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PRODUCTION AND ANALYTICS OF A MULTI-LINKED ROBOTIC SYSTEM USING THE MOVING FRAME METHOD

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ABSTRACT

This paper extends research into flexible robotics through a collaborative, interdisciplinary senior design project. This paper deploys the Moving Frame Method (MFM) to analyze the motion of a relatively high multi-link system, driven by internal servo engines. The MFM describes the dynamics of the system. Lie group theory and Cartan's moving frames are the foundation of this new approach to engineering dynamics. This, together with a restriction on the variation of the angular velocity used in Hamilton's principle, enables an effective way of extracting the equations of motion. The result in a 3D analytical model for the motion of a snake-like robotic system. Furthermore, this project builds a snake-like robot driven by internal servo engines. The multi-linked robot will have two servos in each joint, enabling a three-dimensional movement. An internal microcontroller will compile the equations of motion through a remote computer using a wireless network. The set compiled equations will enable movement of the servos. Finally, a test is performed to compare if the theory and the measurable real-time results match.

NOMENCLATURE

$[B]$	B-matrix
$[D]$	Combined angular velocity matrix
$E^{(\alpha)}$	Frame connection matrix
e_n	Unit vector for n-axis
$\{F^*\}$	Generalized force list
$\{F\}$	Force and moment list
$\{H\}$	Generalized momenta
$J_c^{(\alpha)}$	3x3 Mass moment of inertia matrix
$[M]$	Mass matrix
$[M^*]$	Reduced mass matrix

$\left[N^{*}\right]$ Reduced non-linear velocity matrix

$\{\dot{q}\}$ Generalized velocity variable list

$\{\ddot{q}\}$ Generalized acceleration variable list

$R^{(\alpha)}$ Rotation matrix

$\{\dot{X}\}$ Virtual cartesian velocity

$\{\delta\dot{X}\}$ Variation of cartesian velocity

$\{\delta\tilde{X}\}$ Virtual generalized displacement

δK The virtual system kinetic

δW Virtual work

$\delta\Pi$ Variation of frame connection matrix

$\overleftrightarrow{\delta\pi}$ Virtual rotational displacement

$\Omega^{(\alpha)}$ Time rate of the frame connection matrix

$\omega^{(\alpha)}$ Angular velocity components

$\overleftrightarrow{\omega}^{(\alpha)}$ Skew-symmetric angular velocity matrix

INTRODUCTION

- The *pedagogical* goal of this project is to foster interdisciplinary collaboration on a senior design project.
- The *research* goal of this paper applies the MFM to create a real and virtual robot snake moving in three dimensions.

This paper also presents the MFM. The MFM uses a set of notations that does not distinguish between 2D and 3D problems. This makes solving complex problems at a high academic level lead to undergraduate students completing these tasks.

It is the nature of the MFM, is avoidance of the cross product and its reduction of group theory to matrix multiplications, that makes this accessible to an undergraduate team.

We commence with the foundations of the MFM that exploits the Special Orthogonal Group, SO(3). Then we introduce the robotic linked system under study.

Following this, we introduce the aspects of the MFM for robotics that exploit the Special Euclidean Group, SE(3) and the restriction on the variation of the angular velocity. This is then followed by the building and testing of the real device.

Finally, this work extends previous work by Murakami [1] which was limited to planar motion.

THE MODEL

Figure 1 presents a model of the multi-linked robotic system made in Creo Parametric [2]. The analysis is initiated with the inertial body, represented as the tail with a green frame (the MFM does allow for a free moving system, but this paper, for edification, restricts that free motion). This progresses to the first body. This first body is the engine body with the coupled blue frame. The analysis then progresses to the second body, represented with the red frame. From this, a pattern emerges with increasingly named bodies up to 20 bodies.

