Analysis of a Dual Gyroscope Wave Energy Converter Using the Moving Frame Method in Dynamics

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DUAL GYROSCOPE WAVE ENERGY CONVERTER

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ABSTRACT

This research models the energy extracted by gyroscopic wave energy converters, to assess their use to provide supplementary power to fish farms and lighting on oilrigs. This project implements the Moving Frame Method (MFM) in dynamics to model the power generated from a gyroscopic wave energy converter. The MFM leverages Lie Group Theory, Cartan's moving frames and a new notation from the discipline of geometrical physics. This research extends previous work by incorporating two inertial disks to counter the inducement of yaw, and it improves the numerical integration scheme. Furthermore, it makes use of a coherent data structure founded in the Special Euclidean Group, and it defines the initial disk spin as a prescribed variable. It accounts for the prescribed variables by modifying the equations of motion. Finally, it conducts an analysis of the generated energy which accounts for generator moments. After obtaining the suite of descriptive equations of motion, this project integrates them using the Runge-Kutta method. Finally, a simplified 3D simulation is made using the Web Graphics Library to improve the readers' intuitive understanding of the device.

NOMENCLATURE

a:	Wave amplitude
$[B]$:	Transformation matrix to generalized coordinates
$[C]$:	Transformation matrix for prescribed rates
$d^{(\beta)}$:	Width of the gimbals
$[D]$:	Combined angular velocity matrix
$\mathbf{e}$:	Unit basis vector
E:	Frame connection matrix
$\{F\}$:	Force and moment list
$\{F^*\}$:	Generalized force and moment list
g:	Gravitational acceleration
$\mathbf{H}$:	Angular momentum
I_3:	3×3 Identity matrix
J:	3×3 Mass moment of inertia matrix
K:	Kinetic energy
$\mathbf{L}$:	Linear momentum
L:	Lagrangian
m:	Mass
$[M]$:	Mass matrix
$[M^*]$:	Reduced mass matrix
M_g	Generator Counter Torque
$[N^*]$:	Reduced non-linear velocity matrix
P_{gyro}:	Generated power by the gyroscopes
$q(t)$:	Generalized coordinates
$\{\dot{q}(t)\}$:	Generalized velocity list
$\{\ddot{q}(t)\}$:	Generalized acceleration list
$\{\dot{r}(t)\}$:	Generalized prescribed velocity list
R:	3×3 Rotation matrix
T:	Wave period
$[T^*]$:	Reduced velocity matrix for prescribed rates
W:	Negative work function
δW:	Virtual work
$\{\delta\dot{X}\}$:	Virtual Cartesian velocities
$\{\delta\widetilde{X}\}$:	Virtual Cartesian displacements
$\rho^{(\gamma)}$:	Radius of the disks
Ω:	Time rate of the frame connection matrix
ω:	Angular velocity components
$\overleftrightarrow{\omega}$:	Skew-symmetric angular velocity matrix

INTRODUCTION

Norway's 2030 Agenda presents 17 Sustainable Development Goals (SDGs) as a transformative global roadmap for national and international efforts aimed at eradicating extreme poverty while protecting planetary boundaries and promoting prosperity, peace and justice. This project addresses three of these 17 goals: affordable and clean energy, industry innovation and infrastructure, and sustainable cities and communities.

The world's energy consumption is concentrated along the coastlines, where the population density is highest [1]. Norway's coast line and connection with the North Sea gives ample opportunity for exploiting wave energy. The Norwegian Government has expressed a desire to move Norway in a more environmentally friendly direction over the next 30-50 years [2]. Norway already has a long history of utilizing hydro power for most of its power consumption. This project seeks to expand on the opportunities for environmentally friendly power generation.

One way to extract power from waves is to make use of the gyroscopic effect. The gyroscopic wave energy converter captures this nutation through a generator. Several such connected buoys can create a "farm", with the intention of increasing power output. Possible uses are; fish farms, offshore weather stations, subsea equipment, and supplemental power for oilrigs.

Aaron Goldin in 2004 [3] and Kanki in 2006 [4], demonstrated the validity of the gyroscopic ocean wave energy converter. Aaron, demonstrated a small-scale prototype. Kanki built a full-scale model that captured 20 kW in 2006 and 50 kW in 2012.

This work builds upon previous work [5]. However, here we show a method to explicitly account for the prescribed motion, and conduct a full 3D analysis, while displaying all results in 3D on cell phones.

THE MOVING FRAME METHOD

Élie Cartan (1869-1951) [6] assigned a reference frame to each point of an object under study (a curve, a surface, Euclidean space itself). Then, using an orthonormal expansion, he expressed the rate of change of the frame in terms of the frame. The MFM leverages this by placing a reference frame on every moving link. However, then we need a *method to connect moving frames*. For this, we turn to Sophus Lie.

Marius Sophus Lie (1842-1899) developed the theory of continuous groups and their associated algebras. The MFM adopts the mathematics of rotation groups and their algebras, yet distils them to simple matrix multiplications. However, then we need a simplifying notation. For this, we turn to Frankel.

Theodore Frankel (1929-2017) [7] developed a compact notation in geometrical physics. The MFM adopts this notation to enable a methodology that is identical for both 2D and 3D analyses. The notation is also identical for single bodies and multi-body linked systems. In turn, this uplifts students' understanding from the conceptual to the pragmatic, enabling them to analyze machines of the 3D world.

The reader may find an introduction to the undergraduate and graduate Moving Frame Method, along with a pedagogical assessment in Impelluso [8]. In the following section, we summarize the MFM.

GENERAL PRINCIPLES OF THE MFM

At the center of mass of each body α we place a time-dependent moving frame:

$$\mathbf{e}^{(\alpha)}(t) = \left(\mathbf{e}_1^{(\alpha)}(t)\;\; \mathbf{e}_2^{(\alpha)}(t)\;\; \mathbf{e}_3^{(\alpha)}(t)\right) \tag{1}$$

In the previous, $\mathbf{e}$ is a unit vector and the subscript denotes the direction. Set $t = 0$ to define and deposit an inertial frame from a moving frame:

$$\mathbf{e}^I = \left(\mathbf{e}_1^I\;\; \mathbf{e}_2^I\;\; \mathbf{e}_3^I\right) = \left(\mathbf{e}_1^{(\alpha)}(0)\;\; \mathbf{e}_2^{(\alpha)}(0)\;\; \mathbf{e}_3^{(\alpha)}(0)\right) \tag{2}$$

Define the absolute position vector $\mathbf{r}_C^{(\alpha)}(t)$ of a frame as a translation from the inertial frame $\mathbf{e}^I$:

$$\mathbf{r}_C^{(\alpha)}(t) = \mathbf{e}^I x_C^{(\alpha)}(t) \tag{3}$$

We use $x_C^{(\alpha)}(t)$ to represent the coordinates of the distance from the inertial frame to the center of mass of a body (subscript C), expressed in the inertial frame. Assert the relative position vector of a frame $(\alpha + 1)$ from another frame (α) by $\mathbf{s}_C{}^{(\alpha+1/\alpha)}(t)$. Express this relative translation in the α-frame:

$$\mathbf{s}_C{}^{(\alpha+1/\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) s_C{}^{(\alpha+1/\alpha)}(t) \tag{4}$$

