STABILIZING SHIP MOTION WITH A DUAL SYSTEM INERTIAL DISK

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ABSTRACT

Norwegian industries are constantly assessing new technologies and methods for more efficient and safer maintenance in the aqua cultural, renewable energy, and oil and gas industries. These Norwegian offshore industries share a common challenge: to install new equipment and transport personnel in a safe and controllable way between ships, farms and platforms. This paper deploys the Moving Frame Method (MFM) to analyze ship stability moderated by a dual gyroscopic inertial device. The MFM describes the dynamics of the system using modern mathematics. Lie group theory and Cartan's moving frames are the foundation of this new approach to engineering dynamics. This, together with a restriction on the variation of the angular velocity used in Hamilton's principle, enables an effective way of extracting the equations of motion. This project extends previous work. It accounts for the dual effect of two inertial disk devices, it accounts for the prescribed spin of the disks. It separates out the prescribed variables. This work displays the results in 3D on cell phones. It represents a prelude to testing in a wave tank.

NOMENCLATURE

$[D]$:	Structured angular velocity matrix
E:	Frame Connection matrix
$\mathbf{e}$:	Frame
$\{F\}$:	Force and moment list
$\{F^*\}$:	Essential generalized force
g:	Gravity
H:	Angular momentum
$\{H\}$:	Generalized momenta
$I_3 I_d$:	3x3 identity matrix
$J_c^{(\alpha)}$:	3x3 mass moment of inertia matrix
K:	Kinetic energy
L:	Lagrangian
$\{L\}$:	Linear momentum
$[M]$:	Mass matrix
$[M^*]$:	Reduced mass matrix
$[N^*]$:	Reduced non-linear velocity matrix
q:	Generalized coordinates
$\dot{q}$:	Generalized velocity
$\ddot{q}$:	Generalized acceleration
R:	Rotation matrix
r:	Absolute position vector
s:	Relative position vector
U:	Potential energy
$\{\dot{X}\}$:	List of velocities
δW:	Virtual work
$\delta\Pi$:	Variation of frame connection matrix

$\delta\tilde{X}$: Virtual generalized displacement

$\delta\dot{\tilde{X}}$: Variation of the generalized rates

INTRODUCTION

The largest industries in Norway concern the sale of oil and gas; the second largest concern the associated service and supply industry. These industries have grown significantly since the discovery of oil, *ca.* 1969.

In addition, Norway deploys windfarms and fish farms. These emerging markets have induced a demand for more advanced ships, as shown in Figure 1.

Finally, though the oil and gas supply has stagnated, there still exists in operation, a vast variety of ships outside the Norwegian coastline. These are still in use, but for other purposes such as windmill installation and maintenance, as shown in Figure 2.

The space industry has for years used gyroscopes [1] to adjust the attitude of spacecrafts. In later years this technology has also been used in the maritime sector [2] with great success.

The goal of this research is to demonstrate how the Moving Frame Method can facilitate the analysis of the forces required to stabilize such ships when they are deleteriously affected by wave motion.

FIGURE 1. Ship with a crane for loading and offloading in an offshore environment. [3]

FIGURE 2. Ship servicing a windmill, different methods for stabilization are currently used in todays marked. [4]

This work builds upon previous work [5]. However, here we show a method to explicitly account for the prescribed motion, and conduct a full 3D analysis, while displaying all results in 3D on cell phones.

OVERVIEW OF THE MOVING FRAME METHOD

Élie Cartan (1869-1951) [6] assigned a reference frame to each point of an object under study (a curve, a surface, Euclidean space itself). Then, using an orthonormal expansion, he expressed the rate of change of the frame in terms of the frame. The MFM leverages this by placing a reference frame on every moving link. However, then we need a *method to connect moving frames*. For this, we turn to Sophus Lie.

Marius Sophus Lie (1842-1899) developed the theory of continuous groups and their associated algebras. The MFM adopts the mathematics of rotation groups and their algebras, yet distils them to simple matrix multiplications. However, then we need a simplifying notation. For this, we turn to Frankel.

Ted Frankel [6] developed a compact notation in geometrical physics. The MFM adopts this notation to enable a methodology that is identical for both 2D and 3D analyses. The notation is also identical for single bodies and multi-body linked systems. In turn, this uplifts students' understanding from the conceptual to the pragmatic, enabling them to analyze machines of the 3D world.

The reader may find an introduction to the undergraduate and graduate Moving Frame Method, along with a pedagogical assessment in Impelluso [8]. In the following section, we summarize the MFM.

To the center of mass $C^{(\alpha)}$ of each body-α, we attach a Cartesian coordinate system $s_C^{(\alpha)}(t)=\left\{s_1^{(\alpha)}\ s_2^{(\alpha)}\ s_3^{(\alpha)}\right\}^T$ where the superscript α is defined as a direction of either 1, 2 or 3. We use directional derivatives along the coordinate functions to create an associated basis frame $\mathbf{e}_i^{(\alpha)}=\dfrac{\partial}{\partial s_i^{(\alpha)}}$

To obtain an inertial frame, we select one of the moving frames and define the inertial at the start of the analysis:

$$\mathbf{e}^I=\left\{\mathbf{e}_1^I\quad \mathbf{e}_2^I\quad \mathbf{e}_3^I\right\}=\left\{\mathbf{e}_1^{(\alpha)}(0)\quad \mathbf{e}_2^{(\alpha)}(0)\quad \mathbf{e}_3^{(\alpha)}(0)\right\} \tag{1}$$

The translation of body-α with respect to the inertial frame $\mathbf{e}^I$ is expressed as:

$$\mathbf{r}_c^{(\alpha)}(t)=\mathbf{e}^I x_c^{(\alpha)}(t) \tag{2}$$

Throughout this document we will use "*s*" for coordinates in the moving frame and "*x*" for the coordinates in the inertial frame.

