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Estimation of the interpolation error for semiregular prismatic elements

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A B S T R A C T

In this paper we introduce the semiregularity property for a family of decompositions of a polyhedron into a natural class of prisms. In such a family, prismatic elements are allowed to be very flat or very long compared to their triangular bases, and the edges of quadrilateral faces can be nonparallel. Moreover, the triangular faces of each element are either parallel or skew to each other. To estimate the error of the interpolation operator defined on the finite space whose basis functions are defined on the general prismatic elements, we consider quadratic polynomials as the basis functions for that space which are bilinear on the reference prism. We then prove that under this modification of the semiregularity criterion, the interpolation error is of order $O(h)$ in the H^1-norm.

1. Introduction

The finite element method is one of the most flexible and powerful methods to solve numerically a wide variety of partial differential equations [3,13,11,14]. A fundamental problem is to estimate the error between the exact solution and its computable finite element approximation. This error can be bounded by the best approximation of the exact solution in the finite element space consisting of piecewise polynomial functions (see Céa's lemma [4]). Hence, it is important to estimate the interpolation errors.

In the process of estimation of the interpolation error, some constant times a power of the discretization parameter h appears. It is crucial that this constant does not blow up when h tends to zero. For linear elliptic boundary value problems in 2-dimensional space, Zlámal [15] introduced the minimum angle condition that guarantees a bound on the constant in the final error which comes from the estimation error of the defined interpolation operator. See also Synge [12]. Babuška and Aziz [2] proposed that the minimum angle condition can be relaxed to the maximum angle condition. In 3-dimensional space, the natural extension of the maximum angle condition for tetrahedral elements was proposed by Křížek [9]. Recently, the generalization of the maximum angle condition in d-dimensional spaces ($d \geq 2$), by means of $\sin_d$ [5], for d-simplices is introduced and extended in [7,8] and also the equivalence of the maximum angle condition and its generalized version is proved.

The maximum angle condition enables us to keep an optimal error whereas we are allowed to consider degenerating families of elements in order to cover the narrow or flat parts of a given bounded domain. For instance, in geophysical simulations [10], where the domain consists of horizontal triangles as a base and regular vertical layer, all finite prismatic

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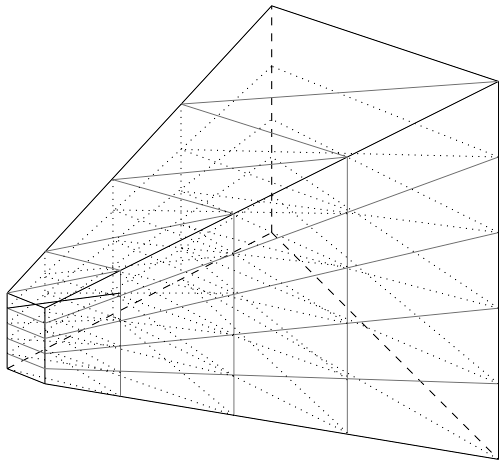

Fig. 1. Partition of frustum into prisms satisfying Definition 3.

elements are produced by the Cartesian product of triangles and the closed intervals called triangular prisms. For such simulation, high aspect-ratio for the elements must be allowed. Therefore, we [6] analyzed the behavior of the interpolation error under the maximum angle condition on the above prisms.

The aim of this paper is to estimate the interpolation error for a more general class of prismatic elements than previously considered in [6]. This class of elements naturally appear e.g. in some standard geometric models. In Fig. 1 an example of a frustum is given. We interpolate a given function by quadratic polynomials which are bilinear on the reference prism. To introduce general prismatic element, similar to [6] we consider the maximum angle condition for all dihedral angles. In addition, we assume that the ratios between the three edges that connect the triangular faces is bounded from below by some positive constant. Note that these ratios for triangular prisms are one. We relax the conditions from [6] to allow e.g. slanted or skew elements. In particular small deformations of the geometry from [6] are covered. We refer to [6] for further motivation and context.

We use the technique of reference element in several parts of the main proof in order to demonstrate that the interpolation error is of order $O(h)$ in the H^1-norm for sufficiently smooth functions and sufficiently small h. In that proof we use a positive lower bound for the Jacobian determinant. In our case, this determinant is a quadratic polynomial in three variables whose coefficients are expressed in terms of volumes of tetrahedra formed by the vertices of the prism.

The outline of the paper is as follows. First, in Section 2, we introduce notations and give some definitions. In addition, we propose an extension of the semiregularity property that allows us to consider some degenerate families of prismatic elements. In Section 3, we obtain a positive lower bound for the Jacobian determinant in Theorem 6, since we use the technique of the reference element to prove the main result. In Section 4 we prove Theorem 7 which states that the interpolation error is of the order $O(h)$ under the extended semiregularity condition, followed by some conclusions in Section 5.

2. Main definitions and geometric preliminaries

We will consider meshes whose elements are defined in this definition:

Definition 1. A straight-side, triangular based prism is a convex polyhedron with six vertices, two triangular faces and three quadrilateral (convex planar) faces. Furthermore, each quadrilateral is incident to the other four faces. The two triangles are not incident. See Fig. 2 (right). In this paper, we will refer to this as a general prism.

We define general prismatic meshes as follows:

Definition 2. A general prismatic mesh $\mathcal{P}_h$ of a bounded polyhedral domain is a face-to-face partition whose elements are general prisms, where h is the maximum diameter of all elements in the mesh.

The following lemma helps us to order the vertices of the prism $\mathcal{P}$. For more details, see also Remark 1.

Lemma 1. *The three edges which connect the two triangular faces of $\mathcal{P}$ are either parallel or if we extend these edges in one direction then they meet each other at some point.*

Proof. The planes containing the three quadrilateral faces intersect in one point or this intersection is empty. □

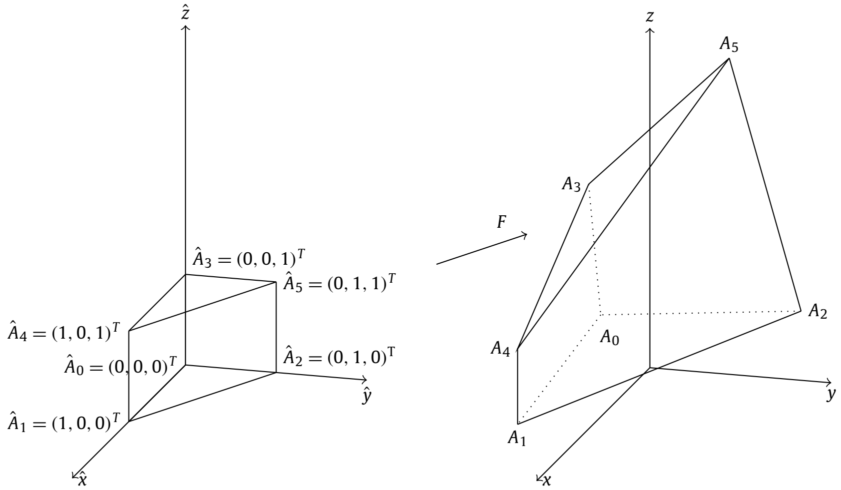

Fig. 2. The reference prism $\hat{\mathcal{P}}$ (left) and an arbitrary prismatic element $\mathcal{P}$ (right). Further, the mapping F is given by formula (9).

Remark 1. We order the vertices of $\mathcal{P}$ similar to [9, pp. 517–518], in such a way that the non-parallel edges $\overline{A_3A_0}$, $\overline{A_4A_1}$ and $\overline{A_5A_2}$ satisfy Lemma 1 and the triangular face $A_3A_4A_5$ is closer to the intersection point than the triangle $A_0A_1A_2$. See Fig. 2 (right). If the edges are parallel, then we do not need to order the vertices similar to the nonparallel case except that the vertices A_0 and A_3 are the end points of the same edge connecting the triangular faces. Furthermore, we assume that in any case the maximum angle for the triangular base is at vertex A_0.

We now define the modification of the semiregularity property from [6] to our setting that will be used throughout the paper.

Definition 3. A family of general prismatic meshes $\mathcal{F} = \{\mathcal{P}_h\}_{h\to 0}$ is semiregular if there exist constants $\overline{\gamma} < \pi$, $c_1 > 0$, and $c_2 > 0$ such that the following conditions hold:

a) **Maximum angle condition** : For any $\mathcal{P} \in \mathcal{P}_h$ and any $\mathcal{P}_h \in \mathcal{F}$ let $\gamma_{\mathcal{P}}$ be the maximum angle of any triangular faces and dihedral angle between any two faces of $\mathcal{P}$. Then

$$\gamma_{\mathcal{P}} \leq \overline{\gamma}. \tag{1}$$

b) **Edge ratio condition** : For any $\mathcal{P} \in \mathcal{P}_h$ and any $\mathcal{P}_h \in \mathcal{F}$ let L_{min} and L_{max} be the minimum and maximum lengths of the three edges connecting the triangular faces. Then

$$\frac{L_{min}}{L_{max}} \geq c_1.$$

c) **Tetrahedra ratio condition** : For any $\mathcal{P} \in \mathcal{P}_h$ and any $\mathcal{P}_h \in \mathcal{F}$ let the vertices of $\mathcal{P}$ be ordered as in Remark 1. Then

$$\frac{\mathrm{Vol}\mathcal{T}(A_0, A_3, A_4, A_5)}{\mathrm{Vol}\mathcal{T}(A_0, A_1, A_2, A_3)} \geq c_2.$$

Lemma 2. *The conditions a), b) and c) from Definition 3 are independent.*

Proof. In Figs. 3-5 we present examples showing the independence. In each figure, all vertices of triangles on the base and on the top of the considered prisms are denoted by $\bullet$ and $\bigcirc$, respectively.

Consider first a case in which *a*) fails, but *b*) and *c*) hold, see Fig. 3. Let $A_0 = (0,0,0)$, $A_1 = (-h, -h^2, 0)$, $A_2 = (h,0,0)$, $A_3 = (0,0,h)$, $A_4 = (-h, -h^2, h)$, and $A_5 = (h,0,h)$ be the vertices of the prism. In this case, $\angle A_1A_0A_2 \to \pi$ as $h \to 0$, so condition *a*) fails. On the other hand, conditions *b*) and *c*) hold with $c_1 = 1$ and $c_2 = 1$.