By adding the absolute position vector of the α-frame $\mathbf{r}_C^{(\alpha)}(t)$ and the relative position vector $\mathbf{s}_C{}^{(\alpha+1/\alpha)}(t)$, we obtain the absolute position vector of the $(\alpha + 1)$ frame:

$$\mathbf{r}_C^{(\alpha+1)}(t) = \mathbf{r}_C^{(\alpha)}(t) + \mathbf{e}^{(\alpha)}(t) s_C^{(\alpha+1/\alpha)}(t) \tag{5}$$

Let us now turn our attention to frame orientations. We use a rotation matrix, a member of the Special Orthogonal Group $R \in SO(3)$, to relate the orientation of a moving frame to an inertial frame:

$$\mathbf{e}^{(\alpha)}(t) = \mathbf{e}^I R^{(\alpha)}(t) \tag{6}$$

The relative rotation of a frame $(\alpha + 1)$ from another frame (α) can be written as:

$$\mathbf{e}^{(\alpha+1)}(t) = \mathbf{e}^{(\alpha)}(t) R^{(\alpha+1/\alpha)}(t) \tag{7}$$

The orientation of body $(\alpha + 1)$ can be expressed in the inertial frame by inserting equation (6) into (7) and exploiting the closure property of the SO(3) Group:

$$\mathbf{e}^{(\alpha+1)}(t) = \mathbf{e}^{I} R^{(\alpha)}(t) R^{(\alpha+1/\alpha)}(t) = \mathbf{e}^{I} R^{(\alpha+1)}(t) \tag{8}$$

The inverse of a rotation matrix is the transpose (a property of SO(3)):

$$\left(R^{(\alpha)}(t)\right)^{-1} = \left(R^{(\alpha)}(t)\right)^{T} \tag{9}$$

The time rate of frame rotation is:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{I} \dot{R}^{(\alpha)}(t) \tag{10}$$

We use (9) in (6) to formulate the inertial frame in terms of the moving frame and then substitute the result into (10) to obtain:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) \left(R^{(\alpha)}(t)\right)^{T} \dot{R}^{(\alpha)}(t) \tag{11}$$

The time rate of frame rotation is now expressed in its own frame.

We define the skew-symmetric angular velocity matrix. We note that this element is a member of the associated algebra, so(3):

$$\overleftrightarrow{\omega^{(\alpha)}(t)} = \left(R^{(\alpha)}(t)\right)^{T} \dot{R}^{(\alpha)}(t) = \begin{bmatrix} 0 & -\omega_3^{(\alpha)}(t) & \omega_2^{(\alpha)}(t) \\ \omega_3^{(\alpha)}(t) & 0 & -\omega_1^{(\alpha)}(t) \\ -\omega_2^{(\alpha)}(t) & \omega_1^{(\alpha)}(t) & 0 \end{bmatrix} \tag{12}$$

We can now rewrite equation (11) as:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) \overleftrightarrow{\omega^{(\alpha)}(t)} \tag{13}$$

The skew-symmetric angular velocity matrix is isomorphic to the same frame to the angular velocity vector of that frame:

$$\boldsymbol{\omega}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) \begin{pmatrix} \omega_1^{(\alpha)}(t) \\ \omega_2^{(\alpha)}(t) \\ \omega_3^{(\alpha)}(t) \end{pmatrix} \tag{14}$$

Kinematics of SE(3)

Continuing, we desire to combine the rotational and translational data of a frame (α), in one structure. We define the 4×4 absolute *frame connection matrix* (a member of the Special Euclidean Group). We note that $E \in$ SE(3).

$$E^{(\alpha)}(t) = \begin{bmatrix} R^{(\alpha)}(t) & x_C^{(\alpha)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{15}$$

We define an *inertial frame connection*. This consists of the frame and its position, represented as:

$$\left(\mathbf{e}^{I}\ \ \mathbf{0}\right) = \left(\mathbf{e}_1^{I}\ \ \mathbf{e}_2^{I}\ \ \mathbf{e}_3^{I}\ \ \mathbf{0}\right) \tag{16}$$

Similarly, we represent the *moving frame connection* as:

$$\left(\mathbf{e}^{(\alpha)}(t)\ \ \mathbf{r}_C^{(\alpha)}(t)\right) = \left(\mathbf{e}_1^{(\alpha)}(t)\ \ \mathbf{e}_2^{(\alpha)}(t)\ \ \mathbf{e}_3^{(\alpha)}(t)\ \ \mathbf{r}_C^{(\alpha)}(t)\right) \tag{17}$$

We relate the inertial frame connection (16) and the moving frame connection (17) by utilizing the absolute frame connection matrix (15):

$$\left(\mathbf{e}^{(\alpha)}(t)\ \ \mathbf{r}_C^{(\alpha)}(t)\right) = \left(\mathbf{e}^{I}\ \ \mathbf{0}\right) E^{(\alpha)}(t) \tag{18}$$

The *relative* frame connection matrix is defined as:

$$E^{(\alpha+1/\alpha)}(t) = \begin{bmatrix} R^{(\alpha+1/\alpha)}(t) & s_C^{(\alpha+1/\alpha)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{19}$$

We use equation (19) to express the relationship between two moving frames, $(\alpha + 1)$ and (α):

$$\left(\mathbf{e}^{(\alpha+1)}(t)\ \ \mathbf{r}_C^{(\alpha+1)}(t)\right) = \left(\mathbf{e}^{(\alpha)}(t)\ \ \mathbf{r}_C^{(\alpha)}(t)\right) E^{(\alpha+1/\alpha)}(t) \tag{20}$$

Equation (20), with its defining element (19), recapitulates equations (5) and (7).

The absolute frame connection matrix of body $(\alpha + 1)$ can be found as the product of the absolute frame connection matrix of body (α) and the relative frame connection matrix that relates them (as a result of the closure property of the SE(3) group):

$$E^{(\alpha+1)}(t) = E^{(\alpha)}(t) E^{(\alpha+1/\alpha)}(t) \tag{21}$$

KINEMATICS OF THE WAVE ENERGY CONVERTER

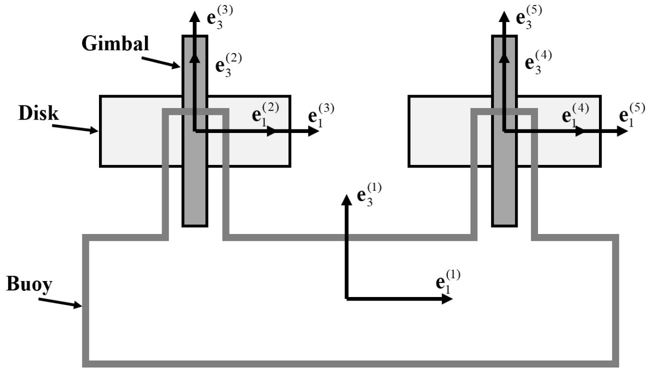

Figure 1 – Schematic of the GWEC

First Frame – The Buoy

We place the first moving frame $\mathbf{e}^{(1)}(t)$ at the center of mass of the buoy. At $t = 0$ we deposit an inertial frame from the first frame:

$$\mathbf{e}^{I} = \mathbf{e}^{(1)}(0) \tag{22}$$

The frame connection matrix for the first frame contains the data for rotation and translation from the inertial frame:

$$E^{(1)}(t)=\begin{bmatrix} R^{(1)}(t) & x_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{23}$$