The rotation of the moving frame $\mathbf{e}^{\alpha}$ in relation to the inertial frame $\mathbf{e}^{I}$ is expressed as:

$$\mathbf{e}^{(\alpha)}(t) = \mathbf{e}^{I} R^{(\alpha)}(t) \tag{3}$$

Where $R^{(\alpha)}(t)$ is a 3x3 matrix, and a member of the Special Orthogonal Group SO(3), thus:

$$\left(R^{(\alpha)}(t)\right)^{-1} = \left(R^{(\alpha)}(t)\right)^{T} \tag{4}$$

Assert the rate of change of Eqn (3) as:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{I} \dot{R}^{(\alpha)}(t) \tag{5}$$

Substituting $\mathbf{e}^{I}$ into (5) by using Orthogonality, provides:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t)\left(R^{(\alpha)}(t)\right)^{T} \dot{R}^{(a)}(t) \tag{6}$$

Define angular velocity matrix $\overleftrightarrow{\omega^{(\alpha)}(t)}$ as the skew form:

$$\overleftrightarrow{\omega^{(\alpha)}(t)} = \left(R^{(\alpha)}(t)\right)^{T} \dot{R}^{(\alpha)}(t) \tag{7}$$

Thus:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t)\overleftrightarrow{\omega^{(\alpha)}(t)} \tag{8}$$

Next, we combine this information and exploit the Special Euclidean Group, SE(3). This approach was first used by Murakami [9, 10].

We create two frame connections by structuring together the frame and its location and relating them as per:

$$\left(\mathbf{e}^{(\alpha)}(t) \quad \mathbf{r}_{C}^{(\alpha)}(t)\right) = \left(\mathbf{e}^{I} \quad \mathbf{0}\right) E^{(\alpha)}(t) \tag{9}$$

In (9) we assert the relationship between the frame connections, through a frame connection matrix, $E^{(\alpha)}(t)$, where:

$$E^{(\alpha)}(t) = \begin{bmatrix} R^{(\alpha)}(t) & x_{C}^{(\alpha)}(t) \\ 0_{1x3}^{T} & 1 \end{bmatrix} \tag{10}$$

Equation (9, 10) recapitulates Eqn. (2) and (3)

Continuing to *relative* motion, the rotation of body-(α+1) in relation to the previous body-(α) becomes:

$$\mathbf{e}^{(\alpha+1)}(t) = \mathbf{e}^{(\alpha)} R^{(\alpha+1/\alpha)}(t) \tag{11}$$

Furthermore, we can assert the position by first moving to the (α) frame and, in the (α) frame, to the (α+1) frame :

$$\mathbf{r}_{c}^{(\alpha+1)}(t) = \mathbf{r}_{c}^{(\alpha)}(t) + \mathbf{s}^{(\alpha+1/\alpha)}(t) \tag{12}$$

$$\mathbf{r}_{c}^{(\alpha+1)}(t) = \mathbf{r}_{c}^{(\alpha)}(t) + \mathbf{e}^{(\alpha)}(t) s^{(\alpha+1/\alpha)}(t) \tag{13}$$

As an aside, assert the closure property of the SO(3) group. Thus:

$$\mathbf{e}^{(\alpha+1)}(t) = \mathbf{e}^{I} R^{(\alpha)}(t) R^{(\alpha+1/\alpha)}(t) = \mathbf{e}^{I} R^{(\alpha+1)}(t) \tag{14}$$

We now assemble this information we have gathered so far and structure a 4x4 relative frame connection matrix $E^{(\alpha+1/\alpha)}(t)$:

$$E^{(\alpha+1/\alpha)}(t) = \begin{bmatrix} R^{(\alpha+1/\alpha)}(t) & s^{(\alpha+1/\alpha)}(t) \\ 0 & 1 \end{bmatrix} \tag{15}$$

With this structure, we can assert:

$$\left(\mathbf{e}^{(\alpha+1)}(t) \quad \mathbf{r}_{C}^{(\alpha+1)}(t)\right) = \left(\mathbf{e}^{(\alpha)}(t) \quad \mathbf{r}_{C}^{(\alpha)}(t)\right) \begin{bmatrix} R^{(\alpha+1/\alpha)}(t) & s^{(\alpha+1/\alpha)}(t) \\ 0 & 1 \end{bmatrix} \tag{16}$$

This equation recapitulates (11) and (13).

Finally, the closure property of the SE(3) group still holds:

$$E^{(\alpha+1)}(t) = E^{(\alpha)}(t) E^{(\alpha+1/\alpha)}(t) \tag{17}$$

Also note that this matrix is a member of the SE(3) and its inverse is known analytically.

We now apply our foundation to the analysis of a ship stabilizer, repeating some of this work as we traverse the links.

KINEMATICS OF THE STABILIZATION SYSTEM

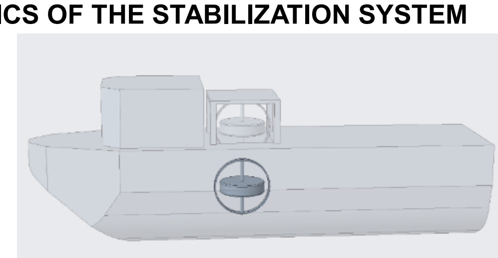

FIGURE 3. Ships with dual gyro

Figure 3 presents the system modeled for this task. There are two gyroscopic assemblies located at an equal distance from the center of mass of the ship.

The gyroscopic assembly consists of a spinning rotor that is driven around its vertical 3-axis. The spinning rotor is placed in a gyro gimbal that nutates the gyro around the port-to-starboard 2-axis. The shared 1-axis runs stern to bow.

Figure 4 presents a close up of the two gyro assemblies, with moving frames.

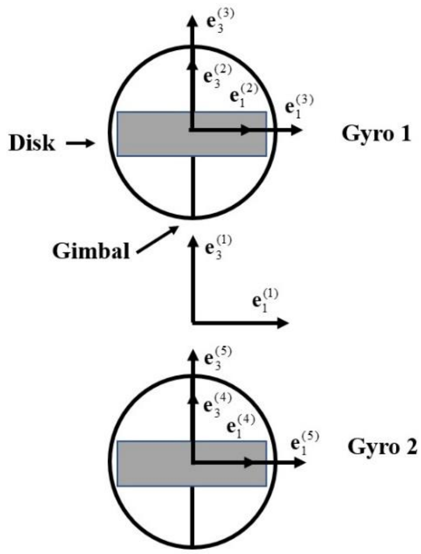

FIGURE 4. Dual Gyro with Fames

Kinematics of body-1: The ship

We use the first body frame (that for the ship) to assert the inertial frame at the start of the analysis (this is merely a pedantic assertion).

$$\mathbf{e}^I = \mathbf{e}^{(1)}(0) \tag{18}$$

We define a full rotation matrix for body(1) (ship) rotation:

$$R^{(1)}(t) = \begin{bmatrix} R_{11}^{(1)}(t) & R_{12}^{(1)}(t) & R_{13}^{(1)}(t) \\ R_{21}^{(1)}(t) & R_{22}^{(1)}(t) & R_{23}^{(1)}(t) \\ R_{31}^{(1)}(t) & R_{32}^{(1)}(t) & R_{33}^{(1)}(t) \end{bmatrix} \tag{19}$$