Consider next a case in which *c*) fails but *a*) and *b*) hold, see Fig. 4. Let $A_0 = (0, \frac{\sqrt{3}}{3}h, 0)$, $A_1 = (-\frac{1}{2}h, -\frac{\sqrt{3}}{6}h, 0)$, $A_2 = (\frac{1}{2}h, -\frac{\sqrt{3}}{6}h, 0)$, $A_3 = (0, \frac{\sqrt{3}}{3}h^2, h)$, $A_4 = (-\frac{1}{2}h^2, -\frac{\sqrt{3}}{6}h^2, h)$, and $A_5 = (\frac{1}{2}h^2, -\frac{\sqrt{3}}{6}h^2, h)$. The triangles on the base and top of the prism are equilateral. Now, if h tends to zero, this family degenerates into a regular tetrahedron, so clearly condition *a*) holds. Further, condition *b*) with $c_1 = 1$ is fulfilled, meanwhile condition *c*) is violated, since the ratio of the volume of the two tetrahedra $\mathcal{T}(A_0, A_3, A_4, A_5)$ and $\mathcal{T}(A_0, A_1, A_2, A_3)$ is h^2.

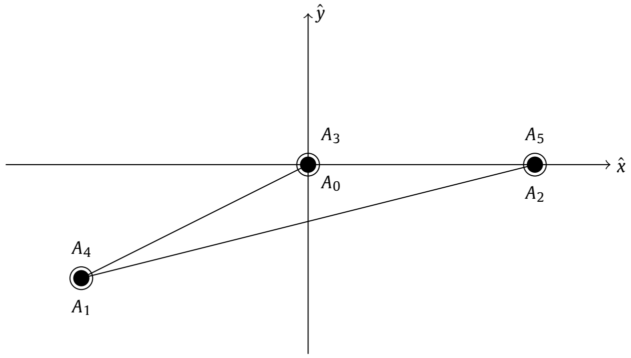

Fig. 3. Orthogonal projection of prism onto $\hat{x}\hat{y}$-plane with vertices $A_0 = (0, 0, 0)$, $A_1 = (-h, -h^2, 0)$, $A_2 = (h, 0, 0)$, $A_3 = (0, 0, h)$, $A_4 = (-h, -h^2, h)$, $A_5 = (h, 0, h)$.

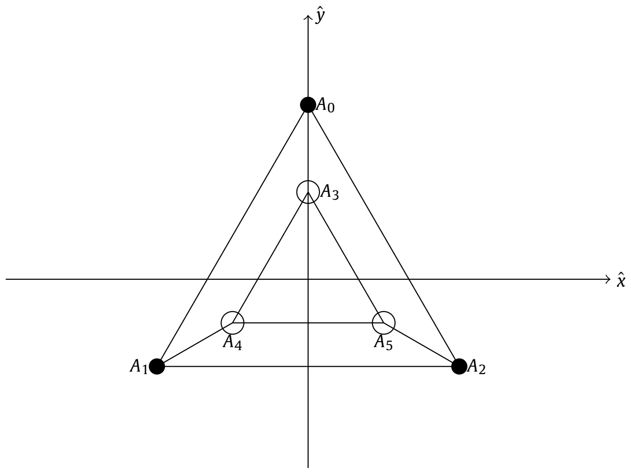

Fig. 4. Orthogonal projection of the prism onto $\hat{x}\hat{y}$-plane with vertices $A_0 = (0, \frac{\sqrt{3}}{3}h, 0)$, $A_1 = (-\frac{1}{2}h, -\frac{\sqrt{3}}{6}h, 0)$, $A_2 = (\frac{1}{2}h, -\frac{\sqrt{3}}{6}h, 0)$, $A_3 = (0, \frac{\sqrt{3}}{3}h^2, h)$, $A_4 = (-\frac{1}{2}h^2, -\frac{\sqrt{3}}{6}h^2, h)$, $A_5 = (\frac{1}{2}h^2, -\frac{\sqrt{3}}{6}h^2, h)$.

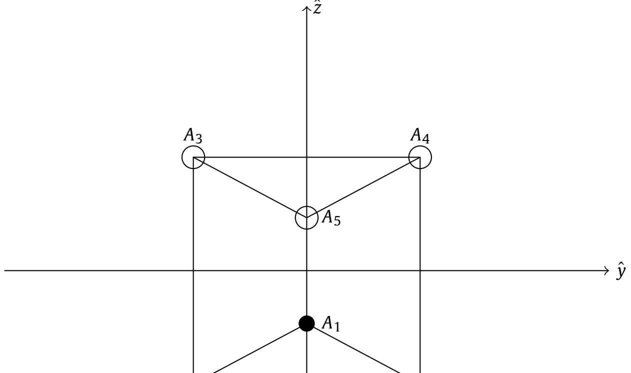

Fig. 5. Orthogonal projection of the prism onto $\hat{y}\hat{z}$-plane with vertices $A_0 = (0, -h, -h)$, $A_1 = (h, 0, -h^2)$, $A_2 = (0, h, -h)$, $A_3 = (0, -h, h)$, $A_4 = (0, h, h)$, $A_5 = (h, 0, h^2)$.

Finally, we consider a case in which b) fails, but the two other conditions hold. Let $A_0 = (0, -h, -h)$, $A_1 = (h, 0, -h^2)$, $A_2 = (0, h, -h)$, $A_3 = (0, -h, h)$, $A_4 = (0, h, h)$, and $A_5 = (h, 0, h^2)$, see Fig. 5. Now assume that $h \to 0$. Then the family degenerates into a pyramid, so clearly condition a) holds. Moreover, the family satisfies condition c) with $c_2 = 1$. But condition b) is violated, since $L_{min}/L_{max} = h$. □

The condition c) in Definition 3 implies bounds on ratios of the volumes of other tetrahedra as well. We will see this in Lemma 5.

To prove Lemma 5, we need the following lemmas from [9].

Lemma 3. *[9] Let $\zeta \le \eta \le \tau$ be angles of an arbitrary face of an arbitrary tetrahedron. Assume furthermore that $\tau \le \overline{\gamma}$. Then $\tau \ge \pi/3$ and*

$$\eta, \tau \in \Big[\frac{\pi - \overline{\gamma}}{2}, \overline{\gamma}\Big].$$

Lemma 4. *[9] Let A be an arbitrary vertex of an arbitrary tetrahedron $\mathcal{T}$ and let $\chi \le \psi \le \varphi$ be angles between faces passing through A. Assume furthermore that $\varphi \le \overline{\gamma}$. Then $\varphi > \pi/3$ and*

$$\psi, \varphi \in \Big(\frac{\pi - \overline{\gamma}}{2}, \overline{\gamma}\Big].$$

Lemma 5. *There exist positive constants $C_i(c_1, m)$, $i = 0, \ldots, 3$, which depend only on c_1 and m such that*

i)
$$\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big) \ge C_0(c_1, m) a b L_{max},$$
where
$$m := \min\Big(\sin(\frac{\pi - \overline{\gamma}}{2}), \sin(\overline{\gamma})\Big),$$
$a = | A_1A_0 |$, and $b = | A_2A_0 |$.

ii) *The ratio of the volumes of the tetrahedra $\mathcal{T}(A_0, A_1, A_2, A_4)$ and $\mathcal{T}(A_0, A_1, A_2, A_3)$ is bounded from below by $C_1(c_1, m)$.*

iii) *The ratio of the volumes of the tetrahedra $\mathcal{T}(A_0, A_1, A_2, A_5)$ and $\mathcal{T}(A_0, A_1, A_2, A_3)$ is bounded from below by $C_2(c_1, m)$.*

iv) *The ratio of the volumes of the tetrahedra $\mathcal{T}(A_0, A_1, A_2, A_3)$ and $\mathcal{T}(A_0, A_3, A_4, A_5)$ is bounded from below by $C_3(c_1, m)$.*

Proof. i) The rays $\overrightarrow{A_4A_1}$ and $\overrightarrow{A_3A_0}$ meet each other at some point or are parallel. One is depicted in Fig. 6, where the angle between the lines $\overrightarrow{A_4A_3}$ and $\overrightarrow{A_0A_3}$, denoted by θ, is not the smallest angle in the triangle $A_0A_3A_4$. Note that for other possibilities we have similar results. Lemma 3 implies that $\sin(\theta)$ is bounded from below by the positive constant m as in [9]. Then

$$\sin(\theta) = \sin(\pi - \theta) = \frac{| A_4M |}{| A_4A_3 |} \ge m,$$

and consequently

$$\frac{| A_1A_0 |}{| A_4A_3 |} \ge \frac{| A_4M |}{| A_4A_3 |} \ge m.$$

Hence,

$$| A_1A_0 | \ge m \, | A_4A_3 |, \tag{2}$$

and similarly

$$| A_2A_0 | \ge m \, | A_5A_3 | . \tag{3}$$

We denote the angles between the edges A_3A_0 and A_1A_0, and the edges A_3A_1 and A_1A_0, by α and β, respectively, see Fig. 6. Now, according to [9, pp. 517–518], Lemmas 3–4, and condition b), if α is greater than or equal to β, we get

$$\begin{aligned}\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big) &\ge \frac{1}{6} m^3 \, | A_1A_0 || A_2A_0 || A_3A_0 | \\ &\ge \frac{1}{6} c_1 m^3 \, | A_1A_0 || A_2A_0 | \, L_{max}.\end{aligned}$$

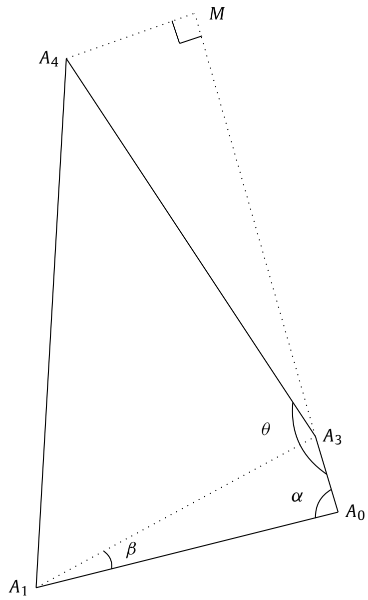

Fig. 6. Quadrilateral face of prism $\mathcal{P}$ made by vertices A_0, A_1, A_3, and A_4.