We express the relationship between the buoy frame and the inertial frame as:

$$\left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right)=\left(\mathbf{e}^I\ \ \mathbf{0}\right)E^{(1)}(t) \tag{24}$$

Next, we take the time rate of the frame connection:

$$\left(\dot{\mathbf{e}}^{(1)}(t)\ \ \dot{\mathbf{r}}_C^{(1)}(t)\right)=\left(\mathbf{e}^I\ \ \mathbf{0}\right)\dot{E}^{(1)}(t) \tag{25}$$

The time rate of the frame connection matrix $\dot{E}^{(1)}(t)$ is found by taking the time derivative of each data block:

$$\dot{E}^{(1)}(t)=\begin{bmatrix} \dot{R}^{(1)}(t) & \dot{x}_c^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{26}$$

We will also need the inverse of the frame connection matrix, which is expressed as (due to $E \in SE(3)$):

$$\left(E^{(1)}(t)\right)^{-1}=\begin{bmatrix} \left(R^{(1)}(t)\right)^T & -\left(R^{(1)}(t)\right)^T x_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{27}$$

We use (27) in (24) to formulate the inertial frame connection in terms of the moving frame connection and then substitute the result into (25) to obtain:

$$\left(\dot{\mathbf{e}}^{(1)}(t)\ \ \dot{\mathbf{r}}_C^{(1)}(t)\right)=\left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right)\left(E^{(1)}(t)\right)^{-1}\dot{E}^{(1)}(t) \tag{28}$$

We define the absolute time rate of frame connection matrix for the first body $\Omega^{(1)}$ as the product of $\left(E^{(1)}(t)\right)^{-1}$ and $\dot{E}^{(1)}(t)$. We note that $\Omega \in se(3)$ (the algebra associated with the SE(3) group):

$$\Omega^{(1)}=\left(E^{(1)}(t)\right)^{-1}\dot{E}^{(1)}(t) \tag{29}$$

As a result, we can rewrite equation (28) as:

$$\left(\dot{\mathbf{e}}^{(1)}(t)\ \ \dot{\mathbf{r}}_C^{(1)}(t)\right)=\left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right)\Omega^{(1)}(t) \tag{30}$$

$\Omega^{(1)}$ multiplied out in matrix from:

$$\Omega^{(1)}=\begin{bmatrix} \left(R^{(1)}(t)\right)^T\dot{R}^{(1)}(t) & \left(R^{(1)}(t)\right)^T\dot{x}_C^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{31}$$

We recognize the expression in the upper left corner to be the same as (12), so we rewrite:

$$\Omega^{(1)}=\begin{bmatrix} \overleftrightarrow{\omega^{(1)}(t)} & \left(R^{(1)}(t)\right)^T\dot{x}_C^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{32}$$

From equation (30), we extract:

$$\dot{\mathbf{e}}^{(1)}(t)=\mathbf{e}^{(1)}(t)\overleftrightarrow{\omega^{(1)}(t)} \tag{33}$$

The angular velocity vector of the first frame is then, as shown earlier with (14):

$$\boldsymbol{\omega}^{(1)}(t)=\mathbf{e}^{(1)}(t)\begin{pmatrix} \omega_1^{(1)}(t) \\ \omega_2^{(1)}(t) \\ \omega_3^{(1)}(t) \end{pmatrix} \tag{34}$$

The second equation extracted from equation (30) is:

$$\dot{\mathbf{r}}_C^{(1)}(t)=\mathbf{e}^{(1)}(t)\left(R^{(1)}(t)\right)^T\dot{x}_C^{(1)}(t) \tag{35}$$

Thus, assert the translational velocity as:

$$\dot{\mathbf{r}}_C^{(1)}(t)=\mathbf{e}^I\dot{x}_C^{(1)}(t) \tag{36}$$

Second and Fourth Frame - The Gimbals

The buoy, rocked by waves, provides the precession. The disk (in the next sub-section) provides the spin. We now focus on the nutation of the gimbals that the generator captures. First, to reduce this paper's length, the following equations represent both gimbals. Here (β) represents frame 2 (first gimbal frame; e.g., left in figure 1) and 4 (second gimbal frame; right).
At the center of mass of the gimbals, we place a moving frame $\mathbf{e}^{(\beta)}(t)$. To get from the first frame (the buoy frame) to the (β)-frame, move a distance l in the $\mathbf{e}_1^{(1)}$- direction, then a distance h in the $\mathbf{e}_3^{(1)}$- direction. The relative position vector $\mathbf{s}_C^{(\beta/1)}(t)$ of the (β)-frame is expressed in the first frame as:

$$\mathbf{s}_C^{(\beta/1)}(t)=\mathbf{e}^{(1)}(t)s_C^{(\beta/1)}(t)=\mathbf{e}^{(1)}(t)\begin{pmatrix} l^{(\beta/1)} \\ 0 \\ h^{(\beta/1)} \end{pmatrix} \tag{37}$$

The orientation of this second frame is obtained from the first buoy frame by a nutation $\theta^{(\beta)}(t)$ about the common 2-axis (it is this nutation we desire to capture):

$$\mathbf{e}^{(\beta)}(t)=\mathbf{e}^{(1)}(t)R^{(\beta/1)}(t)=\mathbf{e}^{(1)}(t)\begin{bmatrix} \cos\theta^{(\beta)}(t) & 0 & \sin\theta^{(\beta)}(t) \\ 0 & 1 & 0 \\ -\sin\theta^{(\beta)}(t) & 0 & \cos\theta^{(\beta)}(t) \end{bmatrix} \tag{38}$$

We now have the information we need to construct the relative frame connection matrix:

$$E^{(\beta/1)}(t)=\begin{bmatrix} R^{(\beta/1)}(t) & s_C^{(\beta/1)} \\ 0_3^T & 1 \end{bmatrix} \tag{39}$$

$E^{(\beta/1)}(t)$ relates the first frame and gimbal frames as shown:

$$\left(\mathbf{e}^{(\beta)}(t)\ \ \mathbf{r}_C^{(\beta)}(t)\right)=\left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right)E^{(\beta/1)}(t) \tag{40}$$

To relate the gimbal frames to the inertial frame, we need the absolute frame relation matrix $E^{(\beta)}(t)$, which is found as shown in (21), but expressed fully as:

$$E^{(\beta)}(t) = E^{(1)}(t)E^{(\beta/1)}(t) = \begin{bmatrix} R^{(1)}(t) & x_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} R^{(\beta/1)}(t) & s_C^{(\beta/1)} \\ 0_3^T & 1 \end{bmatrix} \tag{41}$$

Multiplied out:

$$E^{(\beta)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(\beta/1)}(t) & R^{(1)}(t)s_C^{(\beta/1)} + x_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{42}$$

The inverse:

$$\left(E^{(\beta)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(\beta/1)}(t)\right)^T\left(R^{(1)}(t)\right)^T & -\left(R^{(\beta/1)}(t)\right)^T\left(R^{(1)}(t)\right)^T\left(R^{(1)}(t)s_C^{(\beta/1)} + x_C^{(1)}(t)\right) \\ 0_3^T & 1 \end{bmatrix} \tag{43}$$