The frame connection matrix for the first body contains data for both rotation and translation from the inertial frame:

$$E^{(1)}(t) = \begin{bmatrix} R^{(1)}(t) & x_C^{(1)}(t) \\ 0_{1x3}^T & 1 \end{bmatrix} \tag{20}$$

Next, we obtain the derivative of the connection matrix:

$$\dot{E}^{(1)}(t) = \begin{bmatrix} \dot{R}^{(1)}(t) & \dot{x}_c^{(1)}(t) \\ 0_{1x3}^T & 0 \end{bmatrix} \tag{21}$$

The inverse of the connection matrix is analytically known:

$$\left(E^{(1)}(t)\right)^{-1} = \begin{bmatrix} R^{(1)}(t) & x_C^{(1)}(t) \\ 0_{1x3}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \left(R^{(1)}(t)\right)^T & -\left(R^{(1)}(t)\right)^T x_C^{(1)}(t) \\ 0_{1x3}^T & 1 \end{bmatrix} \tag{22}$$

Next, we desire to express the rate of change of the frame and location in terms of the same frame connection structure:

$$\begin{pmatrix} \dot{\mathbf{e}}^{(1)}(t) & \dot{\mathbf{r}}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} \left(E^{(1)}(t)\right)^{-1} \dot{E}^{(1)}(t) \tag{23}$$

For notational simplification, the product of $\left(E^{(1)}(t)\right)^{-1}$ and $\dot{E}^{(1)}(t)$ is defined as the time rate of the frame connection matrix $\Omega^{(1)}$:

$$\Omega^{(1)}(t) \equiv \left(E^{(1)}(t)\right)^{-1} \dot{E}^{(1)}(t) \tag{24}$$

Thus:

$$\begin{pmatrix} \dot{\mathbf{e}}^{(1)}(t) & \dot{\mathbf{r}}_c^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(1)}(t) & \mathbf{r}_c^{(1)}(t) \end{pmatrix} \Omega^{(1)}(t) \tag{25}$$

$$\Omega^{(1)}(t) = \begin{bmatrix} \left(R^{(1)}(t)\right)^T \dot{R}^{(1)}(t) & \left(R^{(1)}(t)\right)^T \dot{x}_c^{(1)}(t) \\ 0_{1x3}^T & 0 \end{bmatrix} \tag{26}$$

This matrix provides information about the linear and angular velocities of the coordinate frame attached to the ship. The sub-matrix in the upper left corner of $\Omega^{(1)}$ is a skew-symmetric angular velocity of the frame:

$$\overleftrightarrow{\omega^{(\alpha)}}(t) = \left(R^{(1)}(t)\right)^T \dot{R}^{(1)}(t) \tag{27}$$

This matrix represents the time rate of the coordinate frame in its own frame:

$$\dot{\mathbf{e}}^{(1)}(t) = \mathbf{e}^{(1)}(t)\,\overleftrightarrow{\omega^{(\alpha)}}(t) = \mathbf{e}^{(1)}(t) \begin{bmatrix} 0 & -\omega_3^{(1)}(t) & \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) & 0 & -\omega_1^{(1)}(t) \\ -\omega_2^{(1)}(t) & \omega_1^{(1)}(t) & 0 \end{bmatrix} \tag{28}$$

Associate the column components to the moving frame to assert the angular velocity vector:

$$\boldsymbol{\omega}^{(1)}(t) = \mathbf{e}^{(1)}(t) \begin{pmatrix} \omega_1^{(1)}(t) \\ \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) \end{pmatrix} \tag{29}$$

Extract the linear velocity for the ship is expressed in the inertial frame:

$$\dot{\mathbf{r}}_C^{(1)}(t) = \mathbf{e}^I \dot{x}_C^{(1)}(t) \tag{30}$$

Kinematics of the body-2: The first gimbal

At this point, there will be two gyroscopic structures. They will differ only insofar as their placement. The formulation, however, is the same. To reduce the length of this paper, we focus on the first system: the first gimbal and first rotor

We attach a coordinate frame $\mathbf{e}^{(2)}(t)$ to the center of mass of gyro 1, $C^{(1)}$ of the link:

$$\mathbf{e}^{(2)}(t) = \left(\mathbf{e}_1^{(2)}(t)\, \mathbf{e}_2^{(2)}(t)\, \mathbf{e}_3^{(2)}(t)\right) \tag{31}$$

We find the relative position from $\mathbf{e}^{(1)}(t)$ to $\mathbf{e}^{(2)}(t)$ by first translating from the center of mass $C^{(s)}$ of the ship, to the center of mass of gyro 1 $C^{(1)}$. We will also put a frame, $\mathbf{e}^{(2)}(t)$ at the frame holding gyro 1. The frame will rotate to induce a moment on the boat. The translation is in the 3-direction (Naturally, for the second stability disk the coordinates will simply negate):

$$\mathbf{s}_{c_1} = \mathbf{e}^{(1)}(t) s_{c_1} = \mathbf{e}^{(1)}(t)\begin{pmatrix} 0 \\ 0 \\ l^{(1)} \end{pmatrix} \tag{32}$$

The frame that holds the gyro, $\mathbf{e}^{(2)}(t)$ rotates around the second axis. Which gives the following frame relations:

$$\mathbf{e}^{(2)}(t) = \mathbf{e}^{(1)}(t) R^{(2/1)}(t) = \mathbf{e}^{(1)}(t)\begin{bmatrix} \cos\phi(t) & 0 & \sin\phi(t) \\ 0 & 1 & 0 \\ -\sin\phi(t) & 0 & \cos\phi(t) \end{bmatrix} \tag{33}$$

The relation between the first and second frame connections is expressed using the relative frame connection matrix $E^{(2/1)}(t)$ and:

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_C^{(2)}(t)\right) = \left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_C^{(1)}(t)\right) E^{(2/1)}(t) \tag{34}$$

We obtain this frame connection matrix by taking each of the steps described above: translating distance $l^{(1)}$ from the ship frame without rotation, then rotating the gyro 1 frame:

$$E^{(2/1)}(t) = \begin{bmatrix} R^{(2/1)}(t) & s_C^{(2/1)} \\ 0_{1x3}^T & 1 \end{bmatrix} = \begin{bmatrix} I_3 & s_{C1} \\ 0_3^T & 1 \end{bmatrix} \begin{bmatrix} R^{(2/1)}(t) & 0_3 \\ 0_{1x3}^T & 1 \end{bmatrix} \tag{35}$$