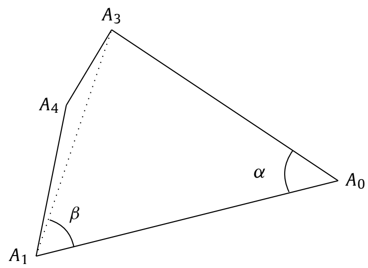

Fig. 7. Quadrilateral face of prism $\mathcal{P}$, where $\beta \geq \alpha$ and $|A_4A_1| \leq |A_3A_1|$.

Otherwise β is greater than α. In this case, either $|A_3A_1| \geq |A_4A_1|$ or $|A_4A_1| > |A_3A_1|$. First, we assume that $|A_3A_1| \geq |A_4A_1|$. Then

$$\begin{aligned} \text{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big) &\geq \frac{1}{6} m^2 \sin(\alpha) \mid A_1A_0 \mid\mid A_2A_0 \mid\mid A_3A_0 \mid \\ &\geq \frac{1}{6} c_1 m^2 \sin(\alpha) \mid A_1A_0 \mid\mid A_2A_0 \mid L_{max}. \end{aligned}$$

To obtain a lower bound for $\sin(\alpha)$, it suffices to use the law of sines for the triangle $A_0A_1A_3$ (see Fig. 7), conditions *a*) and *b*), which implies

$$\sin(\alpha) = \sin(\beta) \frac{|A_3A_1|}{|A_3A_0|} \geq m \frac{|A_4A_1|}{|A_3A_0|} \geq mc_1, \tag{4}$$

and therefore

$$\text{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big) \geq \frac{1}{6} c_1^2 m^3 \sin(\alpha) \mid A_1A_0 \mid\mid A_2A_0 \mid L_{max}.$$

Now, if $|A_4A_1| > |A_3A_1|$, we consider the triangle A_0A_1M, see Fig. 8 (which also defines $\delta = \pi - \alpha - \beta$). Then

$$\begin{aligned} \sin(\alpha) &= \sin(\beta + \beta_1) \frac{|MA_1|}{|MA_0|} \\ &\geq m \frac{|A_4A_1|}{|A_3A_0| + |A_4A_3| \cos(\pi - \theta)} \\ &\geq m \frac{|A_4A_1|}{|A_3A_0| + m^{-1} |A_1A_0|}. \end{aligned} \tag{5}$$

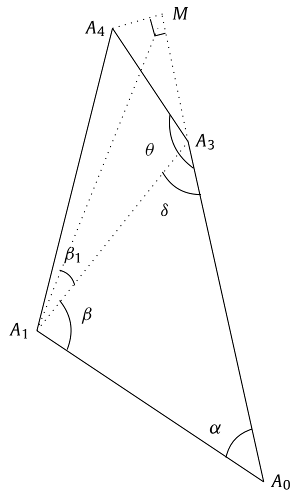

Fig. 8. Quadrilateral face of prism $\mathcal{P}$, where $\beta \geq \alpha$ and $| A_4A_1 |>| A_3A_1 |$.

Note that for the above inequalities we used Lemma 3, since $\alpha < \beta + \beta_1$, and (2), respectively. Writing the law of sines for the triangle $A_0A_1A_3$ once again, leads to

$$| A_1A_0 |= \frac{\sin(\delta)}{\sin(\beta)} | A_3A_0 | \leq m^{-1} | A_3A_0 | . \tag{6}$$

Substitute the right-hand side of (6) into (5), we have

$$\sin(\alpha) \geq \frac{m^3}{(1+m^2)} \frac{| A_4A_1 |}{| A_3A_0 |} \geq \frac{c_1 m^3}{(1+m^2)} \geq \frac{1}{2} c_1 m^3,$$

and consequently

$$\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big) \geq \frac{1}{12} c_1^2 m^5 | A_1A_0 || A_2A_0 | L_{max}.$$

ii) To estimate a lower bound for the ratio of the volumes of the tetrahedra $\mathcal{T}(A_0, A_1, A_2, A_4)$ and $\mathcal{T}(A_0, A_1, A_2, A_3)$, if $\angle A_4A_1A_0$ is greater than or equal to $\angle A_4A_0A_1$, condition b) implies

$$\frac{\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_4)\big)}{\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)} \geq \frac{m^3 | A_1A_0 || A_2A_0 || A_4A_1 |}{| A_1A_0 || A_2A_0 | L_{max}} \geq c_1 m^3.$$

Otherwise, exchanging the indices of the vertices A_0, A_3, A_4, A_1 in Fig. 8 into $1, 4, 3, 0$, respectively, and following the same proof as in part i), there exists a positive constant $C^*(c_1, m)$ such that

$$\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_4)\big) \geq C^*(c_1, m) | A_1A_0 || A_2A_0 | L_{max}.$$

iii) The proof is same as in parts i) and ii).
iv) From part i), (2) and (3), we have

$$\frac{\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)}{\mathrm{Vol}\big(\mathcal{T}(A_0, A_3, A_4, A_5)\big)} \geq \frac{C_0(c_1, m) | A_1A_0 || A_2A_0 | L_{max}}{| A_4A_3 || A_5A_3 | L_{max}} \geq m^2 C_0(c_1, m). \quad \square$$

In what follows, we use the standard denotation $W_p^k(\Omega)$, $k = 0, 1, \ldots, p \geq 1$, for Sobolev spaces with norms $\|.\|_{k,p} = \|.\|_{k,p,\Omega}$ and seminorms $|.|_{k,p} = |.|_{k,p,\Omega}$. The symbol $C(\overline{\Omega})$ stands for the space of continuous functions over $\overline{\Omega}$.

To prove the main result of the paper we will employ the technique based on a transfer of the prism $\mathcal{P} \in \mathcal{P}_h$ onto the reference prism $\hat{\mathcal{P}} = \hat{\mathcal{K}} \times \hat{\mathcal{I}}$, where $\hat{\mathcal{K}}$ is the triangular base and $\hat{\mathcal{I}}$ is the altitude of $\hat{\mathcal{P}}$.

The vertices $\hat{A}_0, \ldots, \hat{A}_5$ of the prism $\hat{\mathcal{P}}$ are given in Fig. 2 (left). The associated basis functions $\hat{\phi}_0, \ldots, \hat{\phi}_5$ for bilinear functions are

$$\begin{aligned}
\hat{\phi}_0(\hat{x},\hat{y},\hat{z}) &= (1-\hat{x}-\hat{y})(1-\hat{z}),\\
\hat{\phi}_1(\hat{x},\hat{y},\hat{z}) &= \hat{x}(1-\hat{z}),\\
\hat{\phi}_2(\hat{x},\hat{y},\hat{z}) &= \hat{y}(1-\hat{z}),\\
\hat{\phi}_3(\hat{x},\hat{y},\hat{z}) &= (1-\hat{x}-\hat{y})\hat{z},\\
\hat{\phi}_4(\hat{x},\hat{y},\hat{z}) &= \hat{x}\hat{z},\\
\hat{\phi}_5(\hat{x},\hat{y},\hat{z}) &= \hat{y}\hat{z}.
\end{aligned} \tag{7}$$

The prismatic interpolant $\hat{\pi}_{\hat{\mathcal{P}}}$ of the function $\hat{u}$ defined on $\hat{\mathcal{P}}$ is constructed as follows:

$$\hat{\pi}_{\hat{\mathcal{P}}}\hat{u} = \sum_{i=0}^{5} \hat{u}(\hat{A}_i)\hat{\phi}_i. \tag{8}$$

By definition, $\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}(\hat{A}_i) = \hat{u}(\hat{A}_i)$, $i = 0,\dots,5$, for any $\hat{u} \in C(\hat{\mathcal{P}})$.

Let

$$F(\hat{x},\hat{y},\hat{z}) = \sum_{i=0}^{5} A_i\hat{\phi}_i(\hat{x},\hat{y},\hat{z}). \tag{9}$$

Equation (9) defines a mapping $F : \hat{\mathcal{P}} \to \mathcal{P}$, which is a bijection from the prism $\hat{\mathcal{P}}$ onto the prism $\mathcal{P}$. Hence we can define ϕ_i on $\mathcal{P}$ such that

$$\phi_i(A) = \hat{\phi}_i(\hat{A}) = \hat{\phi}_i(F^{-1}(A)), \text{ for all points } A \text{ of } \mathcal{P} \in \mathcal{P}_h.$$

With any prismatic mesh $\mathcal{P}_h$ we associate the finite element space

$$V_h = \{u \in C(\overline{\Omega}) \mid u|_{\mathcal{P}} \in Q(\mathcal{P}) \quad \forall \mathcal{P} \in \mathcal{P}_h\},$$

where $Q(\mathcal{P}) = \{\varphi \mid \varphi = \sum_{i=0}^{5} c_i\phi_i\}$. For similar cases, see [1], Section 5.3. Then the interpolation operator $\pi_h : C(\overline{\Omega}) \to V_h$ is uniquely determined by the requirement

$$\pi_h u(A_i) = u(A_i) \text{ for } A_i, i = 0,\dots,5 \text{ of } \mathcal{P} \in \mathcal{P}_h. \tag{10}$$

Consider $\mathcal{B}$ be a (3×5) matrix whose entries are denoted by B_{ij},

$$\mathcal{B} = \left[\, \mathcal{B}_{1:\mathcal{P}} \;\middle|\; \mathcal{B}_{2:\mathcal{P}} \,\right],$$

where

$$\begin{aligned}
\mathcal{B}_{1:\mathcal{P}} &= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}\\
&= \begin{bmatrix} A_{1,x}-A_{0,x} & A_{2,x}-A_{0,x} & A_{3,x}-A_{0,x} \\ A_{1,y}-A_{0,y} & A_{2,y}-A_{0,y} & A_{3,y}-A_{0,y} \\ A_{1,z}-A_{0,z} & A_{2,z}-A_{0,z} & A_{3,z}-A_{0,z} \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_{2:\mathcal{P}} &= \begin{bmatrix} B_{14} & B_{15} \\ B_{24} & B_{25} \\ B_{34} & B_{35} \end{bmatrix}\\
&= \begin{bmatrix} A_{4,x}-A_{0,x}-(B_{11}+B_{13}) & A_{5,x}-A_{0,x}-(B_{12}+B_{13}) \\ A_{4,y}-A_{0,y}-(B_{21}+B_{23}) & A_{5,y}-A_{0,y}-(B_{22}+B_{23}) \\ A_{4,z}-A_{0,z}-(B_{31}+B_{33}) & A_{5,z}-A_{0,z}-(B_{32}+B_{33}) \end{bmatrix}.
\end{aligned}$$