The rate of change:

$$\dot{E}^{(\beta)}(t) = \begin{bmatrix} \dot{R}^{(1)}(t)R^{(\beta/1)}(t) + R^{(1)}(t)\dot{R}^{(\beta/1)}(t) & \dot{R}^{(1)}(t)s_C^{(\beta/1)} + \dot{x}_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{44}$$

We can now find the absolute time rate of frame connection matrix $\Omega^{(\beta)}(t)$ by multiplying (43) and (44). We then use the result to relate the time rate of change to the moving frame:

$$\left(\dot{\mathbf{e}}^{(\beta)}(t)\ \ \dot{\mathbf{r}}_C^{(\beta)}(t)\right) = \left(\mathbf{e}^{(\beta)}(t)\ \ \mathbf{r}_C^{(\beta)}(t)\right)\Omega^{(\beta)}(t) \tag{45}$$

From (45) we extract the angular velocity vectors, and linear velocity vectors of the gimbals in the inertial frame, respectively:

$$\boldsymbol{\omega}^{(\beta)}(t) = \mathbf{e}^{(\beta)}(t)\left(\left(R^{(\beta/1)}(t)\right)^T\omega^{(1)}(t) + \omega^{(\beta/1)}(t)\right) \tag{46}$$

$$\dot{\mathbf{r}}_C^{(\beta)}(t) = \mathbf{e}^I\left(-R^{(1)}(t)\overleftrightarrow{s_C^{(\beta/1)}}\omega^{(1)}(t) + \dot{x}_C^{(1)}(t)\right) \tag{47}$$

Third and Fifth Frame – The Disks

In this section, (γ) represents frames 3 and 5: the body frames for the disks. The (γ)-frame is located at the center of mass of the spinning disk, which is placed at the same location as the gimbal frame. Hence, there is no translation, and $\mathbf{s}_C^{(\gamma/\beta)}(t)$ is zero. The (γ)-frame will rotate from the (β)-frame around the third axis, by the angle $\phi^{(\gamma)}(t)$:

$$\mathbf{e}^{(\gamma)}(t) = \mathbf{e}^{(\beta)}(t)R^{(\gamma/\beta)}(t) = \mathbf{e}^{(\beta)}(t)\begin{bmatrix} \cos\phi^{(\gamma)}(t) & -\sin\phi^{(\gamma)}(t) & 0 \\ \sin\phi^{(\gamma)}(t) & \cos\phi^{(\gamma)}(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{48}$$

We construct the relative frame connection matrix:

$$E^{(\gamma/\beta)}(t) = \begin{bmatrix} R^{(\gamma/\beta)}(t) & 0 \\ 0_3^T & 1 \end{bmatrix} \tag{49}$$

$E^{(\gamma/\beta)}(t)$ relates the disk and gimbal frames as shown:

$$\left(\mathbf{e}^{(\gamma)}(t)\ \ \mathbf{r}_C^{(\gamma)}(t)\right) = \left(\mathbf{e}^{(\beta)}(t)\ \ \mathbf{r}_C^{(\beta)}(t)\right)E^{(\gamma/\beta)}(t) \tag{50}$$

At this point we need to establish the absolute frame connection matrix $E^{(\gamma)}(t)$. This is done using the procedure found in (21).

$$E^{(\gamma)}(t) = E^{(\beta)}(t)E^{(\gamma/\beta)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(\beta/1)}(t) & R^{(1)}(t)s_C^{(\beta/1)} + x_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} R^{(\gamma/\beta)}(t) & 0 \\ 0_3^T & 1 \end{bmatrix} \tag{51}$$

Multiplying these matrices yields the following result:

$$E^{(\gamma)}(t) = \begin{bmatrix} R^{(1)}(t)R^{(\beta/1)}(t)R^{(\gamma/\beta)}(t) & R^{(1)}(t)s_C^{(\beta/1)} + x_C^{(1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{52}$$

Now we take the inverse of $E^{(\gamma)}(t)$:

$$\left(E^{(\gamma)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(\gamma/\beta)}(t)\right)^T\left(R^{(\beta/1)}(t)\right)^T\left(R^{(1)}(t)\right)^T & -\left(R^{(\gamma/\beta)}(t)\right)^T\left(R^{(\beta/1)}(t)\right)^T\left(R^{(1)}(t)\right)^T\left(R^{(1)}(t)s_C^{(\beta/1)} + x_C^{(1)}(t)\right) \\ 0_3^T & 1 \end{bmatrix} \tag{53}$$

Next, we find the rate of change of $E^{(\gamma)}(t)$:

$$\dot{E}^{(\gamma)}(t) = \begin{bmatrix} \dot{R}^{(\gamma)}(t) & \dot{R}^{(1)}(t)s_C^{(\beta/1)} + \dot{x}_C^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix}. \tag{54}$$

Where $\dot{R}^{(\gamma)}(t)$ is given as:

$$\begin{aligned} \dot{R}^{(\gamma)}(t) &= \dot{R}^{(1)}(t)R^{(\beta/1)}(t)R^{(\gamma/\beta)}(t) + \\ &R^{(1)}(t)\dot{R}^{(\beta/1)}(t)R^{(\gamma/\beta)}(t) + R^{(1)}(t)R^{(\beta/1)}(t)\dot{R}^{(\gamma/\beta)}(t) \end{aligned} \tag{55}$$

We find the absolute time rate of frame connection for the third and fifth body $\Omega^{(\gamma)}(t)$ by mulitplying (53) and (54):

$$\left(\dot{\mathbf{e}}^{(\gamma)}(t)\ \ \dot{\mathbf{r}}_C^{(\gamma)}(t)\right) = \left(\mathbf{e}^{(\gamma)}(t)\ \ \mathbf{r}_C^{(\gamma)}(t)\right)\Omega^{(\gamma)}(t) \tag{56}$$

From (56) we extract the angular and linear velocity vectors of the disks, respectively:

$$\boldsymbol{\omega}^{(\gamma)}(t) = \mathbf{e}^{(\gamma)}(t)\begin{pmatrix} \left(R^{(\gamma/\beta)}(t)\right)^T\left(R^{(\beta/1)}(t)\right)^T\omega^{(1)}(t) + \\ \left(R^{(\gamma/\beta)}(t)\right)^T\omega^{(\beta/1)}(t) + \omega^{(\gamma/\beta)}(t) \end{pmatrix} \tag{57}$$

$$\dot{\mathbf{r}}_C^{(\gamma)}(t) = \mathbf{e}^I\left(-R^{(1)}(t)\overleftrightarrow{s_C^{(\beta/1)}}\omega^{(1)}(t) + \dot{x}_C^{(1)}(t)\right) \tag{58}$$

Generalized Coordinates

To simplify the dynamics for the GWEC, we relate the already established Cartesian coordinates to certain generalized coordinates needed for the kinetics. Before continuing, we make some additional assumptions. First, we assume the buoy is stationary; i.e., the center of mass of the buoy does not translate in any direction. Second, we assume the disks are already spinning at a constant rate when analysis begins. Third, we assume there is no resistance on the disks meaning there is no motor needed to maintain the constant velocity of the disks.