The second frame is related to the inertial through the absolute connection matrix $E^{(2)}(t)$:

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_C^{(2)}(t)\right) = \left(\mathbf{e}^I \quad \mathbf{0}\right) E^{(1)}(t) E^{(2/1)}(t) = \left(\mathbf{e}^I \quad \mathbf{0}\right) E^{(2)}(t) \tag{36}$$

We assert:

$$E^{(2)}(t) = \begin{bmatrix} R^{(2)}(t) & x_C^{(2)}(t) \\ 0_{1x3}^T & 1 \end{bmatrix} \tag{37}$$

Multiplying together the rotation matrix from frame 1, $R^{(1)}(t)$ and frame 2, $R^{(2/1)}(t)$ gives us the absolute orientation of frame 2, $\mathbf{e}^{(2)}(t)$:

$$R^{(2)}(t) = R^{(1)}(t) R^{(2/1)}(t) \tag{38}$$

Adding the translations from the first and second frame provides the absolute translation from the inertial:

$$x_C^{(2)} = R^{(1)}(t) s_C^{(2)} + x_C^{(1)} \tag{39}$$

The frame connection matrix, its inverse and derivative are used to calculate the time rate of the frame connection matrix, $\Omega^{(2)}$. From this we extract the angular velocity vector of the gyro 1 frame:

$$\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + \omega^{(2/1)}(t) \tag{40}$$

The tower rotates about the shared common axis of the ship: $\mathbf{e}_2^{(1)}(t) \equiv \mathbf{e}_2^{(2)}(t)$. Thus, the *relative* angular velocity vector $\omega^{(2/1)}(t)$ can be expressed by $\dot{\phi}(t)\mathbf{e}_2$, where $e_2 = (0\ 1\ 0)^T$:

$$\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + \dot{\phi}^{(2)}(t) e_2 \tag{41}$$

The linear velocity vector of the second body from the inertial frame is extracted from the expression for the time derivative of the frame:

$$\left(\dot{\mathbf{e}}^{(2)}(t) \quad \dot{\mathbf{r}}_{\mathrm{C}}^{(2)}(t)\right) = \left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_{\mathrm{C}}^{(2)}(t)\right) \Omega^{(2)}(t) \tag{42}$$

This provides:

$$\begin{aligned} \dot{x}_C^{(2)}(t) &= R^{(1)}(t) R^{(2/1)}(t) \left(\overleftrightarrow{s_{C_1}}\right)^T \omega^{(2)}(t) \\ &+ R^{(1)}(t) \left(\overleftrightarrow{s_{C_1}}\right)^T \omega^{(1)}(t) + \dot{x}_C^{(1)}(t) \end{aligned} \tag{43}$$

Kinematics of body-3: The first rotor

We construct the relative frame connection matrix. The third frame has the same center as the second frame, and therefore we find simply the following:

$$E^{(3/2)}(t) = \begin{bmatrix} R^{(3/2)}(t) & 0 \\ 0_{1x3}^{T} & 1 \end{bmatrix} \tag{44}$$

$E^{(3/2)}(t)$, we use this to relate the precession and spin of the gyro disc, as shown here:

$$\begin{pmatrix} \mathbf{e}^{(3)}(t) & \mathbf{r}_c^{(3)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(2)}(t) & \mathbf{r}_c^{(2)}(t) \end{pmatrix} E^{(3/2)}(t) \tag{45}$$

Now that we have the relative connection matrix, we assert the absolute frame connection using the relations:

$$\begin{gathered} E^{(3)}(t) = E^{(2)}(t) E^{(3/2)}(t) = \\ \begin{bmatrix} R^{(1)}(t) R^{(2/1)}(t) & R^{(1)}(t) s_C^{(1)}(t) \\ 0_{1x3}^{T} & 1 \end{bmatrix} \begin{bmatrix} R^{(3/2)}(t) & 0 \\ 0_{1x3}^{T} & 1 \end{bmatrix} \end{gathered} \tag{46}$$

We compute the equation above and obtain:

$$E^{(3)}(t) = \begin{bmatrix} R^{(1)}(t) R^{(2/1)}(t) R^{(3/2)}(t) & R^{(1)}(t) s_C^{(2/1)} + x_C^{(1)}(t) \\ 0_{1x3}^{T} & 1 \end{bmatrix} \tag{47}$$

We assert the inverse:

$$\begin{gathered} \left(E^{(3)}(t)\right)^{-1} = \\ \begin{bmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T & -\left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \left(R^{(1)}(t)\right)^T \left(R^{(1)}(t) s_C^{(2/1)} + x_C^{(1)}(t)\right) \\ 0_{1x3}^{T} & 1 \end{bmatrix} \end{gathered} \tag{48}$$

We assert the rate of change of $E^{(3)}(t)$:

$$\dot{E}^{(3)}(t) = \begin{bmatrix} \dot{R}^{(3)}(t) & \dot{R}^{(1)}(t) s_C^{(2/1)} + \dot{x}_C^{(1)}(t) \\ 0_{1x3}^{T} & 0 \end{bmatrix} \tag{49}$$

We know that $\dot{R}^{(3)}(t)$ is the same as:

$$\begin{gathered} \dot{R}^{(3)}(t) = \dot{R}^{(1)}(t) R^{(2/1)}(t) R^{(3/2)}(t) + \\ R^{(1)}(t) \dot{R}^{(2/1)}(t) R^{(3/2)}(t) + R^{(1)}(t) R^{(2/1)}(t) \dot{R}^{(3/2)}(t) \end{gathered} \tag{50}$$

Now we have all the elements that are necessary to find the absolute time rate of frame connection for the spin of the gyro disc, "the third body", $\Omega^{(3)}(t)$:

$$\begin{pmatrix} \dot{\mathbf{e}}^{(3)}(t) & \dot{\mathbf{r}}_C^{(3)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{(3)}(t) & \mathbf{r}_{\mathbf{C}}^{(\mathbf{3})}(t) \end{pmatrix} \Omega^{(3)}(t) \tag{51}$$

Out from the above we can extract out the formulas for angular and linear acceleration:

$$\boldsymbol{\omega}^{\mathbf{(3)}}(t) = \mathbf{e}^{\mathbf{(3)}}(t) \begin{pmatrix} \left(R^{(3/2)}(t)\right)^T \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + \\ \left(R^{(3/2)}(t)\right)^T \omega^{(2/1)}(t) + \omega^{(3/2)}(t) \end{pmatrix} \tag{52}$$

$$\dot{\boldsymbol{x}}_{\boldsymbol{C}}^{\boldsymbol{(3)}}(t) = \mathbf{e}^{\boldsymbol{I}} \left(-R^{(1)}(t) \overleftrightarrow{s_C^{(2/1)}} \omega^{(1)}(t) + \dot{x}_C^{(1)}(t) \right) \tag{53}$$

Relationship between Gyro-1 and Gyro-2

We have now found the movement and all the equations needed from the gyro to continue with generalized coordinates, B-matrix and to move on to kinetics.