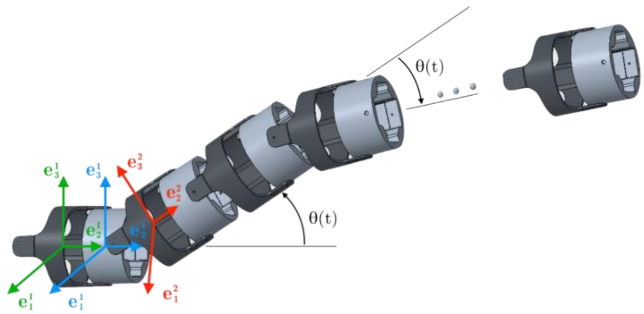

Figure 1. Model and frames.

OVERVIEW OF THE MOVING FRAME METHOD

History

The Moving Frame Method (MFM) presents a powerful pedagogy for dynamics and a more efficient means to extract the equations of motion for analysis. In the MFM, 3D, and 2D analyses manifest the same notation. Furthermore, multi-body systems and single body systems also manifest the same notation.

Élie Cartan (1869-1951) [3] assigned a reference frame to each point of an object under study (a curve, a surface, Euclidean space itself). Then, using an orthonormal expansion, he expressed the rate of change of the frame in terms of the frame. The MFM recognizes this and leverages this by placing a reference frame on every moving link. However, then we need a *method to connect moving frames*. For this, we turn to Lie.

Marius Sophus Lie (1842-1899) developed the theory of continuous groups and their associated algebras. The MFM adopts the mathematics of rotation groups and their algebras, yet distils them to simple matrix multiplications that avoid the non-associative properties of vectors. However, then we need a simplifying notation. For this, we turn to Frankel.

Ted Frankel [4] developed a compact notation in his work on geometrical physics. The MFM adopts this notation to enable a methodology that is identical for both 2D and 3D analyses. The notation is also identical for single bodies and multi-body linked systems.

The method has been pedagogically assessed by Impelluso [5], and a summary can be found there. Allow us first to introduce the underlying and modernized mathematics of the MFM, distinct from the problem under study.

FOUNDATION: MFM AND SO(3) FOR SINGLE BODIES

At the center of mass of each body α we place a moving frame:

$$\mathbf{e}^{(\alpha)}(t) = \left(\mathbf{e}_1^{(\alpha)}(t)\ \mathbf{e}_2^{(\alpha)}(t)\ \mathbf{e}_3^{(\alpha)}(t)\right) \tag{1}$$

In the previous, **e** is a unit vector and the subscript denotes the direction. Set t = 0 to deposit an inertial frame from a moving frame:

$$\mathbf{e}^I = \left\{\mathbf{e}_1^I \quad \mathbf{e}_2^I \quad \mathbf{e}_3^I\right\} = \left\{\mathbf{e}_1^{(\alpha)}(0) \quad \mathbf{e}_2^{(\alpha)}(0) \quad \mathbf{e}_3^{(\alpha)}(0)\right\} \tag{2}$$

Define the absolute position vector $\mathbf{r}_C^{(\alpha)}(t)$ of a frame as a translation $x_C^{(\alpha)}(t)$ formulated in the inertial frame $\mathbf{e}^I$:

$$\mathbf{r}_C^{(\alpha)}(t) = \mathbf{e}^I x_C^{(\alpha)}(t) \tag{3}$$

We use $x_C^{(\alpha)}(t)$ to represent the distance from the inertial frame to the center of mass of a body, thus the subscript C.

The relative position vector of a frame $(\alpha+1)$ from another frame (α) is represented by $\mathbf{s}_C{}^{(\alpha+1/\alpha)}(t)$, and is expressed as a translation, yet formulated in the α-frame:

$$\mathbf{s}_C{}^{(\alpha+1/\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) s_C{}^{(\alpha+1/\alpha)}(t) \tag{4}$$

By adding the absolute position vector of the α-frame $\mathbf{r}_C^{(\alpha)}(t)$ and the relative position vector $\mathbf{s}_C{}^{(\alpha+1/\alpha)}(t)$, we obtain the absolute position vector of the $(\alpha + 1)$ frame:

$$\mathbf{r}_C^{(\alpha+1)}(t) = \mathbf{r}_C^{(\alpha)}(t) + \mathbf{e}^{(\alpha)}(t) s_C^{(\alpha+1/\alpha)}(t) \tag{5}$$