$\{\dot{X}(t)\}$ denotes the Cartesian velocities, $\{\dot{q}(t)\}$ denotes the generalized velocities, while $\{\dot{r}(t)\}$ denotes the generalized velocities that are prescribed.

$$\left\{\dot{X}(t)\right\} \equiv \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \dot{x}_C^{(3)}(t) \\ \omega^{(3)}(t) \\ \dot{x}_C^{(4)}(t) \\ \omega^{(4)}(t) \\ \dot{x}_C^{(5)}(t) \\ \omega^{(5)}(t) \end{Bmatrix}, \quad \{\dot{q}(t)\} \equiv \begin{Bmatrix} \omega^{(1)}(t) \\ \dot{\theta}^{(2)}(t) \\ \dot{\theta}^{(4)}(t) \end{Bmatrix}, \quad \{\dot{r}(t)\} \equiv \begin{Bmatrix} \dot{\phi}^{(3)} \\ \dot{\phi}^{(5)} \end{Bmatrix} \tag{59}$$

The relationship between these data structures is obtained through the *B*-matrix and *C*-matrix as follows:

$$\left\{\dot{X}(t)\right\} = \left[B(t)\right]\{\dot{q}(t)\} + \left[C(t)\right]\{\dot{r}(t)\} \tag{60}$$

The number of rows of the B-matrix is equal to the number of Cartesian velocities, while the number of columns is equal to the number of generalized velocities. In this case, the block size of the B-matrix will be 9×3 (time dependencies are omitted due to space restrictions):

$$[B] \equiv \begin{bmatrix} I_3 & \underset{3\times1}{0} & \underset{3\times1}{0} \\ -R^{(1)}\overleftrightarrow{s_C^{(2/1)}} & \underset{3\times1}{0} & \underset{3\times1}{0} \\ \left(R^{(2/1)}\right)^T & e_2 & \underset{3\times1}{0} \\ -R^{(1)}\overleftrightarrow{s_C^{(2/1)}} & \underset{3\times1}{0} & \underset{3\times1}{0} \\ \left(R^{(3/2)}\right)^T\left(R^{(2/1)}\right)^T & \left(R^{(3/2)}\right)^T e_2 & \underset{3\times1}{0} \\ -R^{(1)}\overleftrightarrow{s_C^{(4/1)}} & \underset{3\times1}{0} & \underset{3\times1}{0} \\ \left(R^{(4/1)}\right)^T & \underset{3\times1}{0} & e_2 \\ -R^{(1)}\overleftrightarrow{s_C^{(4/1)}} & \underset{3\times1}{0} & \underset{3\times1}{0} \\ \left(R^{(5/4)}\right)^T\left(R^{(4/1)}\right)^T & \underset{3\times1}{0} & \left(R^{(5/4)}\right)^T e_2 \end{bmatrix} \tag{61}$$

Where I_3, $\underset{3\times1}{0}$, e_2 and e_3 (found in the C-matrix) is given as:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underset{3\times1}{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{62}$$

The C-matrix is constructed in a similar manner. Number of rows corresponds to number of prescribed Cartesian velocities, while the number of columns is equal to the number of generalized velocities. In this case a 9×2 matrix:

$$\left[C\right] \equiv \begin{bmatrix} \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ e_3 & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & \underset{3\times1}{0} \\ \underset{3\times1}{0} & e_3 \end{bmatrix} \tag{63}$$

KINETICS OF THE WAVE ENERGY CONVERTER

Application of Analytical Mechanics

We begin by defining a Lagrangian as the difference between the kinetic and potential energy:

$$L^{(\alpha)}\left(q(t),\dot{q}(t),t\right) = K^{(\alpha)}\left(q(t),\dot{q}(t),t\right) - U^{(\alpha)}\left(q(t),t\right) \tag{64}$$

Define the Action as the definite integral of the Lagrangian function over time:

$$A = \int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt \tag{65}$$

Hamilton's principle states that "*the motion of a system occurs in such a way that the definite integral A becomes a minimum for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed"* [9]. To obtain the equations of motion, we would set the variation of the Action equal to zero:

$$\delta\int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt = 0 \tag{66}$$

However, we must first deal with the non-conservative forces. To include the non-conservative forces, we exploit the extension of Hamilton's Principle, known as the Principle of Virtual Work. Here, we formulate the Lagrangian as dependent only on the kinetic energy. We will account for all other forces (conservative or non-conservative) as work, on the right side. From this point onwards, we omit the dependencies of position and velocity for ease of notation.

$$\int_{t_0}^{t_1} \delta K^{(\alpha)}(t)dt = -\int_{t_0}^{t_1} \delta W^{(\alpha)}(t)dt \tag{67}$$

The kinetic energy of each body in the system is expressed by the angular momentum $\mathbf{H}_C^{(\alpha)}(t)$, and linear momentum $\mathbf{L}_C^{(\alpha)}(t)$:

$$\mathbf{H}_C^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t)\,H_C^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t)\,J_C^{(\alpha)}\omega^{(\alpha)}(t) \tag{68}$$

$$\mathbf{L}_C^{(\alpha)}(t) = \mathbf{e}^I L_C^{(\alpha)}(t) = \mathbf{e}^I m^{(\alpha)}\dot{x}_C^{(\alpha)}(t) \tag{69}$$

Here, $J_C^{(\alpha)}$ represents the moment of inertia matrix for body α. The total kinetic energy of a body α with the frame placed at the center of mass is defined as:

$$K^{(\alpha)}(t) = \frac{1}{2}\left\{\dot{\mathbf{r}}_C^{(\alpha)} \cdot \mathbf{L}_C^{(\alpha)} + \boldsymbol{\omega}^{(\alpha)} \cdot \mathbf{H}_C^{(\alpha)}\right\} \tag{70}$$

For the whole system, the total kinetic energy is expressed in matrix form as:

$$K(t) = \frac{1}{2}\left\{\dot{X}(t)\right\}^T \left[M\right]\left\{\dot{X}(t)\right\} \tag{71}$$

Where the generalized mass matrix $[M]$ contains the masses and moments of inertia for each body:

$$\left[M\right] \equiv \begin{bmatrix} J_C^{(1)} & \underset{3\times3}{0} & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & m^{(2)}I_3 & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & J_C^{(2)} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \vdots & \vdots & \vdots & \ddots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & m^{(5)}I_3 & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & J_C^{(5)} \end{bmatrix} \tag{72}$$

The variation of the kinetic energy is expressed as:

$$\delta K(t) = \left\{\delta\dot{X}(t)\right\}^T \left[M\right]\left\{\dot{X}(t)\right\} \tag{73}$$

At this point, we need to find an appropriate expression for the virtual Cartesian velocities $\{\delta \dot{X}(t)\}$:

$$\left\{\delta\dot{X}(t)\right\} \equiv \begin{Bmatrix} \delta\omega^{(1)}(t) \\ \delta\dot{x}_C^{(2)}(t) \\ \delta\omega^{(2)}(t) \\ \vdots \\ \delta\dot{x}_C^{(5)}(t) \\ \delta\omega^{(5)}(t) \end{Bmatrix} \tag{74}$$