To cancel out the moment created when accelerating and decelerating the disc we use 2 gyroscopes with opposite spin. They will move in a mirrored way to create the precession around the same axis. Beside the gyros being inverse of each other they are identical. As a result of this we can use the same equations for gyro 2 as for gyro 1 but with new frame numbers. So $E^{(2)}$ becomes $E^{(4)}$, $E^{(3)} = E^{(5)}$, $E^{(2/1)} = E^{(4/1)}$ and $E^{(3/2)} = E^{(5/4)}$.

The difference between the two gyros is found in the coordinate values, but the work up is the same:

$$\mathbf{s}_{\mathbf{c}_1} = \mathbf{e}^{\mathbf{(1)}}(t) s_{c_1} = \mathbf{e}^{\mathbf{(1)}}(t) \begin{pmatrix} 0 \\ 0 \\ -l^{(1)} \end{pmatrix}$$

GENERALIZED COORDINATES

We now relate the established Cartesian coordinates to certain Generalized coordinates. Before continuing, we make some assumptions to simplify this process. First, we assume the boat is stationary; i.e., the center of mass of the boat does not translate in any direction. Second, we assume the discs are already spinning at a constant rate when analysis begins. Third, we assume there is no resistance on the disk, meaning there is no motor needed to maintain the constant velocity of the rotors. These assumptions reduce the number of Generalized coordinates and allows us to use the Method of Prescribed Rates. $\{\dot{\mathrm{x}}(\mathrm{t})\}$ denotes the Cartesian velocities, $\{\dot{\mathrm{q}}(\mathrm{t})\}$ denotes the essential generalized velocities, while $\{\dot{\mathrm{r}}(\mathrm{t})\}$ denotes the generalized velocities that are not essential:

$$\left\{\dot{X}(t)\right\} \equiv \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix} \quad \left\{\dot{\mathrm{q}}(\mathrm{t})\right\} \equiv \begin{Bmatrix} \omega^{(1)}(\mathrm{t}) \\ \dot{\theta}^{(2)}(\mathrm{t}) \\ \dot{\theta}^{(4)}(\mathrm{t}) \end{Bmatrix} \quad \left\{\dot{\mathrm{r}}(\mathrm{t})\right\} \equiv \begin{Bmatrix} \dot{\phi}^{(3)} \\ \dot{\phi}^{(5)} \end{Bmatrix} \tag{54}$$

The relationship between these data structures is obtained through the B-matrix and C-matrix as follows:

$$\left\{\dot{X}(t)\right\} = \left[B(t)\right]\left\{\dot{q}(t)\right\} + \left[C(t)\right]\left\{\dot{r}(t)\right\} \tag{55}$$

The number of rows of the B-matrix is equal to the number of Cartesian velocities, while the number of columns is equal to the number of Essential Generalized velocities. In this case, the size of the B-matrix will be 9 × 3:

$$\left[B\right] = \begin{bmatrix} I_{3x3} & 0 & 0 \\ -R^{(1)}(t)\overleftrightarrow{S_C^{(2/1)}} & 0 & 0 \\ (R^{(2/1)}(t))^T & \mathbf{e}_2 & 0 \\ -R^{(1)}(t)\overleftrightarrow{S_C^{(2/1)}} & 0 & 0 \\ (R^{(3/2)}(t))^T(R^{(2/1)}(t))^T & (R^{(3/2)}(t))^T\mathbf{e}_2 & 0 \\ -R^{(1)}(t)\overleftrightarrow{S_C^{(4/1)}(t)} & 0 & 0 \\ (R^{(4/1)}(t))^T & 0 & \mathbf{e}_2 \\ -R^{(1)}(t)\overleftrightarrow{S_C^{(4/1)}} & 0 & 0 \\ (R^{5/4)}(t))^T(R^{(4/1)}(t))^T & 0 & (R^{(5/4)}(t))^T\mathbf{e}_2 \end{bmatrix} \tag{56}$$

Where I_{3x3}, 0_{3x3}, $\mathbf{e}_2$ and $\mathbf{s}_c^{(\alpha+1)}(t) = \mathbf{e}^{(\alpha)}s_c^{(\alpha+1/\alpha)}(t)$ (which is found in the C-matrix) is given as:

$$I_{3x3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \underset{3x1}{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The C-matrix is constructed in a similar manner. Number of rows corresponds to number of Cartesian velocities, while the number of columns is equal to the number of Generalized velocities. In this case a 9 × 2 matrix:

$$\left[C\right] \equiv \begin{bmatrix} \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ e_3 & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & e_3 \end{bmatrix} \tag{57}$$

KINETICS

Application of Analytical Mechanics

We begin by defining a Lagrangian as the difference between the kinetic and potential energy:

$$L^{(\alpha)}\left(q(t),\dot{q}(t)\right) = K^{(\alpha)}\left(q(t),\dot{q}(t)\right) - U^{(\alpha)}\left(q(t)\right) \tag{58}$$

Define the Action as the definite integral of the Lagrangian function over time:

$$A = \int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt \tag{59}$$

Hamilton's principle states that "the motion of a system occurs in such a way that the definite integral A becomes a minimum for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed" [11]. This means that the equations of motion can be obtained by setting the variation of the Action equal to zero:

$$\delta\int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt = 0 \tag{60}$$

To include the non-conservative forces, we exploit the extension of Hamilton's Principle, known as the Principle of Virtual Work. Here, we formulate the Lagrangian as dependent only on the kinetic energy. We will account for all other forces (conservative or non-conservative) as work, on the right side. From this point onwards, we omit the dependencies of position and velocity for ease of notation.

$$\int_{t_0}^{t_1} \delta K^{(\alpha)}(t)dt = -\int_{t_0}^{t_1} \delta W^{(\alpha)}(t)dt \tag{61}$$