We use a rotation matrix a member of the Special Orthogonal Group, $R \in SO(3)$, to relate the orientation of a moving frame to an inertial frame:

$$\mathbf{e}^{(\alpha)}(t) = \mathbf{e}^I R^{(\alpha)}(t) \tag{6}$$

The relative rotation of a frame $(\alpha + 1)$ from another frame (α) can be written as:

$$\mathbf{e}^{(\alpha+1)}(t) = \mathbf{e}^{(\alpha)}(t) R^{(\alpha+1/\alpha)}(t) \tag{7}$$

The orientation of the body $(\alpha + 1)$ can be expressed in the inertial frame by inserting an equation (6) into (7) and exploiting the closure property of Groups:

$$\mathbf{e}^{(\alpha+1)}(t) = \mathbf{e}^I R^{(\alpha)}(t) R^{(\alpha+1/\alpha)}(t) = \mathbf{e}^I R^{(\alpha+1)}(t) \tag{8}$$

In SO(3), the inverse of a rotation matrix is the transpose:

$$\left(R^{(\alpha)}(t)\right)^{-1} = \left(R^{(\alpha)}(t)\right)^T \tag{9}$$

The time rate of frame rotation:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^I \dot{R}^{(\alpha)}(t) \tag{10}$$

By using orthogonality, we can replace $\mathbf{e}^I$ in (10) and get:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) \left(R^{(\alpha)}(t)\right)^T \dot{R}^{(\alpha)}(t) \tag{11}$$

The time rate of frame rotation is now expressed in its own frame.

We define the skew-symmetric angular velocity matrix. We note that this element is a member of the associated algebra, so(3)

$$\overleftrightarrow{\omega^{(\alpha)}(t)} = \left(R^{(\alpha)}(t)\right)^T \dot{R}^{(\alpha)}(t) = \begin{bmatrix} 0 & -\omega_3^{(\alpha)}(t) & \omega_2^{(\alpha)}(t) \\ \omega_3^{(\alpha)}(t) & 0 & -\omega_1^{(\alpha)}(t) \\ -\omega_2^{(\alpha)}(t) & \omega_1^{(\alpha)}(t) & 0 \end{bmatrix} \tag{12}$$

We may write equation (11) as:

$$\dot{\mathbf{e}}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) \overleftrightarrow{\omega^{(\alpha)}(t)} \tag{13}$$

The skew-symmetric angular velocity matrix is isomorphic to the angular velocity vector of that frame:

$$\boldsymbol{\omega}^{(\alpha)}(t) = \mathbf{e}^{(\alpha)}(t) \begin{pmatrix} \omega_1^{(\alpha)}(t) \\ \omega_2^{(\alpha)}(t) \\ \omega_3^{(\alpha)}(t) \end{pmatrix} \tag{14}$$

THE MFM FOR MULTI-BODIES (SE3)

4×4 matrices of *homogeneous transformations* can relate orientations and positions. Denavit and Hartenberg [6] presented these in 1955, and they now are widely used in computer science and robotics. However, at that time, they did not recognize that such matrices were elements of a Lie group and were equipped with an associated Lie algebra. The MFM recognizes this and exploits the algebraic power.

Define a frame connection as a structure that contains a frame, $\mathbf{e}^{(\alpha)}(t)$ and its location. $\mathbf{r}_C^{(\alpha)}(t)$.

$$\left(\mathbf{e}^{(\alpha)}(t)\ \ \mathbf{r}_C^{(\alpha)}(t)\right) = \left(\mathbf{e}_1^{(\alpha)}(t) \quad \mathbf{e}_2^{(\alpha)}(t) \quad \mathbf{e}_3^{(\alpha)}(t) \quad \mathbf{r}_C^{(\alpha)}(t)\right) \tag{15}$$

The inertial frame connection may be written as $(\mathbf{e}^I\ \mathbf{0})$ where no translation and rotation are present. We may relate the two as:

$$\left(\mathbf{e}^{(\alpha)}(t)\ \ \mathbf{r}_C^{(\alpha)}(t)\right) = (\mathbf{e}^I\ \mathbf{0})E^{(\alpha)}(t) \tag{16}$$

Thus, we define the absolute frame connection matrix

$$E_{4\times 4}^{(\alpha)}(t) = \begin{bmatrix} R^{(\alpha)}(t) & x_C^{(\alpha)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{17}$$

With this, we recover equations (6) and (3). Rather than continue with this introduction of how the MFM handles multi-bodies, allow us to proceed to the problem under study and present the rest of the MFM in the context of this problem.