Let $\hat{J}$ denote the Jacobian of the mapping F. Then

$$\hat{J} = \begin{bmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} & \frac{\partial x}{\partial \hat{z}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} & \frac{\partial y}{\partial \hat{z}} \\ \frac{\partial z}{\partial \hat{x}} & \frac{\partial z}{\partial \hat{y}} & \frac{\partial z}{\partial \hat{z}} \end{bmatrix}$$

$$= \begin{bmatrix} B_{11} + B_{14}\hat{z} & B_{12} + B_{15}\hat{z} & B_{13} + B_{14}\hat{x} + B_{15}\hat{y} \\ B_{21} + B_{24}\hat{z} & B_{22} + B_{25}\hat{z} & B_{23} + B_{24}\hat{x} + B_{25}\hat{y} \\ B_{31} + B_{34}\hat{z} & B_{32} + B_{35}\hat{z} & B_{33} + B_{34}\hat{x} + B_{35}\hat{y} \end{bmatrix}. \tag{11}$$

In order to obtain the rate of convergence of the interpolation operator, we will estimate an upper bound for $|\det(\hat{J})|^{-1}$, which plays the key role in the proof of Theorem 7. We will show that the lower bound of $|\det(\hat{J})|$ depends on the volumes of tetrahedra in the prism $\mathcal{P}$.

3. Jacobian determinant

For prisms [6] the determinant of the Jacobian is a constant. We see that for the general prisms, according to (11), $\det(\hat{J})$ is a polynomial in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$. To show that $\det(\hat{J}) \neq 0$, using the linearity property of the determinant, the Jacobian determinant has the explicit form

$$\det(\hat{J}) = \mathbf{A} + \mathbf{B}\hat{x} + \mathbf{C}\hat{y} + \mathbf{D}\hat{z} - \mathbf{E}\hat{x}\hat{z} - \mathbf{F}\hat{y}\hat{z} + \mathbf{G}\hat{z}^2, \tag{12}$$

where

$$\begin{aligned}
\mathbf{A} &= \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} \\
&= 6\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big), \\
\mathbf{B} &= \begin{vmatrix} B_{11} & B_{12} & B_{14} \\ B_{21} & B_{22} & B_{24} \\ B_{31} & B_{32} & B_{34} \end{vmatrix} \\
&= 6\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_4)\big) - \mathbf{A} \\
&= \mathbf{B}_1 - \mathbf{A}, \\
\mathbf{C} &= \begin{vmatrix} B_{11} & B_{12} & B_{15} \\ B_{21} & B_{22} & B_{25} \\ B_{31} & B_{32} & B_{35} \end{vmatrix} \\
&= 6\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_5)\big) - \mathbf{A} \\
&= \mathbf{C}_1 - \mathbf{A}, \\
\mathbf{D} &= \begin{vmatrix} B_{12} & B_{13} & B_{14} \\ B_{22} & B_{23} & B_{24} \\ B_{32} & B_{33} & B_{34} \end{vmatrix} - \begin{vmatrix} B_{11} & B_{13} & B_{15} \\ B_{21} & B_{23} & B_{25} \\ B_{31} & B_{33} & B_{35} \end{vmatrix} \\
&= 6\big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_3, A_4)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_5, A_3)\big)\big\} - 2\mathbf{A} \\
&= \mathbf{D}_1 + \mathbf{D}_2 - 2\mathbf{A}, \\
\mathbf{E} &= \begin{vmatrix} B_{11} & B_{14} & B_{15} \\ B_{21} & B_{24} & B_{25} \\ B_{31} & B_{34} & B_{35} \end{vmatrix} \\
&= 6\big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_3, A_1, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big)\big\} + \mathbf{B} \\
&= \mathbf{D}_2 - \mathbf{E}_1 + \mathbf{B}, \\
\mathbf{F} &= \begin{vmatrix} B_{12} & B_{14} & B_{15} \\ B_{22} & B_{24} & B_{25} \\ B_{32} & B_{34} & B_{35} \end{vmatrix} \\
&= 6\big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_3, A_4)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_5, A_4)\big)\big\} + \mathbf{C} \\
&= \mathbf{D}_1 - \mathbf{F}_1 + \mathbf{C}, \\
\mathbf{G} &= \begin{vmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \\ B_{33} & B_{34} & B_{35} \end{vmatrix} \\
&= 6\big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_3, A_4, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_3, A_4, A_2)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_3, A_1, A_5)\big)\big\} + \mathbf{A} \\
&= \mathbf{G}_1 - \mathbf{D}_1 - \mathbf{D}_2 + \mathbf{A}.
\end{aligned}$$

Therefore,

$$\det(\hat{J}) = \mathbf{A}(1-\hat{x}-\hat{y})(1-\hat{z}) + \mathbf{B}_1\hat{x}(1-\hat{z}) + \mathbf{C}_1\hat{y}(1-\hat{z}) + \mathbf{D}_1\hat{z}(1-\hat{y}) \\ + \mathbf{D}_2\hat{z}(1-\hat{x}) + \mathbf{E}_1\hat{x}\hat{z} + \mathbf{F}_1\hat{y}\hat{z} + \mathbf{G}_1\hat{z}^2 + \mathbf{A}\hat{z}^2 - \mathbf{A}\hat{z} - (\mathbf{D}_1+\mathbf{D}_2)\hat{z}^2. \tag{13}$$

Theorem 6. *Let $\mathcal{F} = \{\mathcal{P}_h\}_{h\to 0}$ be a semiregular family of general prisms of a bounded polygonal domain. Then, there exists a positive constant $\bar{C}(c_1, c_2, m)$ which depends on c_1, c_2 and m, such that*

$$|\det(\hat{J})|^{-1} \le \bar{C}(c_1, c_2, m)\big(abL_{max}\big)^{-1}. \tag{14}$$

Proof. For a fixed $\hat{z} = \hat{z}_0$, $\det(\hat{J})$ is linear, and thus attains its maximum and minimum at vertices of the triangle $0 \le \hat{x}, \hat{y} \le 1, \hat{x}+\hat{y} \le 1, \hat{z} = \hat{z}_0$. Therefore it is enough to consider the restriction of $\det(\hat{J})$ to the three vertical lines. Then the extremal values of $\det(\hat{J})$ can be found at one of these points: the six vertices of the prism $\hat{A}_i, i = 0, \dots, 5$, as well as if points $(0, 0, -\mathbf{D}/2\mathbf{G})$, $(1, 0, (\mathbf{E}-\mathbf{D})/2\mathbf{G})$ and $(0, 1, (\mathbf{F}-\mathbf{D})/2\mathbf{G})$ are in the domain of definition.

Now, if the minimum value of $\det(\hat{J})$ occurs at one of the vertices $\hat{A}_i, i = 0, \dots, 5$, then

$$\min_{\{\hat{A}_0,\dots,\hat{A}_5\}} \det(\hat{J}) = \min\{\mathbf{A}, \mathbf{B}_1, \mathbf{C}_1, \mathbf{G}_1, \mathbf{G}_1+\mathbf{E}_1-\mathbf{D}_2, \mathbf{G}_1+\mathbf{F}_1-\mathbf{D}_1\}. \tag{15}$$

On the right-hand side of (15), all terms are six times the volume of a tetrahedron. Indeed,

$$\begin{aligned}\mathbf{G}_1+\mathbf{E}_1-\mathbf{D}_2 &= 6\Big\{\mathrm{Vol}\big(\mathcal{P}(A_0,\dots,A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0,A_1,A_2,A_5)\big)\Big\} - \mathbf{D}_2 \\ &= 6\Big\{\mathrm{Vol}\big(\mathcal{P}(A_0,\dots,A_5)\big) \\ &\quad - \Big\{\mathrm{Vol}\big(\mathcal{T}(A_0,A_1,A_2,A_5)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0,A_1,A_5,A_3)\big)\Big\}\Big\} \\ &= 6\mathrm{Vol}\big(\mathcal{T}(A_1,A_4,A_3,A_5)\big),\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathbf{G}_1+\mathbf{F}_1-\mathbf{D}_1 &= 6\Big\{\mathrm{Vol}\big(\mathcal{P}(A_0,\dots,A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0,A_1,A_2,A_4)\big)\Big\} - \mathbf{D}_1 \\ &= 6\Big\{\mathrm{Vol}\big(\mathcal{P}(A_0,\dots,A_5)\big) \\ &\quad - \Big\{\mathrm{Vol}\big(\mathcal{T}(A_0,A_1,A_2,A_4)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0,A_2,A_3,A_4)\big)\Big\}\Big\} \\ &= 6\mathrm{Vol}\big(\mathcal{T}(A_3,A_4,A_5,A_2)\big).\end{aligned} \tag{17}$$

Now, Lemma 5 provides the lower bounds for $\mathbf{A}$, $\mathbf{B_1}$, and $\mathbf{C_1}$. In addition, condition *c*) and part *i*) of Lemma 5 imply that

$$\mathbf{G}_1 \ge c_2 C_0(c_1, m) abL_{max}.$$

Using the same proof as in Lemma 5 for (16) and (17), we obtain the lower bounds which consist of constants in terms of c_1 and m, times abL_{max}.