The kinetic energy of each body in the system is expressed by the angular momentum $\mathbf{H}_C^{(\alpha)}(t)$, and linear momentum $\mathbf{L}_C^{(\alpha)}(t)$:

$$\mathbf{H}_C^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t)\,H_C^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t)\,J_C^{(\alpha)}\omega^{(\alpha)}(t) \tag{62}$$

$$\mathbf{L}_C^{(\alpha)}(t) = \mathbf{e}^I L_C^{(\alpha)}(t) = \mathbf{e}^I m^{(\alpha)} \dot{x}_C^{(\alpha)}(t) \tag{63}$$

Here, $J_C^{(\alpha)}$represents the moment of inertia matrix for body α. The total kinetic energy of a body α with the frame placed at the center of mass is defined as:

$$K^{(\alpha)}(t) = \frac{1}{2}\left\{\dot{\mathbf{r}}_C^{(\alpha)}(t)\cdot\mathbf{L}_C^{(\alpha)}(t) + \boldsymbol{\omega}^{(\alpha)}(t)\cdot\mathbf{H}_C^{(\alpha)}(t)\right\} \tag{64}$$

For the whole system, the total kinetic energy is expressed in matrix form as:

$$K(t) = \frac{1}{2}\left\{\dot{X}(t)\right\}^T [M]\left\{\dot{X}(t)\right\} \tag{65}$$

The masses and moments of inertia for each body is contained in the generalized mass matrix $[M]$:

$$[M] = \begin{bmatrix}
J_C^{(1)} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & m^{(2)}I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & J_C^{(2)} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & m^{(3)}I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & J_C^{(3)} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & m^{(4)}I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & J_C^{(4)} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & m^{(5)}I_3 & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & J_C^{(5)}
\end{bmatrix} \tag{66}$$

To continue, define virtual rotational displacement $\delta\pi^{(\alpha)}(t)$ is the un-skewed form of $\overleftrightarrow{\delta\pi^{(\alpha)}(t)}$, which is defined as the product of the transpose and the variation of the rotation matrix:

$$\overleftrightarrow{\delta\pi^{(\alpha)}(t)} = \left(R^{(\alpha)}(t)\right)^T \delta R^{(\alpha)}(t) \tag{67}$$

Compare this Eqn. (74) with Eqn. (12). With this, we structure the virtual generalized displacements $\{\delta \tilde{X}(t)\}$:

$$\left\{\delta\tilde{X}(t)\right\} = \begin{pmatrix} \delta\pi^{(1)}(t) \\ \delta x_C^{(2)}(t) \\ \delta\pi^{(2)}(t) \\ \vdots \\ \delta x_C^{(5)}(t) \\ \delta\pi^{(5)}(t) \end{pmatrix} \tag{68}$$

Next, the variation of the velocities is called the virtual generalized velocities $\{\delta \dot{X}(t)\}$:

$$\left\{\delta\dot{X}(t)\right\} = \begin{pmatrix} \delta\omega^{(1)}(t) \\ \delta\dot{x}_C^{(2)}(t) \\ \delta\omega^{(2)}(t) \\ \vdots \\ \delta\dot{x}_C^{(5)}(t) \\ \delta\omega^{(5)}(t) \end{pmatrix} \tag{69}$$

For the linear displacement, the variation of the derivative is equal to the derivative of the variation:

$$\delta\dot{x}_C^{(\alpha)}(t) = \frac{d}{dt}\delta x_C^{(\alpha)}(t) \tag{70}$$

However, there is a restriction on the variation of the angular velocity, as proven by Murakami [11]:

$$\delta\omega^{(\alpha)}(t) = \frac{d}{dt}\delta\pi^{(\alpha)}(t) + \overleftrightarrow{\omega^{(\alpha)}(t)}\,\delta\pi^{(\alpha)}(t) \tag{71}$$

The last two equations are written in compact form as:

$$\left\{\delta\dot{X}(t)\right\} = \left\{\delta\dot{X}(t)\right\} + [D]\left\{\delta\tilde{X}(t)\right\} \tag{72}$$

Where $[D]$ is a skew symmetric matrix that contains the angular velocity matrices for each frame:

$$[D] = \begin{bmatrix}
\overleftrightarrow{\omega^{(1)}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & \overleftrightarrow{\omega^{(2)}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & \overleftrightarrow{\omega^{(3)}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & \overleftrightarrow{\omega^{(4)}} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & \overleftrightarrow{\omega^{(5)}}
\end{bmatrix} \tag{73}$$

The variation of the kinetic energy is thus expressed as:

$$\delta K(t) = \left\{\delta\dot{X}(t)\right\}^T [M]\left\{\dot{X}(t)\right\} \tag{74}$$

Next, we need the forces and moments acting on the different bodies of the system. They are expressed in a single column matrix $\{Q(t)\}$. We note that the rotors spin in opposite directions, such that the moments acting on the buoy from the generator will be cancelled out. This is a major advantage of having two gyroscopes instead of one.

$$\{Q(t)\}=\begin{pmatrix} F_C^{(1)I}(t) \\ M_C^{(1)}(t) \\ F_C^{(2)I}(t) \\ M_C^{(2)}(t) \\ F_C^{(3)I}(t) \\ M_C^{(3)}(t) \\ F_C^{(4)I}(t) \\ M_C^{(4)}(t) \\ F_C^{(5)I}(t) \\ M_C^{(5)}(t) \end{pmatrix}=\begin{pmatrix} F_w^I(t)+F_b^I - m^{(1)}ge_3 \\ M_w(t)+M_g^{(2)}(t)e_2+M_g^{(4)}(t)e_2 \\ -m^{(2)}ge_3 \\ -M_g^{(2)}(t)e_2 \\ -m^{(3)}ge_3 \\ 0 \\ -m^{(4)}ge_3 \\ -M_g^{(4)}(t)e_2 \\ -m^{(5)}ge_3 \\ 0 \end{pmatrix} \tag{75}$$

Here, $-m^{(\alpha)}ge_3$ is the gravitational force at the center of mass. $F_w^I(t)$ and $F_b^I(t)$ are the wave forces and buoyancy forces, respectively. $M_w(t)$ is the moment induced by waves, and $M_g^{(\alpha)}(t)e_2$ is the moment from the motors that are attached to the discs.