MULTI-LINK KINEMATICS

Kinematics for Body 1

In this restricted analysis, we deposit an inertial frame $\mathbf{e}^I$ on the tail body known as body-I, at the center of mass. A moving frame, $\mathbf{e}^{(1)}(t)$ is placed on the body-1 with the rotation between the bodies is about the 1st axis:

$$\mathbf{e}^{(1)}(t) = \mathbf{e}^I R^{(1)}(t) = \mathbf{e}^I(t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^{(1)}(t) & -\sin\theta^{(1)}(t) \\ 0 & \sin\theta^{(1)}(t) & \cos\theta^{(1)}(t) \end{bmatrix} \tag{18}$$

This first inertial body is equipped with a frame at the center of mass. For the first body-1, we add a frame at the center of mass. To reach the first body, we first translate in the 2-axis direction of the inertial frame a distance l_I; then we rotate about 1-axis of the inertial frame; finally we translate a distance l_I to reach the center of mass of the first body. $l_I = \begin{pmatrix} 0 & L & 0 \end{pmatrix}^T$ In terms of frame connection matrices, we assert:

$$E^{(1)}(t) = \begin{bmatrix} I_{3x3} & l_I \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} R^{(1)}(t) & 0 \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} I_{3x3} & l_1 \\ 0_3^T & 1 \end{bmatrix} = \begin{bmatrix} R^{(1)}(t) & R^{(1)}(t)l_1 + l_I \\ 0_3^T & 1 \end{bmatrix} \tag{19}$$

The relationship between the first frame and the inertial frame is then expressed as:

$$\left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right) = \left(\mathbf{e}^I\ \ \mathbf{0}\right)E^{(1)}(t) \tag{20}$$

Next, take the time rate of the frame connection:

$$\left(\dot{\mathbf{e}}^{(1)}(t)\ \ \dot{\mathbf{r}}_C^{(1)}(t)\right) = \left(\mathbf{e}^I\ \ \mathbf{0}\right)\dot{E}^{(1)}(t) \tag{21}$$

The time rate of the frame connection matrix, $\dot{E}^{(1)}(t)$ is found by taking the time derivative of each element. Due rigid bodies, the derivative for the translation will be equal to zero.

$$\dot{E}^{(1)}(t) = \begin{bmatrix} \dot{R}^{(1)}(t) & \dot{R}^{(1)}(t)l_1 \\ 0_3^T & 0 \end{bmatrix} \tag{22}$$

Express the inverse of the frame connection matrix (a member of SE(3)) as:

$$\left(E^{(1)}(t)\right)^{-1} = \begin{bmatrix} \left(R^{(1)}(t)\right)^T & -\left(R^{(1)}(t)\right)^T\left(R^{(1)}(t)l_1 + l_I\right) \\ 0_3^T & 1 \end{bmatrix} \tag{23}$$

Multiply the results of equation (20) with (23) Define the absolute time rate of the frame connection matrix for the first body $\Omega^{(1)}$as the product of $\left(E^{(1)}(t)\right)^{-1}$ and $\dot{E}^{(1)}(t)$. It is recognized that $\Omega \in se(3)$, the algebra associated with the *SE(3)* group.

$$\Omega^{(1)} \equiv \left(E^{(1)}(t)\right)^{-1}\dot{E}^{(1)}(t) \tag{24}$$

Substitute the result with the inertial frame connection in (21). We obtain:

$$\left(\dot{\mathbf{e}}^{(1)}(t)\ \ \dot{\mathbf{r}}_C^{(1)}(t)\right) = \left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right)\left(E^{(1)}(t)\right)^{-1}\dot{E}^{(1)}(t) \tag{25}$$