Now if the critical point $P_{(c)} = (0, 0, -\mathbf{D}/2\mathbf{G})$ is a point, where $\det(\hat{J})$ has a minimum value, we have

$$\det(\hat{J})(P_{(c)}) = \mathbf{A} - \frac{\mathbf{D}^2}{4\mathbf{G}}. \tag{18}$$

Due to the valid interval of $\hat{z}$, there are two possibilities for $\mathbf{D}$ and $\mathbf{G}$, $\mathbf{D} > 0, \mathbf{G} < 0$ or $\mathbf{D} < 0, \mathbf{G} > 0$. When $\mathbf{G} < 0$, we obtain $\det(\hat{J})(P_{(c)}) \ge \mathbf{A}$. For $\mathbf{D} < 0$,

$$\det(\hat{J})(P_{(c)}) = \frac{1}{4\mathbf{G}}\big\{4\mathbf{A}\mathbf{G}_1 - (\mathbf{D}_1+\mathbf{D}_2)^2\big\}. \tag{19}$$

If $\mathbf{A} \le \mathbf{G}_1$, we get

$$\begin{aligned}\det(\hat{J})(P_{(c)}) &\ge \frac{1}{4\mathbf{G}}\big\{2\mathbf{A} - (\mathbf{D}_1+\mathbf{D}_2)\big\}\big\{2\mathbf{A}+\mathbf{D}_1+\mathbf{D}_2\big\} \\ &\ge \lambda_1\big\{2\mathbf{A}+\mathbf{D}_1+\mathbf{D}_2\big\} \\ &> 2\lambda_1\mathbf{A},\end{aligned} \tag{20}$$

where

$$0 < \lambda_1 = \frac{2\mathbf{A} - (\mathbf{D}_1 + \mathbf{D}_2)}{4\mathbf{G}} = \frac{1}{2} P_{(c)} \leq \frac{1}{4}.$$

When λ_1 tends to zero, consequently $P_{(c)}$ tends to $(0, 0, 0)$, and due to Lemma 5, the family of functions $\det(\hat{\mathrm{J}})(P_{(c)})$ for all $\mathcal{P} \in \mathcal{P}_h \in \mathcal{F}$ is equicontinuous and by (18) we obtain

$$\det(\hat{\mathrm{J}})(P_{(c)}) \to \mathbf{A}.$$

Otherwise, $\mathbf{G}_1 < \mathbf{A}$ and

$$\det(\hat{\mathrm{J}})(P_{(c)}) \geq \frac{1}{4\mathbf{G}} \left\{2\mathbf{G}_1 - (\mathbf{D}_1 + \mathbf{D}_2)\right\}\left\{2\mathbf{G}_1 + \mathbf{D}_1 + \mathbf{D}_2\right\}$$

Since in this case, condition $2\mathbf{G}_1 - (\mathbf{D}_1 + \mathbf{D}_2) < 0$ leads to $P_{(c)}$ be outside of the domain, then the valid condition is $2\mathbf{G}_1 - (\mathbf{D}_1 + \mathbf{D}_2) > 0$. Hence,

$$\begin{aligned}\det(\hat{\mathrm{J}})(P_{(c)}) &\geq \lambda_2 \left\{2\mathbf{G}_1 + \mathbf{D}_1 + \mathbf{D}_2\right\} \\ &> 2\lambda_2 \mathbf{G}_1,\end{aligned} \tag{21}$$

where

$$0 < \lambda_2 = \frac{2\mathbf{G}_1 - (\mathbf{D}_1 + \mathbf{D}_2)}{4\mathbf{G}} < \frac{1}{4}.$$

When λ_2 tends to zero, $\mathbf{G}_1$ and $(\mathbf{D}_1 + \mathbf{D}_2)/2$ tend together, and from definition of $\mathbf{G}$ we have

$$2\mathbf{G} \to 2\mathbf{A} - (\mathbf{D}_1 + \mathbf{D}_2) = -\mathbf{D}. \tag{22}$$

This means that $P_{(c)}$ tends to $(0, 0, 1)$ and by (18) and condition c), the family of functions $\det(\hat{\mathrm{J}})(P_{(c)})$ for all $\mathcal{P} \in \mathcal{P}_h$ and $\mathcal{P}_h \in \mathcal{F}$ is equicontinuous, and $\det(\hat{\mathrm{J}})(P_{(c)})$ tends to

$$A + \frac{1}{2}\mathbf{D} = \frac{1}{2}(\mathbf{D}_1 + \mathbf{D}_2) \to \mathbf{G}_1 \geq c_2 \mathbf{A}. \tag{23}$$

Now, let $P_{(c)} = (1, 0, (\mathbf{E} - \mathbf{D})/2\mathbf{G})$ be a critical point, where the Jacobian matrix has a minimum value. Then

$$\det(\hat{\mathrm{J}})(P_{(c)}) = \mathbf{B}_1 - \frac{(\mathbf{E} - \mathbf{D})^2}{4\mathbf{G}}. \tag{24}$$

Since $(\mathbf{E} - \mathbf{D})/2\mathbf{G} \in (0, 1)$, then we have either $\mathbf{D} > \mathbf{E}, \mathbf{G} < 0$ or $\mathbf{D} < \mathbf{E}, \mathbf{G} > 0$. For the first case, from (24), we get

$$\det(\hat{\mathrm{J}})(P_{(c)}) \geq \mathbf{B}_1. \tag{25}$$

For the case that $\mathbf{D} < \mathbf{E}, \mathbf{G} > 0$, if $\mathbf{B}_1 \leq \mathbf{G}$, we have

$$\begin{aligned}\det(\hat{\mathrm{J}})(P_{(c)}) &= \frac{1}{4\mathbf{G}}\{4\mathbf{B}_1\mathbf{G} - (\mathbf{E} - \mathbf{D})^2\} \\ &\geq \frac{1}{4\mathbf{G}}\{2\mathbf{B}_1 - (\mathbf{E} - \mathbf{D})\}\{2\mathbf{B}_1 + \mathbf{E} - \mathbf{D}\} \\ &\geq \frac{1}{4\mathbf{G}}\{2\mathbf{B}_1 - (\mathbf{E} - \mathbf{D})\}\{\mathbf{E} - \mathbf{D}\} \\ &\geq \lambda_3\{\mathbf{B}_1 + \mathbf{E}_1 + \mathbf{D}_1 - \mathbf{A}\},\end{aligned}$$

where

$$0 < \lambda_3 = \frac{\mathbf{E} - \mathbf{D}}{4\mathbf{G}} < \frac{1}{2}.$$

Furthermore,

$$\begin{aligned}\mathbf{B}_1 + \mathbf{E}_1 + \mathbf{D}_1 - \mathbf{A} &= 6\Big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_4)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big) \\ &\quad + \mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_3, A_4)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)\Big\} \\ &= 6\Big\{\mathrm{Vol}\big(\mathcal{P}(A_0, \ldots, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_2, A_5, A_3, A_4)\big) \\ &\quad + \mathrm{Vol}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)\Big\}.\end{aligned}$$

Using

$$\mathrm{Vol}\big(\mathcal{T}(A_2, A_5, A_3, A_4)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)$$
$$= \mathrm{Vol}\big(\mathcal{P}(A_0, \ldots, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_3, A_4, A_2, A_1)\big),$$

implies

$$\mathbf{B}_1 + \mathbf{E}_1 + \mathbf{D}_1 - \mathbf{A} = 6\Big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big) + \mathrm{Vol}\big(\mathcal{T}(A_3, A_4, A_2, A_1)\big)\Big\}. \tag{26}$$

Hence

$$\det(\hat{\mathrm{J}})(P_{(c)}) \geq 6\lambda_3 \mathrm{Vol}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big), \tag{27}$$

and a same proof as in Lemma 5, parts *i*) and *ii*), to obtain a lower bound for $\mathrm{Vol}_{(3)}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big)$ implies the desirable result. Further, if $\lambda_3 \to 0$, then $\mathbf{E} - \mathbf{D} \to 0$ and $P_{(c)} \to (1, 0, 0)$, and according to Lemma 5, the family of Jacobian determinant at $P_{(c)}$ for all $\mathcal{P} \in \mathcal{P}_h \in \mathcal{F}$ is equicontinuous. Therefore (24) yields

$$\det(\hat{\mathrm{J}})(P_{(c)}) \to \mathbf{B}_1.$$

The other case is $\mathbf{D} < \mathbf{E}, \mathbf{G} > 0$ and $\mathbf{G} < \mathbf{B}_1$. Since the third coordinate of $P_{(c)}$ must be in $(0, 1)$, we have

$$(\mathbf{E} - \mathbf{D}) < 2\mathbf{G}. \tag{28}$$

Hence, we use $(\mathbf{E} - \mathbf{D})^2 < 4\mathbf{G}^2$ to obtain

$$\begin{aligned}\det(\hat{\mathrm{J}})(P_{(c)}) &= \frac{1}{4\mathbf{G}}\{4\mathbf{B}_1\mathbf{G} - (\mathbf{E} - \mathbf{D})^2\}\\ &\geq \mathbf{B}_1 - \mathbf{G}\\ &= (1 - \lambda_4)\mathbf{B}_1,\end{aligned}$$

where

$$0 < \lambda_4 = \frac{\mathbf{G}}{\mathbf{B}_1} < 1.$$

If λ_4 tends to zero, then $\mathbf{G}$ tends to zero. By (28), $\mathbf{D} \to \mathbf{E}$ (or conversely), and from (25) we conclude that $\det(\hat{\mathrm{J}})(P_{(c)}) \to \mathbf{B}_1$. When $\lambda_4 \to 1$, then $\mathbf{G} \to \mathbf{B}_1$ and (24) tends to

$$\begin{aligned}\mathbf{B}_1 - \frac{(\mathbf{E} - \mathbf{D})^2}{4\mathbf{B}_1} &= \frac{1}{4\mathbf{B}_1}\{2\mathbf{B}_1 - (\mathbf{E} - \mathbf{D})\}\{2\mathbf{B}_1 + (\mathbf{E} - \mathbf{D})\}\\ &\geq \frac{1}{2}\{2\mathbf{B}_1 - (\mathbf{E} - \mathbf{D})\}\\ &\geq 3\mathrm{Vol}\big(\mathcal{T}(A_0, A_4, A_1, A_5)\big).\end{aligned} \tag{29}$$

Note that for the last inequality we used (26). Now, the same argument for (27) implies (14).

Finally, for $P_{(c)} = (0, 1, (\mathbf{F} - \mathbf{D})/2\mathbf{G})$,

$$\det(\hat{\mathrm{J}})(P_{(c)}) = \mathbf{C}_1 - \frac{(\mathbf{F} - \mathbf{D})^2}{4\mathbf{G}}. \tag{30}$$

Moreover, $\mathbf{D} > \mathbf{F}$, $\mathbf{G} < 0$, or $\mathbf{D} < \mathbf{F}$, $\mathbf{G} > 0$, since $(\mathbf{F} - \mathbf{D})/2\mathbf{G} \in (0, 1)$.