For linear velocities, the variation of the time derivative is equal to the time derivative of the variation:

$$\delta\dot{x}_C^{(\alpha)}(t) = \frac{d}{dt}\delta x_C^{(\alpha)}(t) \tag{75}$$

For angular velocities, however, there is a restriction as proven by Murakami [7]:

$$\delta\omega^{(\alpha)}(t) = \frac{d}{dt}\left(R^{(\alpha)^T}\delta R^{(\alpha)}\right)_{\text{un}} + \overleftrightarrow{\omega^{(\alpha)}(t)}\left(R^{(\alpha)^T}\delta R^{(\alpha)}\right)_{\text{un}} \tag{76}$$

Here, the subscript "un" means that the matrix is un-skewed into a column vector. Time dependencies on the rotation matrices are omitted due to space restrictions. Moving on, we define the Virtual Cartesian displacements $\{\delta \tilde{X}(t)\}$:

$$\left\{\delta\tilde{X}(t)\right\} \equiv \begin{Bmatrix} \left(R^{(1)^T}\delta R^{(1)}\right)_{\text{un}} \\ \delta x_C^{(2)}(t) \\ \left(R^{(2)^T}\delta R^{(2)}\right)_{\text{un}} \\ \vdots \\ \delta x_C^{(5)}(t) \\ \left(R^{(5)^T}\delta R^{(5)}\right)_{\text{un}} \end{Bmatrix} \tag{77}$$

Making use of previous three equations, (74) may now be written in compact form as:

$$\left\{\delta\dot{X}(t)\right\} = \frac{d}{dt}\left\{\delta\tilde{X}(t)\right\} + [D]\left\{\delta\tilde{X}(t)\right\} \tag{78}$$

Where $[D]$ is a skew symmetric matrix that contains the angular velocity matrices for each frame:

$$\left[D\right] \equiv \begin{bmatrix} \overleftrightarrow{\omega^{(1)}} & \underset{3\times3}{0} & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \overleftrightarrow{\omega^{(2)}} & \cdots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \vdots & \vdots & \vdots & \ddots & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \overleftrightarrow{\omega^{(5)}} \end{bmatrix} \tag{79}$$

With the virtual kinetic energy established, we move on the virtual work. We begin by defining the forces and moments acting on the different bodies of the system as $\{F(t)\}$:

$$\left\{F(t)\right\} \equiv \begin{Bmatrix} M_C^{(1)}(t) \\ F_C^{(2)I}(t) \\ M_C^{(2)}(t) \\ F_C^{(3)I}(t) \\ M_C^{(3)}(t) \\ F_C^{(4)I}(t) \\ M_C^{(4)}(t) \\ F_C^{(5)I}(t) \\ M_C^{(5)}(t) \end{Bmatrix} = \begin{Bmatrix} M_w(t) + M_g^{(2)}(t)e_2 - M_g^{(4)}(t)e_2 \\ -m^{(2)}ge_3 \\ -M_g^{(2)}(t)e_2 \\ -m^{(3)}ge_3 \\ 0 \\ -m^{(4)}ge_3 \\ M_g^{(4)}(t)e_2 \\ -m^{(5)}ge_3 \\ 0 \end{Bmatrix} \tag{80}$$

Here, $-m^{(\alpha)}ge_3$ is the gravitational force at the center of mass. $M_w(t)$ is the moment induced by waves, and $M_g^{(\beta)}(t)e_2$ is the moment from the generators that are attached to the gimbals. We note that the disks spin in opposite directions, such that the moments acting on the gimbals from the generator will be cancelled out. This is a major advantage of having two gyroscopes instead of one.

We express the virtual work done by the forces and moments as:

$$\delta W = \left\{\delta \tilde{X}(t)\right\}^T \left\{F(t)\right\} \tag{81}$$

In so doing, we note that to obtain the variation of the work from the moments, the moments and the virtual rotation terms, are conjugate to each other.

The B-matrix that relates the Cartesian velocities $\{\dot{X}(t)\}$ to the generalized velocities $\{\dot{q}(t)\}$, also relates the virtual Cartesian displacements $\{\delta \tilde{X}(t)\}$ to the virtual generalized displacements $\{\delta q(t)\}$:

$$\left\{\delta \tilde{X}(t)\right\} = \left[B(t)\right]\left\{\delta q(t)\right\} \tag{82}$$

The transpose of the above is used to rewrite equation (81):

$$\delta W = \left\{\delta q(t)\right\}^T \left\{F^*(t)\right\} \tag{83}$$

Where the generalized forces $\left\{F^*(t)\right\}$ are defined as:

$$\left\{F^*(t)\right\} = \left[B(t)\right]^T \left\{F(t)\right\} \tag{84}$$

By inserting the expressions obtained for the variation of the kinetic energy and the virtual work into equation (67), we obtain the basis for the equation of motion:

$$\int_{t_0}^{t_1}\left(\left\{\delta \dot{X}(t)\right\}^T \left[M\right]\left\{\dot{X}(t)\right\} + \left\{\delta q(t)\right\}^T \left\{F^*(t)\right\}\right)dt = 0 \tag{85}$$

Equation of Motion

After performing integration by parts on (85), and accounting for zero virtual displacement at the endpoints, we obtain a second order coupled differential equation:

$$\left[M^*(t)\right]\left\{\ddot{q}(t)\right\} + \left[N^*(t)\right]\left\{\dot{q}(t)\right\} = \left\{F^*(t)\right\} - \left[T^*(t)\right]\left\{\dot{r}(t)\right\} \tag{86}$$

Where the following terms are defined:

$$\left[M^*(t)\right] \equiv \left[B(t)\right]^T \left[M\right]\left[B(t)\right] \tag{87}$$

$$\left[N^*(t)\right] \equiv \left[B(t)\right]^T \left(\left[M\right]\left[\dot{B}(t)\right] + \left[D(t)\right]\left[M\right]\left[B(t)\right]\right) \tag{88}$$

$$\left[T^*(t)\right] \equiv \left[B(t)\right]^T \left[D(t)\right]\left[M\right]\left[C\right] \tag{89}$$

Solving (86) with respect to the list of generalized accelerations $\{\ddot{q}(t)\}$, yields:

$$\left\{\ddot{q}(t)\right\} = \left[M^*(t)\right]^T \left(\left\{F^*(t)\right\} - \left[T^*(t)\right]\left\{\dot{r}(t)\right\} - \left[N^*(t)\right]\left\{\dot{q}(t)\right\}\right) \tag{90}$$

This list of five equations, one for each generalized coordinate, will be integrated numerically using the method of Runge-Kutta.