As a result, we can write (25) as:

$$\left(\dot{\mathbf{e}}^{(1)}(t)\ \ \dot{\mathbf{r}}_C^{(1)}(t)\right) = \left(\mathbf{e}^{(1)}(t)\ \ \mathbf{r}_C^{(1)}(t)\right)\Omega^{(1)}(t) \tag{26}$$

$\Omega^{(1)}$ multiplied out in a matrix form and recognizing that $\omega^{(1)}(t)$ is the same as (12):

$$\Omega^{(1)} = \begin{bmatrix} \overleftrightarrow{\omega^{(1)}(t)} & v^{(1)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{27}$$

From equation (26), we extract:

$$\dot{\mathbf{e}}^{(1)}(t) = \mathbf{e}^{(1)}(t)\overleftrightarrow{\omega^{(1)}(t)} \tag{28}$$

We associate this as the angular velocity vector:

$$\boldsymbol{\omega}^{(1)}(t) = \mathbf{e}^{(1)}(t)\begin{pmatrix} \dot{\theta}^{(1)}(t) \\ 0 \\ 0 \end{pmatrix} \tag{29}$$

The second equation extracted from (27) is

$$\dot{\mathbf{r}}_C^{(1)}(t) = \mathbf{e}^{(1)}(t) v_C^{(1)}(t) = \mathbf{e}^{(1)}(t)\left(\overleftrightarrow{\omega^{(1)}(t)} l_1\right) \tag{30}$$

First, we reverse the effective cross product, and we assert this in the inertial frame

$$\dot{\mathbf{r}}_C^{(1)}(t) = -\mathbf{e}^{(1)}(t)\left(\overleftrightarrow{l}_1 \omega^{(1)}(t)\right) = -\mathbf{e}^I R^{(1)}(t)\left(\overleftrightarrow{l}_1 \omega^{(1)}(t)\right) \tag{31}$$

Thus, we assert:

$$\dot{x}^{(1)}(t) = -R^{(1)}(t)\overleftrightarrow{l}_1 \omega^{(1)}(t) \tag{32}$$

Kinematics for Body 2

Place a moving frame, $\mathbf{e}^{(2)}(t)$, at the center of mass of the body-2. To get from to the 2nd frame from the first frame, move a distance l_1 in the $\mathbf{e}_2$- direction, then rotate about the 3rd axes $R^{(2/1)}(t)$ this time, and move a distance l_2.

The orientation of the second frame is obtained from the first frame by a rotation $\theta^{(2)}(t)$ about the common 3-axis.

$$\mathbf{e}^{(2)}(t) = \mathbf{e}^{(1)}(t) R^{(2/1)}(t) = \mathbf{e}^{(1)}(t)\begin{bmatrix} \cos\theta^{(2)}(t) & -\sin\theta^{(2)}(t) & 0 \\ \sin\theta^{(2)}(t) & \cos\theta^{(2)}(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

Thus, we obtain the following frame connection matrix

$$\begin{aligned} E^{(2/1)}(t) &= \begin{bmatrix} I_{3x3} & l_1 \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} R^{(2/1)}(t) & 0 \\ 0_3^T & 1 \end{bmatrix}\begin{bmatrix} I_{3x3} & l_2 \\ 0_3^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} R^{(2/1)}(t) & R^{(2/1)}(t) l_2 + l_1 \\ 0_3^T & 1 \end{bmatrix} \end{aligned} \tag{34}$$

The relative frame connection matrix becomes:

$$E^{(2/1)}(t) = \begin{bmatrix} R^{(2/1)}(t) & s_C^{(2/1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{35}$$

$E^{(2/1)}(t)$ relates the first frame and second frame connections as:

$$\left(\mathbf{e}^{(2)}(t) \quad \mathbf{r}_C^{(2)}(t)\right) = \left(\mathbf{e}^{(1)}(t) \quad \mathbf{r}_C^{(1)}(t)\right) E^{(2/1)}(t) \tag{36}$$

The inverse:

$$\left(E^{(2/1)}\right)^{-1}(t) = \begin{bmatrix} \left(R^{(2/1)}(t)\right)^T & -\left(R^{(2/1)}(t)\right)^T s_C^{(2/1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{37}$$