Let $\mathbf{D} > \mathbf{F}$ and $\mathbf{G} < 0$. By (30), we then have

$$\det(\hat{\mathrm{J}})(P_{(c)}) \geq \mathbf{C}_1.$$

Now, let $\mathbf{D} < \mathbf{F}$, $\mathbf{G} > 0$. If $\mathbf{C}_1 < \mathbf{G}$ we have

$$\begin{aligned}\det(\hat{\mathrm{J}})(P_{(c)}) &= \frac{1}{4\mathbf{G}}\{4\mathbf{C}_1\mathbf{G} - (\mathbf{F} - \mathbf{D})^2\}\\ &\geq \frac{1}{4\mathbf{G}}\{2\mathbf{C}_1 - (\mathbf{F} - \mathbf{D})\}\{2\mathbf{C}_1 + \mathbf{F} - \mathbf{D}\}\\ &\geq \frac{1}{4\mathbf{G}}\{2\mathbf{C}_1 - (\mathbf{F} - \mathbf{D})\}\{\mathbf{F} - \mathbf{D}\}\\ &\geq \lambda_5\{\mathbf{C}_1 + \mathbf{F}_1 + \mathbf{D}_2 - \mathbf{A}\},\end{aligned} \tag{31}$$

where

$$0 < \lambda_5 = \frac{\mathbf{F} - \mathbf{D}}{4\mathbf{G}} < \frac{1}{2}.$$

Using

$$\begin{aligned}\mathrm{Vol}\big(\mathcal{T}(A_1, A_4, A_5, A_3)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big) \\ = \mathrm{Vol}\big(\mathcal{P}(A_0, \dots, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_1, A_2, A_5, A_3)\big),\end{aligned}$$

implies

$$\begin{aligned}\mathbf{C}_1 + \mathbf{F}_1 + \mathbf{D}_2 - \mathbf{A} &= 6\Big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_5)\big) + \mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_5, A_4)\big) \\ &\quad + \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_5, A_3)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)\Big\} \\ &= 6\Big\{\mathrm{Vol}\big(\mathcal{P}(A_0, \dots, A_5)\big) - \mathrm{Vol}\big(\mathcal{T}(A_1, A_4, A_5, A_3)\big) \\ &\quad + \mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_5, A_4)\big) - \mathrm{Vol}\big(\mathcal{T}(A_0, A_1, A_2, A_3)\big)\Big\} \\ &= 6\Big\{\mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_5, A_4)\big) + \mathrm{Vol}\big(\mathcal{T}(A_1, A_2, A_5, A_3)\big)\Big\}. \end{aligned} \tag{32}$$

Then we can obtain the lower bound for (31) as follows.

$$\det(\hat{\mathrm{J}})(P_{(c)}) \geq 6\lambda_5 \mathrm{Vol}\big(\mathcal{T}(A_0, A_2, A_5, A_4)\big).$$

Extending the proof of Lemma 5, one comes to (14).

Now, if $\lambda_5 \to 0$, then $\mathbf{F} - \mathbf{D} \to 0$, and as a result $P_{(c)} \to (0, 1, 0)$. Similar to previous cases, due to Lemma 5, here the family of Jacobian determinant at $P_{(c)}$ for all prisms belonging to $\mathcal{F}$ is also equicontinuous and we get

$$\det(\hat{\mathrm{J}})(P_{(c)}) \to \mathbf{C}_1. \tag{33}$$

For the case that $\mathbf{G} < \mathbf{C}_1$,

$$\begin{aligned}\det(\hat{\mathrm{J}})(P_{(c)}) &= \frac{1}{4\mathbf{G}}\{4\mathbf{C}_1\mathbf{G} - (\mathbf{F} - \mathbf{D})^2\} \\ &\geq \mathbf{C}_1 - \mathbf{G} \\ &= (1 - \lambda_6)\mathbf{C}_1,\end{aligned}$$

where

$$0 < \lambda_6 = \frac{\mathbf{G}}{\mathbf{C}_1} < 1.$$

If λ_6 approaches to zero, then $\mathbf{G} \to 0$, which implies (33).

To end the proof, let λ_6 tend to 1. Then $\mathbf{G} \to \mathbf{C}_1$, and consequently by (32), (30) tends to

$$\begin{aligned}\mathbf{C}_1 - \frac{(\mathbf{F} - \mathbf{D})^2}{4\mathbf{C}_1} &= \frac{1}{4\mathbf{C}_1}\{2\mathbf{C}_1 - (\mathbf{F} - \mathbf{D})\}\{2\mathbf{C}_1 + (\mathbf{F} - \mathbf{D})\} \\ &\geq \frac{1}{2}\{2\mathbf{C}_1 - (\mathbf{F} - \mathbf{D})\} \\ &\geq 3\mathrm{Vol}_{(3)}\big(\mathcal{T}(A_0, A_2, A_5, A_4)\big). \quad \square \end{aligned} \tag{34}$$

4. Interpolation error

Theorem 7. *Let $u \in W_2^3(\Omega)$ and $\mathcal{F} = \{\mathcal{P}_h\}_{h \to 0}$ be a family of semiregular prismatic partitions of $\bar{\Omega}$. Then, there exists a positive constant C^*, independent of the diameter $h(\mathcal{P})$, such that*

$$|u - \pi_h u|_{1,2,\Omega} \leq C^*\big\{h(\mathcal{P})\,|u|_{2,2,\Omega} + h^2(\mathcal{P})\,|u|_{3,2,\Omega}\big\} \tag{35}$$

Proof. From the definition of semi-norm we have

$$|u-\pi_{\mathcal{P}}u|^2_{1,2,\mathcal{P}}=\int\limits_{\mathcal{P}}\Big(\Big|\frac{\partial}{\partial x}(u-\pi_{\mathcal{P}}u)\Big|^2+\Big|\frac{\partial}{\partial y}(u-\pi_{\mathcal{P}}u)\Big|^2+\Big|\frac{\partial}{\partial z}(u-\pi_{\mathcal{P}}u)\Big|^2\Big)dX. \tag{36}$$

To estimate (36), first we will estimate it on $\hat{\mathcal{P}}$. Then, from equation (40) in [6], we have

$$\int\limits_{\hat{\mathcal{P}}}\Big|\frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u})\Big|^2 d\hat{X}\le 12\int\limits_{\hat{\mathcal{P}}}\Big(\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}^2}\Big|^2+\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}\partial\hat{y}}\Big|^2+\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}\partial\hat{z}}\Big|^2+\Big|\frac{\partial^3\hat{u}}{\partial\hat{x}\partial\hat{y}\partial\hat{z}}\Big|^2\Big)d\hat{X}, \tag{37}$$

where

$$\begin{aligned}\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}^2}\Big|^2 &= \Big|\hat{J}^2_{(11)}\frac{\partial^2 u}{\partial x^2}+\hat{J}^2_{(21)}\frac{\partial^2 u}{\partial y^2}+\hat{J}^2_{(31)}\frac{\partial^2 u}{\partial z^2}\\ &\quad +2\Big\{\hat{J}_{(11)}\hat{J}_{(21)}\frac{\partial^2 u}{\partial x\partial y}+\hat{J}_{(11)}\hat{J}_{(31)}\frac{\partial^2 u}{\partial x\partial z}+\hat{J}_{(21)}\hat{J}_{(31)}\frac{\partial^2 u}{\partial y\partial z}\Big\}\Big|^2\\ &\le 24\Big\{\hat{J}^4_{(11)}\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\hat{J}^4_{(21)}\Big|\frac{\partial^2 u}{\partial y^2}\Big|^2+\hat{J}^4_{(31)}\Big|\frac{\partial^2 u}{\partial z^2}\Big|^2+\hat{J}^2_{(11)}\hat{J}^2_{(21)}\Big|\frac{\partial^2 u}{\partial x\partial y}\Big|^2\\ &\quad +\hat{J}^2_{(11)}\hat{J}^2_{(31)}\Big|\frac{\partial^2 u}{\partial x\partial z}\Big|^2+\hat{J}^2_{(21)}\hat{J}^2_{(31)}\Big|\frac{\partial^2 u}{\partial y\partial z}\Big|^2\Big\}.\end{aligned}$$

For the last inequalities we used the so-called sum of squares inequality

$$\Big(\sum_{j=1}^{s}a_j\Big)^2\le s\sum_{j=1}^{s}a_j^2.$$

In the remaining computations, we will use C as an unspecified positive constant. It is not necessarily the same in two lines of a computation, for instance in equation (38). We get