The virtual work done by the generalized forces can then be expressed as:

$$\delta W=\left\{\delta\tilde{X}(t)\right\}^T\{Q(t)\} \tag{76}$$

The B-matrix that relates the Cartesian velocities $\left\{\dot{X}(t)\right\}$ to the essential generalized velocities $\{\dot{q}(t)\}$, also relates the virtual generalized displacements $\left\{\delta\tilde{X}(t)\right\}$ to the essential virtual displacements $\{\delta q(t)\}$ **:**

$$\left\{\delta\tilde{X}(t)\right\}=[B(t)]\{\delta q(t)\} \tag{77}$$

The transpose of the above is used to rewrite equation (18):

$$\delta W=\{\delta q(t)\}^T\left\{F^*(t)\right\} \tag{78}$$

Where the essential generalized forces $\left\{F^*(t)\right\}$ are defined as:

$$\left\{F^*(t)\right\}=[B(t)]^T\{Q(t)\} \tag{79}$$

We now insert the expressions obtained for the variation of the kinetic energy and the virtual work into the integral equation (61)

$$\int_{t_0}^{t_1}\left(\left\{\delta\dot{X}(t)\right\}^T[M]\left\{\dot{X}(t)\right\}+\{\delta q(t)\}^T\left\{F^*(t)\right\}\right)dt=0 \tag{80}$$

Equation of motion

After performing integration by parts on (80), and accounting for zero virtual displacement at the endpoints, we obtain a second order coupled differential equation:

$$\left[M^*(t)\right]\{\ddot{q}(t)\}+\left[N^*(t)\right]\{\dot{q}(t)\}=\left\{F^*(t)\right\}-\left[T^*(t)\right]\{\dot{r}(t)\} \tag{81}$$

Where the following terms are defined:

$$\left[M^*(t)\right]\equiv[B(t)]^T[M][B(t)] \tag{82}$$

$$\left[N^*(t)\right]\equiv[B(t)]^T\left([M]\left[\dot{B}(t)\right]+[D(t)][M][B(t)]\right) \tag{83}$$

$$\left[T^*(t)\right]\equiv[B(t)]^T\left([M]\left[\dot{C}(t)\right]+[D(t)][M][C(t)]\right) \tag{84}$$

Solving (81) with respect to the list of generalized accelerations $\{\ddot{q}(t)\}$, yields:

$$\{\ddot{q}(t)\}=\left[M^*(t)\right]^{-1}\left(\left\{F^*(t)\right\}-\left[T^*(t)\right]\{\dot{r}(t)\}-\left[N^*(t)\right]\{\dot{q}(t)\}\right) \tag{85}$$

Reconstructing the Rotation Matrix and Numerical Integration

As the boat rotates due to the motion of the ocean, its rotation matrix is changing. Hence, the rotation matrix must be updated at each time-step of the analysis. From the definition of skew-symmetric angular velocity we state the following:

$$\dot{R}^{(1)}(t)=R^{(1)}(t)\overleftrightarrow{\omega^{(1)}(t)} \tag{86}$$

This is a coupled differential equation in matrix form. For the special case of a case of a constant angular velocity matrix, $\overleftrightarrow{\omega_0}$, we propose the following solution:

$$R^{(1)}(t)=R^{(1)}(0)\exp\left(t\overleftrightarrow{\omega_0}\right) \tag{87}$$

From here we can use the Taylor series for sin and cosine:

$$\exp(t\overleftrightarrow{\omega_0})\equiv I+\frac{\overleftrightarrow{\omega_0}}{\|\boldsymbol{\omega}_0\|}\sin\left(t\|\boldsymbol{\omega}_0\|\right)+\left(\frac{\overleftrightarrow{\omega_0}}{\|\boldsymbol{\omega}_0\|}\right)^2 1-\cos\left(t\|\boldsymbol{\omega}_0\|\right) \tag{88}$$

Thus, assuming an initial rotation matrix, $R^{(1)}(0)$, we obtain the following rotation matrix from the angular velocity matrix:

$$R^{(1)}(t) = R^{(1)}(0)\left(I + \frac{\overleftrightarrow{\omega_0}}{\|\boldsymbol{\omega}_0\|}\sin\left(t\|\boldsymbol{\omega}_0\|\right) + \left(\frac{\overleftrightarrow{\omega_0}}{\|\boldsymbol{\omega}_0\|}\right)^2 1-\cos\left(t\|\boldsymbol{\omega}_0\|\right) \right) \tag{89}$$

We will assume that during the numerical integration of the equations of motion t to t + Δt, that the angular velocity is constant. While this is not the case here, it can be applied to each individual time step of the Runge-Kutta (RK4) integration. Therefore, we adopt the mid-point integration method using the mean value of the angular velocity, $\omega(t+\Delta t/2)$:

$$\omega(t+\Delta t/2) = \left(\omega(t) + \omega(t+\Delta t)\right)/2 \tag{90}$$

Essentially, after coming out of each time step, we will have a new omega. We use that newfound expression and compute an assumed constant angular velocity matrix by averaging. We then use that constant value to reconstruct the rotation matrix:

$$R^{(1)}(t+\Delta t) = R^{(1)}(t)\begin{pmatrix} I + \dfrac{\overleftrightarrow{\omega_0(t+\Delta t)}}{\|\boldsymbol{\omega}(t+\Delta t)\|}\sin\left(t\|\boldsymbol{\omega}_0(t+\Delta t)\|\right) \\ +\left(\dfrac{\overleftrightarrow{\omega_0(t+\Delta t)}}{\|\boldsymbol{\omega}(t+\Delta t)\|}\right)^2 \left(1-\cos\left(t\|\boldsymbol{\omega}_0(t+\Delta t)\|\right)\right) \end{pmatrix} \tag{91}$$

The target of this research has been to demonstrate the power of the moving frame method. Furthermore, we showed how the MFM simplifies the calculations of an advanced problem to a level that is accessible to undergraduate students (who conducted this work). It is also another pass at earlier work using MFM to model gyroscopic stabilization [13].

SIMPLIFICATION OF MODEL

Some simplifications are still necessary.

We do not allow the boat to translate on the surface of the sea. We do allow changes its pitch, yaw and roll. As a simplification of the wave moment, we apply a sinusoidal moment around the axis from bow to stern, to represent waves.

We modeled the geometry of the boat as a cuboid, with a diagonal moment of inertia matrix. We neglect buoyancy and gravity.