The rate of change of the frame connection matrix becomes:

$$\dot{E}^{(2/1)}(t) = \begin{bmatrix} \dot{R}^{(2/1)}(t) & \dot{R}^{(2/1)}(t) l_2 \\ 0_3^T & 0 \end{bmatrix} \tag{38}$$

The time rate of the frame connection matrix may be written as:

$$\Omega^{(2/1)}(t) = \begin{bmatrix} \overleftrightarrow{\omega^{(2/1)}(t)} & \overleftrightarrow{\omega^{(2/1)}(t)} l_2 \\ 0_3^T & 0 \end{bmatrix} \tag{39}$$

By unskewing the angular velocity. The vector of the second frame is then obtained:

$$\boldsymbol{\omega}^{(2/1)}(t) = \mathbf{e}^{(2)}(t)\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}^{(2)}(t) \end{pmatrix} \tag{40}$$

The absolute time rate of the frame connection matrix $\Omega^{(2)}(t)$ can be found by multiplying (37), (24) and (35) adding (39), and used to relate the time rate of change to the moving frame in compact notation:

$$\Omega^{(2)}(t) = \left(E^{(2/1)}\right)^{-1}(t)\Omega^{(1)}(t) E^{(2/1)}(t) + \Omega^{(2/1)}(t) \tag{41}$$

Equation (41) yields the following:

$$\begin{aligned} \Omega^{(2)}(t) &= \begin{bmatrix} \left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} R^{(2/1)}(t) + \overleftrightarrow{\omega^{(2/1)}(t)} & \begin{matrix} \left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)}\left(R^{(2/1)}(t) l_2 + l_1\right) \\ +\left(R^{(2/1)}(t)\right)^T \overleftrightarrow{\omega^{(1)}(t)} l_1 + \overleftrightarrow{\omega^{(2/1)}(t)} l_2 \end{matrix} \\ 0_3^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} \overleftrightarrow{\omega^{(2)}(t)} & v^{(2)}(t) \\ 0_3^T & 0 \end{bmatrix} \end{aligned} \tag{42}$$

First, define $R^{(2)}(t)$ as:

$$R^{(2)}(t) = R^{(1)}(t) R^{(2/1)}(t) \tag{43}$$

Extract and manipulate the terms of (42), respectively:

$$\omega^{(2)}(t) = \left(R^{(2/1)}(t)\right)^T \omega^{(1)}(t) + \omega^{(2/1)}(t) \tag{44}$$

$$\begin{aligned} \dot{x}^{(2)}(t) = &-R^{(1)}(t)(\overleftrightarrow{R^{(2/1)}(t) l_2} + 2\overleftrightarrow{l}_1)\omega^{(1)}(t) \\ &-R^{(2)}(t)\overleftrightarrow{l}_2 \omega^{(2/1)}(t) \end{aligned} \tag{45}$$

Kinematics for multi-bodies: Body α

As this method progresses to more bodies, a pattern appears. This section presents the pattern. Thus, we now reduce this to a B-matrix for α-bodies. For this system, the α represents the number 20th body.

For the α-frame, we create a moving coordinate frame using the relative frame connection, $E^{(\alpha/\alpha-1)}(t)$ to represent the relation between two adjacent bodies

$$(\mathbf{e}^{(\alpha)}(t)\,\mathbf{r}_C^{(\alpha)}(t)) = (\mathbf{e}^{(\alpha-1)}(t)\,\mathbf{r}_C^{(\alpha-1)}(t))\,E^{(\alpha/\alpha-1)}(t) \tag{46}$$

Here we define the relative frame connection is defined using the relative rotation $R^{(\alpha/\alpha-1)}(t)$ and the relative coordinates of the body-α frame origin $s_C^{(\alpha/\alpha-1)}(t)$ (which we keep time dependent, for now).

$$E^{(\alpha/\alpha-1)}(t) = \begin{bmatrix} R^{(\alpha/\alpha-1)}(t) & s_C^{(\alpha/\alpha-1)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{47}$$