$$\begin{aligned}\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}\partial\hat{y}}\Big|^2 &\le C\Big\{\hat{J}^2_{(11)}\hat{J}^2_{(12)}\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\hat{J}^2_{(21)}\hat{J}^2_{(22)}\Big|\frac{\partial^2 u}{\partial y^2}\Big|^2+\hat{J}^2_{(31)}\hat{J}^2_{(32)}\Big|\frac{\partial^2 u}{\partial z^2}\Big|^2\\ &\quad +(\hat{J}^2_{(12)}\hat{J}^2_{(21)}+\hat{J}^2_{(11)}\hat{J}^2_{(22)})\Big|\frac{\partial^2 u}{\partial x\partial y}\Big|^2+(\hat{J}^2_{(12)}\hat{J}^2_{(31)}+\hat{J}^2_{(32)}\hat{J}^2_{(11)})\Big|\frac{\partial^2 u}{\partial x\partial z}\Big|^2\\ &\quad +(\hat{J}^2_{(22)}\hat{J}^2_{(31)}+\hat{J}^2_{(21)}\hat{J}^2_{(32)})\Big|\frac{\partial^2 u}{\partial y\partial z}\Big|^2\Big\},\\ \Big|\frac{\partial^2\hat{u}}{\partial\hat{x}\partial\hat{z}}\Big|^2 &\le C\Big\{\hat{J}^2_{(11)}\hat{J}^2_{(13)}\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\hat{J}^2_{(21)}\hat{J}^2_{(23)}\Big|\frac{\partial^2 u}{\partial y^2}\Big|^2+\hat{J}^2_{(31)}\hat{J}^2_{(33)}\Big|\frac{\partial^2 u}{\partial z^2}\Big|^2\\ &\quad +(\hat{J}^2_{(13)}\hat{J}^2_{(21)}+\hat{J}^2_{(11)}\hat{J}^2_{(23)})\Big|\frac{\partial^2 u}{\partial x\partial y}\Big|^2+(\hat{J}^2_{(13)}\hat{J}^2_{(31)}+\hat{J}^2_{(33)}\hat{J}^2_{(11)})\Big|\frac{\partial^2 u}{\partial x\partial z}\Big|^2\\ &\quad +(\hat{J}^2_{(23)}\hat{J}^2_{(31)}+\hat{J}^2_{(21)}\hat{J}^2_{(33)})\Big|\frac{\partial^2 u}{\partial y\partial z}\Big|^2\Big\},\\ \Big|\frac{\partial^3\hat{u}}{\partial\hat{x}\partial\hat{y}\partial\hat{z}}\Big|^2 &\le C\Big\{\hat{J}^2_{(11)}\hat{J}^2_{(12)}\hat{J}^2_{(13)}\Big|\frac{\partial^3 u}{\partial x^3}\Big|^2+\hat{J}^2_{(21)}\hat{J}^2_{(22)}\hat{J}^2_{(23)}\Big|\frac{\partial^3 u}{\partial y^3}\Big|^2\\ &\quad +\hat{J}^2_{(31)}\hat{J}^2_{(32)}\hat{J}^2_{(33)}\Big|\frac{\partial^3 u}{\partial z^3}\Big|^2+\Big(\hat{J}^2_{(12)}\hat{J}^2_{(11)}\hat{J}^2_{(23)}+(\hat{J}^2_{(12)}\hat{J}^2_{(21)}+\hat{J}^2_{(11)}\hat{J}^2_{(22)})\hat{J}^2_{(13)}\Big)\Big|\frac{\partial^3 u}{\partial x^2\partial y}\Big|^2\\ &\quad +\Big(\hat{J}^2_{(11)}\hat{J}^2_{(12)}\hat{J}^2_{(33)}+(\hat{J}^2_{(12)}\hat{J}^2_{(31)}+\hat{J}^2_{(11)}\hat{J}^2_{(32)})\hat{J}^2_{(13)}\Big)\Big|\frac{\partial^3 u}{\partial x^2\partial z}\Big|^2\\ &\quad +\Big((\hat{J}^2_{(12)}\hat{J}^2_{(21)}+\hat{J}^2_{(11)}\hat{J}^2_{(22)})\hat{J}^2_{(23)}+\hat{J}^2_{(22)}\hat{J}^2_{(21)}\hat{J}^2_{(13)}\Big)\Big|\frac{\partial^3 u}{\partial x\partial y^2}\Big|^2\\ &\quad +\Big((\hat{J}^2_{(12)}\hat{J}^2_{(21)}+\hat{J}^2_{(11)}\hat{J}^2_{(22)})\hat{J}^2_{(33)}+(\hat{J}^2_{(12)}\hat{J}^2_{(31)}+\hat{J}^2_{(32)}\hat{J}^2_{(11)})\hat{J}^2_{(23)}+(\hat{J}^2_{(22)}\hat{J}^2_{(31)}+\hat{J}^2_{(32)}\hat{J}^2_{(21)})\hat{J}^2_{(13)}\Big)\Big|\frac{\partial^3 u}{\partial x\partial y\partial z}\Big|^2\\ &\quad +\Big(\hat{J}^2_{(22)}\hat{J}^2_{(21)}\hat{J}^2_{(33)}+(\hat{J}^2_{(22)}\hat{J}^2_{(31)}+\hat{J}^2_{(32)}\hat{J}^2_{(21)})\hat{J}^2_{(23)}\Big)\Big|\frac{\partial^3 u}{\partial y^2\partial z}\Big|^2\end{aligned}$$

$$+\left(\hat{J}^2_{(32)}\hat{J}^2_{(31)}\hat{J}^2_{(13)} + (\hat{J}^2_{(12)}\hat{J}^2_{(31)} + \hat{J}^2_{(32)}\hat{J}^2_{(21)})\hat{J}^2_{(33)}\right)\Big|\frac{\partial^3 u}{\partial x \partial z^2}\Big|^2$$
$$+\left(\hat{J}^2_{(32)}\hat{J}^2_{(31)}\hat{J}^2_{(23)} + (\hat{J}^2_{(22)}\hat{J}^2_{(31)} + \hat{J}^2_{(32)}\hat{J}^2_{(21)})\hat{J}^2_{(33)}\right)\Big|\frac{\partial^3 u}{\partial y \partial z^2}\Big|^2\Big\}.$$

To estimate the upper bounds for $\left|\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{y}}\right|^2$, $\left|\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{z}}\right|^2$, and $\left|\frac{\partial^3 \hat{u}}{\partial \hat{x} \partial \hat{y} \partial \hat{z}}\right|^2$, we denote the length of the segments $\overrightarrow{A_4A_3}$ and $\overrightarrow{A_5A_3}$ by c and d respectively. Now, we have

$$(A_{1,x}-A_{0,x})^2+(A_{1,y}-A_{0,y})^2 \le a^2, \qquad (A_{4,x}-A_{3,x})^2+(A_{4,y}-A_{3,y})^2 \le c^2,$$
$$(A_{2,x}-A_{0,x})^2+(A_{2,y}-A_{0,y})^2 \le b^2, \qquad (A_{5,x}-A_{3,x})^2+(A_{5,y}-A_{3,y})^2 \le d^2,$$

which imply

$$\begin{aligned}
\big|\hat{J}_{(11)}\big| &= \big|B_{11}+(A_{4,x}-A_{0,x}-(B_{11}+B_{13}))\tilde{z}\big| \\
&\le \big|A_{1,x}-A_{0,x}\big|(1-\tilde{z}) + \big|A_{4,x}-A_{3,x}\big|\tilde{z} \\
&\le a+c, \\
\big|\hat{J}_{(21)}\big| &= \big|B_{21}+(A_{4,y}-A_{0,y}-(B_{21}+B_{23}))\tilde{z}\big| \\
&\le \big|A_{1,y}-A_{0,y}\big|(1-\tilde{z}) + \big|A_{4,y}-A_{3,y}\big|\tilde{z} \\
&\le a+c, \\
\big|\hat{J}_{(31)}\big| &= \big|B_{31}+(A_{4,z}-A_{0,z}-(B_{31}+B_{33}))\tilde{z}\big| \\
&\le \big|(A_{1,z}-A_{0,z}\big|(1-\tilde{z}) + \big|A_{4,z}-A_{3,z}\big|\tilde{z} \\
&\le a+c,
\end{aligned}$$

and similarly

$$\big|\hat{J}_{(12)}\big| \le b+d, \qquad \big|\hat{J}_{(22)}\big| \le b+d, \qquad \big|\hat{J}_{(32)}\big| \le b+d.$$

Now the upper bounds of $\mid J_{(i,j)} \mid$, $i=1,2,3$, $j=1,2$ can be expressed in terms of a, b, and m as follows.

$$\left\{\big|\hat{J}_{(11)}\big|, \big|\hat{J}_{(21)}\big|, \big|\hat{J}_{(31)}\big|\right\} \le (1+m^{-1})a,$$
$$\left\{\big|\hat{J}_{(12)}\big|, \big|\hat{J}_{(22)}\big|, \big|\hat{J}_{(32)}\big|\right\} \le (1+m^{-1})b.$$

Moreover

$$\big|\hat{J}_{(13)}\big| \le \big|A_{3,x}-A_{0,x}\big| + \big|A_{4,x}-A_{1,x}\big| + \big|A_{5,x}-A_{2,x}\big| \le 3L_{max},$$
$$\big|\hat{J}_{(23)}\big| \le \big|A_{3,y}-A_{0,y}\big| + \big|A_{4,y}-A_{1,y}\big| + \big|A_{5,y}-A_{2,y}\big| \le 3L_{max},$$
$$\big|\hat{J}_{(33)}\big| \le \big|A_{3,z}-A_{0,z}\big| + \big|A_{4,z}-A_{1,z}\big| + \big|A_{5,z}-A_{2,z}\big| \le 3L_{max}.$$

Therefore,

$$\begin{aligned}
\Big|\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\Big|^2 &\le 24(1+m^{-1})^4a^4\Big\{\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial z^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x \partial y}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x \partial z}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y \partial z}\Big|^2\Big\} \\
&= 24(1+m^{-1})^4a^4 \sum_{|\beta|=2} \mid D^\beta u \mid^2,
\end{aligned}$$

and

$$\Big|\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{y}}\Big|^2 \le 24(1+m^{-1})^4a^2b^2 \sum_{|\beta|=2} \mid D^\beta u \mid^2,$$
$$\Big|\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{z}}\Big|^2 \le 6^3(1+m^{-1})^2a^2L^2_{max} \sum_{|\beta|=2} \mid D^\beta u \mid^2,$$
$$\Big|\frac{\partial^3 \hat{u}}{\partial \hat{x} \partial \hat{y} \partial \hat{z}}\Big|^2 \le 10 \times 18^2(1+m^{-1})^4a^2b^2L^2_{max} \sum_{|\beta|=3} \mid D^\beta u \mid^2 .$$

Using Theorem 6, (37) can be expressed as follows.

$$\begin{aligned}
&\int_{\hat{\mathcal{P}}} \Big|\frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u})\Big|^2 d\hat{X} \\
&\le C\int_{\mathcal{P}} |\det(\hat{\mathrm{J}})|^{-1}(1+m^{-1})^2 a^2\Big\{\big((1+m^{-1})^2(a^2+b^2)+L_{max}^2\big)\sum_{|\beta|=2}\big|D^\beta u\big|^2 \\
&\quad + \big((1+m^{-1})^2 b^2 L_{max}^2\big)\sum_{|\beta|=3}\big|D^\beta u\big|^2\Big\}dX \\
&\le C\int_{\mathcal{P}} |\det(\hat{\mathrm{J}})|^{-1} a^2\Big\{h^2(\mathcal{P})\sum_{|\beta|=2}\big|D^\beta u\big|^2 + h^4(\mathcal{P})\sum_{|\beta|=3}\big|D^\beta u\big|^2\Big\}dX. \qquad (38)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\hat{\mathcal{P}}} \Big|\frac{\partial}{\partial \hat{y}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u})\Big|^2 d\hat{X} \\
&\le C\int_{\mathcal{P}} |\det(\hat{\mathrm{J}})|^{-1}(1+m^{-1})^2 b^2\Big\{\big((1+m^{-1})^2(a^2+b^2)+L_{max}^2\big)\sum_{|\beta|=2}\big|D^\beta u\big|^2 \\
&\quad + \big((1+m^{-1})^2 a^2 L_{max}^2\big)\sum_{|\beta|=3}\big|D^\beta u\big|^2\Big\}dX \\
&\le C\int_{\mathcal{P}} |\det(\hat{\mathrm{J}})|^{-1} b^2\Big\{h^2(\mathcal{P})\sum_{|\beta|=2}\big|D^\beta u\big|^2 + h^4(\mathcal{P})\sum_{|\beta|=3}\big|D^\beta u\big|^2\Big\}dX. \qquad (39)
\end{aligned}$$