PARAMETERS USED

We used the following parameters:

Mass boat:	10000 kg
Mass gyro 1:	500 kg
Mass gyro 2:	500 kg
Length boat:	50 m
Width boat:	5 m
Length from cm to gyro 1, $\mathbf{S}_{\mathbf{c}_1}$:	1 m
Radius gyro 1 ring:	0.5 m
Radius gyro 1 disk:	0.5 m
Length from cm to gyro 2, $\mathbf{S}_{\mathbf{c}_2}$:	1 m
Radius gyro 2 ring:	0.5 m
Radius gyro 2 disk:	0.5 m
Gyro 1 spin rate:	20000 RPM
Gyro 2 spin rate:	-20000 RPM
Numerical integration:	Runge Kutta 4th order

3D VISUALIZATION AND WEBGL

To both give a better visualization of the calculations and to demonstrate how suitable MFM is for programming, we created a 3D simulation with Web Graphics Library (WebGL). WebGL is a JavaScript interface for rendering interactive 2D and 3D computer graphics. WebGL is compatible with most of the major web browsers such as Chrome, Firefox, Safari, and Opera. In addition, it is free and can be used without the need for plugins. It does so by introducing an API which closely conforms to OpenGL ES 2.0, thus being compatible with HTML5.

The reader my visit the website here on cell phones.

http://home.hib.no/prosjekter/dynamics/2019/stability/

Figure 5 presents a view from above the bow, but looking down and toward the stern, with no wave moment and no gyro action.

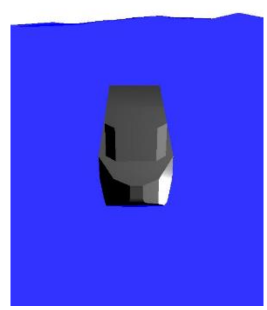

FIGURE 5: Above the bow

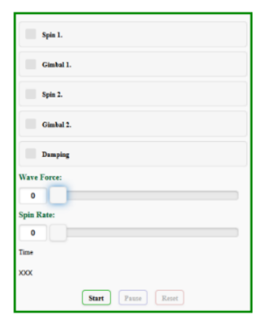

FIGURE 6: WebGL interface

To observe motion of the 3D-models, tick off the checkboxes for Spin 1 & 2, Gyro 1 & 2 and damping on the interface, shown in Figure 6.

Spin 1 and 2 activates the spin of the inertial disks of gyro 1 and 2. Gimbal 1 and 2 activates the gimbal precession torque of gyro 1 and 2.

Adjust the wave force and spin rate anywhere from 0-1, where 1 equal 20000 RPM and then press start to initiate the test. To adjust any of these values, simply press reset and then set them to desired value and then run the code again by pressing start.

Figure 7 presents the behavior of the boat with wave moments turned on, but with no gyroscopic action. The boat rolls.

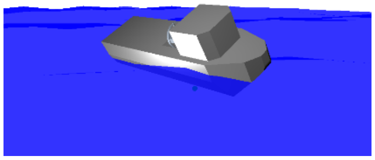

FIGURE 7. No gyro action

FIGURE 8: One gyro acting

Figure 8 presents the results with only one gyro action. The view is directly overhead. It shows a non-existent roll, but a visible yaw. Finally, Figure 9 shows both gyros acting with wave motion. There is both nonexistent roll and yawing.

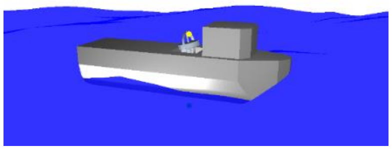

FIGURE 9. Both gyros active.

DISCUSSION AND RESULTS

We have obtained the equations of motion for the boat, under gyroscopic action and wave moments, using the MFM. We have solved them using numerical integration and has thus shown the potential of the MFM to calculate gyroscopic wave stabilization. We have also demonstrated how gyroscopic stabilization can greatly reduce the roll movement created by waves.

Different from last year's project [13], we decided to use a different numerical method due to its proven inaccurate calculations. This inaccuracy caused the ship to move uncontrollably and sink after a few seconds. Based on what we discovered form last year, we decided to go for RK4. This choice has proven to be a much more stable way of executing the calculations used to visualize the spin and moments shown on our WebGL page.

CONCLUSION AND FUTRE WORK

The Moving Frame Method has been proven highly effective to model the movement and forces created by a gyroscopic stabilizing system. We have also discovered how well suited this method is for programming. The future possibilities for MFM in control languages is significant. A stabilizing program that doesn't just react against forces acting on the vessel but one that anticipates the forces before they happen. If one were to add a crane on the boat, one could then simulate the motion from the crane and what effect it will have on the vessel and compensate for this in real time with the inertial disks.

To make the analysis more realistic buoyancy forces can be added. Buoyancy will act as damping force and through the mechanics of hydrostatics and ship stability, it will stabilize the boat at a growing rate as the boat pitches and rolls. This though, requires gravity, but gravity will also add to the realism of the simulation. We will also confirm these calculations with experimental analyses in the HVL wave tank.

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FLOW CHART

At the start of the semster we were tasked with creating a GANTT-form. This is a flow chart that shows wich tasks that are going to be worked at what time, and how much time we assigne the different tasks.
This has helped us tremendously, as it have given us an very helpful overview of the tasks at hand. It have been
helpful to see what work is to be done, and what we are done with.

Since the beginning we have tried to follow the chart as acurate, as we could. We soon realized that there were tasks we had given to little time, or tasks that went more effectively then anticipated.

	uke 3	uke 4	uke 5	uke 6	uke 7	uke 8	uke 9	uke 10	uke 11	uke12	uke 13	uke 14	uke 15	uke 16	uke 17	uke 18	uke 19	uke 20	uke 21	uke 22	uke 23	uke 24
Registrer IMECE	T,G,K									ferie				Påske								
Last opp "abstract"							25.feb															
Oppgave leveres til IMECE																29.apr						
Underveis presentasjon																						
Frist innlevering																			24.mai			
Hovedpresentasjon																					3 - 7 juni	
Expo																						13.jun
Omskrivning av paper til BA																	T,G,K	T,G,K	T,G,K			
Utskrivning av B-matrix by hand		T,K	T,K	T,K																		
Ny abstract			T	T																		
intro MFM			K	K																		
java code			G	G	G	G	G	G														
forbrede presentasjon								T,G,K														
B-matrix paper					T,K	T,K	T,K	T,K	T,K													
skrive artikkel											T,G,K	T,G,K	T,G,K	T,G,K	T,G,K	T,G,K						
Forbredelse expo																				T,G,K	T,G,K	T,G,K
Forbredelse hoved presentasjon																			T,G,K	T,G,K	T,G,K