We compute the relative frame connection as translation-rotation-translation between the two bodies. We apply $l_{\alpha-1}$ with the relative rotation-matrices, $R^{(\alpha/\alpha-1)}(t)$ and the translation from to the next body l_α .

$$\begin{aligned} E^{(\alpha/\alpha-1)} &= \begin{bmatrix} I_{3x3} & l_{\alpha-1} \\ 0_3^T & 1 \end{bmatrix} \begin{bmatrix} R^{(\alpha/\alpha-1)}(t) & 0 \\ 0_3^T & 1 \end{bmatrix} \begin{bmatrix} I_{3x3} & l_\alpha \\ 0_3^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} R^{(\alpha/\alpha-1)}(t) & R^{(\alpha/\alpha-1)}(t) l_\alpha + l_{\alpha-1} \\ 0_3^T & 1 \end{bmatrix} \end{aligned} \tag{48}$$

The inverse of the absolute frame connection matrix (48) becomes:

$$\left(E^{(\alpha/\alpha-1)}\right)^{-1} = \begin{bmatrix} \left(R^{(\alpha/\alpha-1)}(t)\right)^T & -\left(R^{(\alpha/\alpha-1)}(t)\right)^T x_C^{(\alpha)}(t) \\ 0_3^T & 1 \end{bmatrix} \tag{49}$$

The time derivative of the frame connection matrix (48) is obtained by taking the time derivative of each element in the matrix (now recognizing rigid bodies):

$$\dot{E}^{(\alpha/\alpha-1)}(t) = \begin{bmatrix} \dot{R}^{(\alpha/\alpha-1)}(t) & \dot{R}^{(\alpha/\alpha-1)}(t) l_\alpha \\ 0_3^T & 0 \end{bmatrix} \tag{50}$$

The time rate of the frame connection matrix $\Omega^{(\alpha/\alpha-1)}$ in relative form is defined as:

$$\Omega^{(\alpha/\alpha-1)} \equiv \left(E^{(\alpha/\alpha-1)}(t)\right)^{-1} \dot{E}^{(\alpha/\alpha-1)}(t) \tag{51}$$

To acquire the rate of absolute frame connection $\Omega^{(\alpha)}$ for the α-body, mulitply eq. (49) by the previous absolute frame connection matrix $\Omega^{(\alpha-1)}(t)$ and eq. (48). Then adding the time rate of the relative frame connection matrix (51):

$$\Omega^{(\alpha)}(t) = \left(E^{(\alpha/\alpha-1)}\right)^{-1}(t)\Omega^{(\alpha-1)}(t)E^{(\alpha/\alpha-1)}(t) + \Omega^{(\alpha/\alpha-1)}(t) \tag{52}$$

The absolute frame connection matrix (52) for the α-body expanded will become:

$$\Omega^{(\alpha)}(t) = \begin{bmatrix} \overleftrightarrow{\omega}^{(\alpha)}(t) & v^{(\alpha)}(t) \\ 0_3^T & 0 \end{bmatrix} \tag{53}$$

THE GENERALIZED COORDINATES

The velocities and angular velocities for all α-bodies are grouped in a *6n x 1* matrix $\{\dot{X}(t)\}$. These are referred to as *Cartesian velocities*:

$$\{\dot{X}(t)\} \equiv \begin{Bmatrix} \dot{x}_C^{(1)}(t) \\ \omega^{(1)}(t) \\ \dot{x}_C^{(2)}(t) \\ \omega^{(2)}(t) \\ \vdots \\ \dot{x}_C^{(\alpha)}(t) \\ \omega^{(\alpha)}(t) \end{Bmatrix} \tag{54}$$

The generalized essential velocity matrix detonated by $\{\dot{q}(t)\}$ is a set that takes account for the minimal *n*-degrees-of-freedom*. In the model, the angular velocities for each body will be reduced to twenty generalized essential velocities (for a twenty-link system).

$$\{\dot{q}(t)\} \equiv \begin{Bmatrix} \omega^{(1)}(t) \\ \omega^{(2/1)}(t) \\ \omega^{(3/2)}(t) \\ \vdots \\ \omega^{(\alpha/\alpha-1)}(t) \end{Bmatrix} \tag{55}$$