RESULTS

Preparation for Analysis

With the necessary equations of motion secured, we proceed to analyze the power generated by the GWEC. A generator is attached to each of the two gimbals. We will perform the proceeding calculation on the first gyroscope, and assume that it is located in the centre of mass of the buoy, such that $\mathbf{s}_C^{(\beta/1)}(t) = 0$. We begin by investigating the simplified equation of motion for the first gimbal, found in (90):

$$\ddot{\theta}^{(2)} = \frac{2\left(-M_g + J_{3C}^{(3)}\omega_1^{(1)}\cos\left(\theta^{(2)}\right)\dot{\phi}^{(3)} - J_{3C}^{(3)}\frac{1}{4}\left(\omega_1^{(1)}\right)^2 \sin\left(2\theta^{(2)}\right)\right)}{2\left(J_{2C}^{(2)}\right) + J_{3C}^{(3)}} \tag{91}$$

The numerator on the right-hand side of (91) consists of three terms. M_g is the counter torque in the generator, and is assumed to be constant. The next term depends on the angular velocity of the disk and is therefore a result of the gyroscopic effect. The third term accounts for oscillation effects of the buoy. In this pass we ignore the last term, as we are only interested in the gyroscopically generated power, and not the total generated power. Moving on, we define $\ddot{\theta}_{\text{gyro}}^{(2)}$ as the gimbal's angular acceleration caused by gyroscopic effects:

$$\ddot{\theta}_{\text{gyro}}^{(2)} \equiv \frac{2\left(-M_g + J_{3C}^{(3)}\omega_1^{(1)}\cos\left(\theta^{(2)}\right)\dot{\phi}^{(3)}\right)}{2\left(J_{2C}^{(2)}\right) + J_{3C}^{(3)}} \tag{92}$$

The mass moments of inertia are given by:

$$J_{2C}^{(2)} = \frac{m^{(2)}d^{(2)}}{12}, \quad J_{3C}^{(3)} = \frac{m^{(3)}\rho^{(3)}\rho^{(3)}}{2} \tag{93}$$

Where $d^{(2)}$ is the width of the gimbal, and $\rho^{(3)}$ is the radius of the disk. In order to approximate the angular velocity of the buoy $\omega_1^{(1)}$, we assume that its motion is tangent to the wave profile, and that deep-water wave conditions apply. Thus, we obtain the following simplified angular velocity, in accordance with Rios & Murakami [10]:

$$\omega_1^{(1)} = -\left(\frac{2\pi}{T}\right)^3 \left(\frac{a}{g}\right)\cos\left(\frac{2\pi t}{T}\right) \tag{94}$$

Where a is the wave amplitude, T is the wave period, and g is the gravitational acceleration. Next, we turn to the generation of power. The power produced by the first gyroscope is given by:

$$P_{\text{gyro},1}(t) = M_g\,\dot{\theta}_{\text{gyro}}^{(2)}(t) \tag{95}$$

Where $\dot{\theta}_{\text{gyro}}^{(2)}(t)$ is found as the integral of (92) with respect to time. The power produced by both gyroscopes is the sum of the power generated in each gyroscope:

$$P_{\mathrm{gyro}}(t) = M_{\mathrm{g}}\dot{\theta}^{(2)}_{\mathrm{gyro}}(t) + M_{\mathrm{g}}\dot{\theta}^{(4)}_{\mathrm{gyro}}(t) = 2 \cdot M_{\mathrm{g}}\dot{\theta}^{(2)}_{\mathrm{gyro}}(t) \qquad (96)$$

Here we notice the advantage of having two gyroscopes instead of one. The total power output is doubled. This shows how gyroscopic wave energy can easily be scaled to produce large amounts of power. Also, the two gyroscopes are spinning in equal but opposite directions. This eliminiates the unwanted yaw-motion that arises due to the gyroscopic effect.

Numerical Integration

In order to solve the equations of motion, we apply numerical methods. More specifically, we use a Matlab function called *ode45*. This integration scheme is based on the method of Runge-Kutta. The results will shortly be presented through plots. The parameters used in this process are listed in the table below:

Mass of the gimbals	$m^{(\beta)}$	75 kg
Width of the gimbals	$d^{(\beta)}$	0.85 m
Mass of the disks	$m^{(\gamma)}$	750 kg
Radius of the disks	$\rho^{(\gamma)}$	0.60 m
Angular velocity of disks	$\dot{\phi}^{(\gamma)}$	5000 rpm
Gravitational acceleration	g	9.81 m/s^2
Wave amplitude	a	3 m
Wave period	T	4 s
Counter torque in the generator	M_g	8250 Nm

Table 1 – Integration Parameters

Only the parameters that are directly influencing the plots are listed in this table. The parameters have been determined to maximize power output, while staying within reasonable limits.

DISCUSSION

In order to create reasonable plots we had to find the optimal relationship between disk angular velocity and counter torque. We chose an angular velocity of 5000 rpm, and adjusted the generator moment subsequently. From (96) we see that the generated power is proportional to the counter torque. Furthermore, a large counter torque is needed to constrain the angular displacement of the gimbals. On the other hand, if the counter torque is bigger than the torque created by the gyroscope, the gimbals are unable to move. The figure below shows that the chosen counter torque is close, but not equal, to the maximum values of the gyroscopically induced moment. The sinusoidal curve is the gyroscopically generated moment, corresponding to the second term of the numerator in (92). The stippled line is the counter torque M_g in the generator.

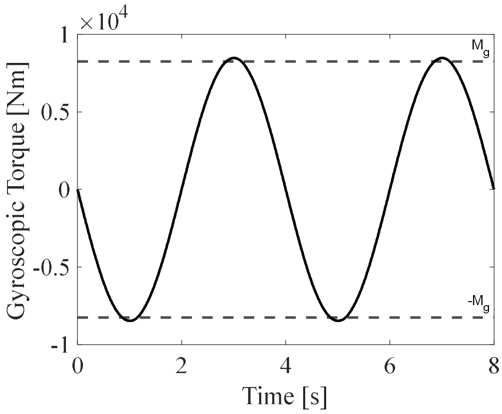

Figure 2 – Gyroscopic Torque

Next, we move on to the angular displacement of the gimbals. Here you can see a perioidic oscillating curve. As the counter torque increases, the smoothness of the curve increases, but at the same time the angular displacement is decreased. This curve is obtained by numerical integration of the gimbal equation of motion, found in (92).

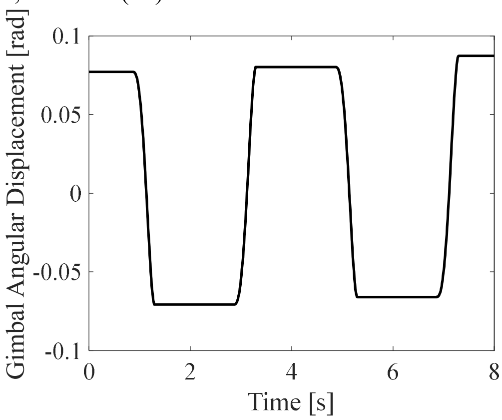

Figure 3 – Gimbal Angular Displacement

The angular velocity is also found by numerical integration of the gimbal equation of motion. The angular velocity of the gimbal is very important in relation to the generated power.

