol              = zeroMatrix(3, 3)

var time = dt

presc[0] = myBlade.zetad
presc[1] = myUpper.phid
presc[2] = myLower.psid
```

```
pn[0] = myResults.w1[0]
pn[1] = myResults.w2[0]
pn[2] = myResults.w3[0]

for(var icount = 1; icount < nstop; icount++) {

        time = time + dt
        formB(time)

        formBdot(time)
        formD()
        formF(time)

        MatTranspose(BT, myGenData.B, 27, 3)

        MatMatMult(DM, myGenData.D, myGenData.M, 27, 27, 27)
        MatMatMult(BTDM, BT, DM, 3, 27, 27)
        MatMatMult(BTDMB, BTDM, myGenData.B, 3, 3, 27)

        MatMatMult(MB, myGenData.M, myGenData.B, 27, 3, 27)
        MatMatMult(MBdot, myGenData.M, myGenData.Bdot, 27, 3, 27)
        MatMatMult(BTMBdot, BT, MBdot, 3, 3, 27)

        MatMatAdd(Nx, BTMBdot, BTDMB, 3, 3)
        MatMatMult(Mx, BT, MB, 3, 3, 27)
        MatMatMult(Tx, BTDM, myGenData.C, 3, 3, 27)
        MatVecMult(Fx, BT, myGenData.F, 3, 27)
        MxI= inverse(Mx)

        MatVecMult(Txr, Tx, presc, 3, 3)
        MatVecMult(Nxpn, Nx, pn, 3, 3)

        Viscous[0] = damp * pn[0] * 1000
        Viscous[1] =  0* pn[1] * 1000
        Viscous[2] = 0 * pn[2] * 1000
```

```
                    for(i = 0; i < 3; i ++) {
          FxmTxrmNxp[i] = (Fx[i] - Txr[i] - Nxpn[i] - Viscous[i] ) * dt
}



MatVecMult(ptemp, MxI, FxmTxrmNxp, 3, 3)
for(i = 0; i < 3; i ++) {
          pnp1[i] = pn[i] + ptemp[i]
          pn[i] = pnp1[i]
}
var w1 = pnp1[0]
var w2 = pnp1[1]
var w3 = pnp1[2]

dR = expMRodriguez(w1, w2, w3, dt)
MatByCons(R11dot, dR, dti, 3, 3)
MatMatMult(R11, myFloat.R1, dR, 3, 3, 3)

for(i = 0; i <3; i++)  {
          for(j = 0; j <3; j++)  {
                    myFloat.R1d[i][j] = R11dot[i][j]
                    myFloat.R1[i][j]  = R11[i][j]
          }
}

var w1 = pnp1[0]
var w2 = pnp1[1]
var w3 = pnp1[2]

myResults.time[icount] = time
myResults.w1[icount] = w1
myResults.w2[icount] = w2
myResults.w3[icount] = w3
myResults.dw1[icount] = ptemp[0]
myResults.dw2[icount] = ptemp[1]
myResults.dw3[icount] = ptemp[2]
```

```
        MatVecMult(x1, myGenData.B, pnp1, 27, 3)
        MatVecMult(x2, myGenData.C, presc, 27, 3)
        for(i = 0; i < 27; i++) {
            x3[i] = x2[i] + x1[i]
        }


        for(i = 0; i < 3; i++) {
            of[i] = x3[i+0]
            oc[i] = x3[i+6]
            ob[i] = x3[i+12]
            ou[i] = x3[i+18]
            ol[i] = x3[i+24]
        }

        skewVector2Matrix(myFloat.os, of)
        skewVector2Matrix(myCombi.os, oc)
        skewVector2Matrix(myBlade.os, ob)
        skewVector2Matrix(myUpper.os, ou)
        skewVector2Matrix(myLower.os, ol)
    }

}
```