From equation (45) in [6], we get

$$\begin{aligned}
&\int_{\hat{\mathcal{P}}} \Big|\frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u})\Big|^2 d\hat{X} \\
&\quad \le 2\int_{\hat{\mathcal{P}}}\Big|\frac{\partial^2 \hat{u}}{\partial \hat{z}^2}\Big|^2 d\hat{X} + C\int_{\hat{\mathcal{P}}}\left(\Big|\frac{\partial^3 \hat{u}}{\partial \hat{x}^2\partial \hat{z}}\Big|^2 + \Big|\frac{\partial^3 \hat{u}}{\partial \hat{x}\partial \hat{y}\partial \hat{z}}\Big|^2 + \Big|\frac{\partial^3 \hat{u}}{\partial \hat{y}^2\partial \hat{z}}\Big|^2\right)d\hat{X}.
\end{aligned}$$

Since

$$\begin{aligned}
\Big|\frac{\partial^2 \hat{u}}{\partial \hat{z}^2}\Big|^2 &\le 6\times 18^2 L_{max}^4 \sum_{|\beta|=2}\big|D^\beta u\big|^2, \\
\Big|\frac{\partial^3 \hat{u}}{\partial \hat{x}^2\partial \hat{z}}\Big|^2 &\le 10\times 18^2(1+m^{-1})^4 a^4 L_{max}^2 \sum_{|\beta|=3}\big|D^\beta u\big|^2, \\
\Big|\frac{\partial^3 \hat{u}}{\partial \hat{y}^2\partial \hat{z}}\Big|^2 &\le 10\times 18^2(1+m^{-1})^4 b^4 L_{max}^2 \sum_{|\beta|=3}\big|D^\beta u\big|^2,
\end{aligned}$$

we get

$$\begin{aligned}
&\int_{\hat{\mathcal{P}}} \Big|\frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u})\Big|^2 d\hat{X} \\
&\le C\int_{\mathcal{P}} |\det(\hat{\mathrm{J}})|^{-1} L_{max}^2\Big\{L_{max}^2\sum_{|\beta|=2}\big|D^\beta u\big|^2 + (1+m^{-1})^4\big(a^4+b^4+a^2b^2\big)\sum_{|\beta|=3}\big|D^\beta u\big|^2\Big\}dX \\
&\le C\int_{\mathcal{P}} |\det(\hat{\mathrm{J}})|^{-1} L_{max}^2\Big\{h^2(\mathcal{P})\sum_{|\beta|=2}\big|D^\beta u\big|^2 + h^4(\mathcal{P})\sum_{|\beta|=3}\big|D^\beta u\big|^2\Big\}dX. \qquad (40)
\end{aligned}$$

Now, we estimate (36) as follows.

$$\begin{aligned}
&|u-\pi_{\mathcal{P}}u|^2_{1,2,\mathcal{P}} \\
&= \int\limits_{\hat{\mathcal{P}}} |\det(\hat{\mathrm{J}})| \Bigg(\Big| \mathrm{J}^{-1}_{(11)} \frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) + \mathrm{J}^{-1}_{(21)} \frac{\partial}{\partial \hat{y}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) + \mathrm{J}^{-1}_{(31)} \frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \\
&\quad + \Big| \mathrm{J}^{-1}_{(12)} \frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) + \mathrm{J}^{-1}_{(22)} \frac{\partial}{\partial \hat{y}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) + \mathrm{J}^{-1}_{(32)} \frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \\
&\quad + \Big| \mathrm{J}^{-1}_{(13)} \frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) + \mathrm{J}^{-1}_{(23)} \frac{\partial}{\partial \hat{y}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) + \mathrm{J}^{-1}_{(33)} \frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \Bigg) d\hat{X} \\
&\le 3 \int\limits_{\hat{\mathcal{P}}} |\det(\hat{\mathrm{J}})| \Bigg(\Big(|\mathrm{J}^{-1}_{(11)}|^2 + |\mathrm{J}^{-1}_{(12)}|^2 + |\mathrm{J}^{-1}_{(13)}|^2 \Big) \Big| \frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \\
&\quad + \Big(|\mathrm{J}^{-1}_{(21)}|^2 + |\mathrm{J}^{-1}_{(22)}|^2 + |\mathrm{J}^{-1}_{(23)}|^2 \Big) \Big| \frac{\partial}{\partial \hat{y}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \\
&\quad + \Big(|\mathrm{J}^{-1}_{(31)}|^2 + |\mathrm{J}^{-1}_{(32)}|^2 + |\mathrm{J}^{-1}_{(33)}|^2 \Big) \Big| \frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \Bigg) d\hat{X}.
\end{aligned} \tag{41}$$

Theorem 6 and computations of cofactors lead to

$$\begin{aligned}
\{ |\mathrm{J}^{-1}_{(11)}|, |\mathrm{J}^{-1}_{(12)}|, |\mathrm{J}^{-1}_{(13)}| \} &\le \frac{6}{|\det(\hat{\mathrm{J}})|}(1+m^{-1}) b L_{max} \le C\big(\frac{1}{a}\big), \\
\{ |\mathrm{J}^{-1}_{(21)}|, |\mathrm{J}^{-1}_{(22)}|, |\mathrm{J}^{-1}_{(23)}| \} &\le \frac{6}{|\det(\hat{\mathrm{J}})|}(1+m^{-1}) a L_{max} \le C\big(\frac{1}{b}\big), \\
\{ |\mathrm{J}^{-1}_{(31)}|, |\mathrm{J}^{-1}_{(32)}|, |\mathrm{J}^{-1}_{(33)}| \} &\le \frac{2}{|\det(\hat{\mathrm{J}})|}(1+m^{-1})^2 ab \le C\big(\frac{1}{L_{max}}\big).
\end{aligned} \tag{42}$$

Using (42) for (41) yields

$$|u-\pi_{\mathcal{P}}u|^2_{1,2,\mathcal{P}} \le C \int\limits_{\hat{\mathcal{P}}} \Big(\frac{1}{a^2} \Big| \frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 + \frac{1}{b^2} \Big| \frac{\partial}{\partial \hat{y}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 + \frac{1}{L^2_{max}} \Big| \frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\hat{\mathcal{P}}}\hat{u}) \Big|^2 \Big) d\hat{X},$$

and by (38), (39), and (40) we deduce

$$|u-\pi_{\mathcal{P}}u|^2_{1,2,\mathcal{P}} \le C \big\{ \big(h(\mathcal{P})\big)^2 |u|^2_{2,2,\mathcal{P}} + \big(h(\mathcal{P})\big)^4 |u|^2_{3,2,\mathcal{P}} \big\},$$

which implies (35). □

5. Conclusion

In this paper, we proposed the combination of the edge and tetrahedra ratio conditions with the maximum angle condition in three dimensional space, as the natural version of semiregularity for possibly degenerating families of prismatic elements. We have shown that the new semiregularity condition property guarantees that an optimal order of interpolation error is preserved.

In future work, we plan to estimate interpolation errors for pyramidal elements under similar conditions.

References

[1] O. Axelsson, V.A. Barker, Finite Element Solution of Boundary Value Problems: Theory and Computation, Academic Press, New York, 1984.
[2] I. Babuška, A.K. Aziz, On the angle condition in the finite element method, SIAM J. Numer. Anal. 13 (1976) 214–226.
[3] I. Babuška, R. Tempone, G. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, SIAM J. Numer. Anal. 42 (2004) 800–825.
[4] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[5] F. Eriksson, The law of sines for tetrahedra and n-simplices, Geom. Dedic. 7 (1979) 71–80.
[6] A. Khademi, S. Korotov, J.E. Vatne, On interpolation error on degenerating prismatic elements, Appl. Math. 63 (2018) 237–258.
[7] A. Khademi, S. Korotov, J.E. Vatne, On equivalence of maximum angle conditions for tetrahedral finite element meshes, in: V. Garanzha, L. Kamenski, H. Si (Eds.), Proceedings of the 9th International Conference on Numerical Geometry, Grid Generation and Scientific Computing, NUMGRID 2018, Moscow, Russia, in: Lecture Notes in Computional Science and Engineering, vol. 131, Springer, 2019, pp. 101–108.
[8] A. Khademi, S. Korotov, J.E. Vatne, On the generalization of the Synge-Křížek maximum angle condition for d-simplices, J. Comput. Appl. Math. 358 (2019) 29–33.

[9] M. Křížek, On the maximum angle condition for linear tetrahedral elements, SIAM J. Numer. Anal. 29 (1992) 513–520.
[10] A.T.T. Mcrae, G.-T. Bercea, L. Mitchell, D.A. Ham, C.J. Cotter, Automated generation and symbolic manipulation of tensor product finite elements, SIAM J. Sci. Comput. 38 (2016).
[11] P. Šolín, K. Segeth, I. Doležel, Higher-Order Finite Element Methods, Chapman & Hall/CRC, New York, 2004.
[12] J.L. Synge, The Hypercircle in Mathematical Physics: A Method for the Approximate Solution of Boundary Value Problems, Cambridge University Press, New York, 1957.
[13] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer, New York, 1997.
[14] Y. Yan, Galerkin finite element methods for stochastic parabolic partial differential equations, SIAM J. Numer. Anal. 43 (4) (2005) 1363–1384.
[15] M. Zlámal, On the finite element method, Numer. Math. 12 (1968) 394–409.