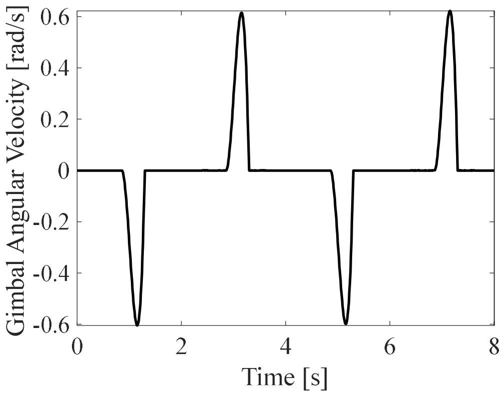

Figure 4 – Gimbal Angular Velocity

After the kinematical plots we turn our attention to the generated power. As mentioned, we assigned a previously selected value for M_g, and got $\dot{\theta}^{(2)}_{\text{gyro}}(t)$ from the numerical integration of the gimbal equation of motion. The plot below shows the absolute value of the power produced using equation (96), as both clockwise and counter-clockwise rotation of the gimbal will be assumed to generate power. The stippled line shows the average generated power. With the selected integration parameters, this GWEC will on average produce 1250 W, with a peak power output of roughly 10 kW.

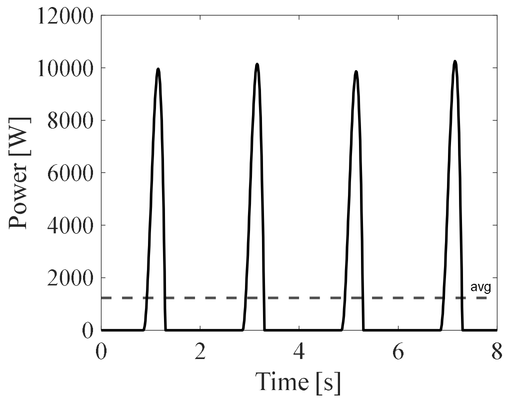

Figure 5 – Generated Power

Integration of the generated power with respect to time gives us the generated energy. We use a Matlab function called *cumtrapz*, which approximates the accumulative integral of a function using the trapezoidal method. This cumulative plot shows how more and more energy is generated as the buoy is oscillating on the waves.

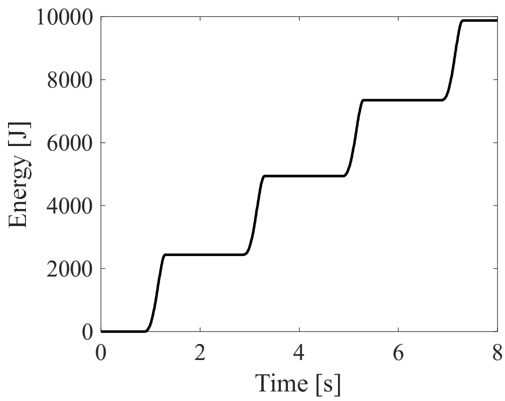

Figure 6 – Generated Energy

WebGL and visualization

To help visualize the GWEC, a simplified simulation has been made using WebGL. The simulation is available at: http://home.hib.no/prosjekter/dynamics/2019/gyro/. Figure 7 shows the GWEC from the same perspective as figure 1.

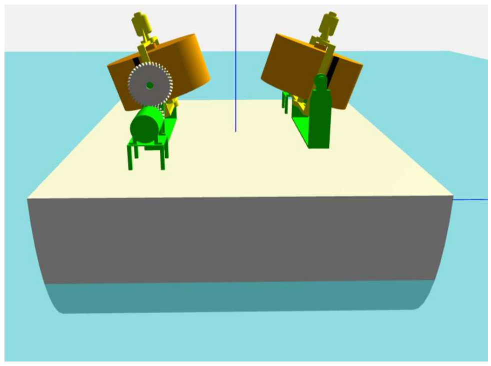
Figure 7 – Illustration of the 3D Simulation

CONCLUSION AND FUTURE WORK

In this project the power of the Moving Frame Method has been demonstrated by deriving the equations of motion for a gyroscopic wave generator. The equations were solved numerically and the results represented as plots. This project builds on, and extends previous work [5], by incorporating two spinning disks to reduce yaw-motion and increase power output, doing a full 3D analysis, while accounting for the prescribed rotations, and visualizing the results on a 3D web page. Additionally, a more compact and coherent notation was achieved by utilizing the Special Euclidean Group SE(3) to handle rotations and translations. Moreover, the angular rates of the disks were prescribed, which greatly simplified the analysis as we did not have to account for the time the disks would take to spin up. A simplified 3D-model was created in WebGL to illustrate the motion, and concept of the GWEC.

In future work there would be of interest to account for hydrostatic bouancy, added mass, and translation of the buoy. There is also great potential in improving the behavior of the generator to allow for a more efficient extraction of power, in addition to more accurate control of the gimbals movements.

REFERENCES

[1] H. Bendix, "forskning.no," 2016. [Online]. Available: https://forskning.no/alternativ-energi-havforskning-spor-en-forsker/spor-en-forsker-hvor-blir-det-av-bolgeenergien/378374. [Accessed: Jan. 11, 2019]

[2] N. Regjering, "Regjeringen.no," 2014. [Online]. Available: https://www.regjeringen.no/no/tema/klima-og-miljo/klima/innsiktsartikler-klima/gront-skifte/id2076832/. [Accessed: Jan. 11, 2019]

[3] Goldin, A., 2004, "Autonomous Gyroscopic Ocean-Wave Powered Generator: Invention of a New Energy Conversion Technology, "www.siemensfoundation.org/en/competition/2004-winners/aaron_goldin.html.

[4] Kanki, H, Arii, S., Fukui, K., and Tsukuo, K., 2007, "Ocean experiment of wave-power generation system by gyro effect," Proceedings for the 2007 Japan Society of

Mechanical Engineers Conference, Suita, Japan, abstract 0511, 2 pages (in Japanese).
[5] H. Murakami, O. Rios and A. Amini, "A Mathematical Model with Preliminary Experiments of A Gyroscopic Ocean Wave Energy Converter." Proceedings of the 2015 ASME International Mechanical Engineering Congress and Exposition, paper ID IMECE2015-51163.
[6] É. Cartan, 1986, On Manifolds with an Affine Connection and the Theory of General Relativity, translated by A. Magnon and A. Ashtekar, Napoli, Italy, Bibiliopolis.
[7] T. Frankel, 2012, "The Geometry of Physics, an Introduction," third edition, Cambridge University Press, New York; First edition published in 1997.
[8] T. Impelluso, "The moving frame method in dynamics: Reforming a curriculum and assessment," International Journal of Mechanical Engineering Education, pp. 158-191, 2018.
[9] C. Lanczos, The Variational Principles of Mechanics, Toronto: Dover Publications, 1970.
[10] H. Murakami and O. Rios, "A Mathematical Model for A Gyroscopic Ocean-Wave Energy Converter," Proceedings of the 2013 ASME International Mechanical Engineering Congress and Exposition, paper ID IMECE2013-62834, pp. 3-8.
[11] H. Murakami, O. Rios and A. Amini, "A Mathematical Model with Preliminary Experiments of A Gyroscopic Ocean Wave Energy Converter." Proceedings of the 2015 ASME International Mechanical Engineering Congress and Exposition, paper ID IMECE2015-51163.

FLOW CHART

Here we present the Gantt-form we created at the start of the semester. The green cells show what we planned to work on each week. The red cells show what we had to finish that week, while the yellow cells indicate an individual study week. Due to unforeseen incidents, we had some difficulties with following the planned progress in February and March, but with increased we effort we caught up with the schedule and finished in time.

This form helped us structure our workflow and was of great use throughout the project.