We now relate the Cartesian velocity (54) with the generalized essential velocity matrix (55). The B-matrix is used:

$$\{\dot{X}(t)\} = \left[B(t)\right]\{\dot{q}(t)\} \tag{56}$$

The B-matrix is devoted to the appendix, where a thorough description is given. Here one can find an algorithm for α-bodies.

$$[B(t)]=\begin{bmatrix} B_{1,1} & 0_{3x1} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{2,1} & 0_{3x1} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{3,1} & B_{3,2} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{4,1} & B_{4,2} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{5,1} & B_{5,2} & B_{5,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{6,1} & B_{6,2} & B_{6,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{7,1} & B_{7,2} & B_{7,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{8,1} & B_{8,2} & B_{8,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{(m-1),1} & B_{(m-1),2} & B_{(m-1),3} & \cdots & B_{(m-1),(n-2)} & B_{(m-1),(n-1)} & B_{(m-1),n} \\ B_{m,1} & B_{m,2} & B_{m,3} & \cdots & B_{m,(n-2)} & B_{m,(n-1)} & B_{m,n} \end{bmatrix} \tag{57}$$

KINETICS

Application of Analytical Mechanics

We begin by defining a Lagrangian as the difference between the kinetic and potential energy:

$$L^{(\alpha)}\left(q(t),\dot{q}(t)\right)=K^{(\alpha)}\left(q(t),\dot{q}(t)\right)-U^{(\alpha)}\left(q(t)\right) \tag{58}$$

Define the Action as the definite integral of the Lagrangian function over time:

$$A=\int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt \tag{59}$$

Hamilton's principle states that "the motion of a system occurs in such a way that the definite integral (59) becomes a minimum for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed" [7]. This means that the equations of motion can be obtained by setting the variation of the Action equal to zero:

$$\delta\int_{t_0}^{t_1} L^{(\alpha)}\left(q(t),\dot{q}(t),t\right)dt=0 \tag{60}$$

To include the non-conservative forces, we exploit the extension of Hamilton's Principle, known as the Principle of Virtual Work. Here, we formulate the Lagrangian as dependent only on the kinetic energy. We will account for all other forces (conservative or non-conservative) as work, on the right side. From this point onwards, we omit the dependencies of position and velocity for ease of notation.

$$\int_{t_0}^{t_1}\delta K^{(\alpha)}(t)dt=-\int_{t_0}^{t_1}\delta W^{(\alpha)}(t)dt \tag{61}$$

The kinetic energy of each body in the system is expressed by the angular momentum $H_C^{(\alpha)}(t)$, and linear momentum $L_C^{(\alpha)}(t)$:

$$\mathbf{H}_C^{(\alpha)}(t)=\mathbf{e}^{(\alpha)}(t)\,H_C^{(\alpha)}(t)=\mathbf{e}^{(\alpha)}(t)\,J_C^{(\alpha)}\omega^{(\alpha)}(t) \tag{62}$$

$$\mathbf{L}_C^{(\alpha)}(t)=\mathbf{e}^I L_C^{(\alpha)}(t)=\mathbf{e}^I m^{(\alpha)}\dot{x}_C^{(\alpha)}(t) \tag{63}$$

Here, $J_C^{(\alpha)}$ represents the moment of inertia matrix for body α. The total kinetic energy of a body α with the frame placed at the center of mass is defined as:

$$K^{(\alpha)}(t)=\frac{1}{2}\left\{\dot{\mathbf{r}}_C^{(\alpha)}\cdot\mathbf{L}_C^{(\alpha)}+\boldsymbol{\omega}^{(\alpha)}\cdot\mathbf{H}_C^{(\alpha)}\right\} \tag{64}$$

For the whole system, the total kinetic energy is expressed in matrix form as:

$$K(t)=\frac{1}{2}\left\{\dot{X}(t)\right\}^T\left[M\right]\left\{\dot{X}(t)\right\} \tag{65}$$

The masses and moments of inertia for each body is contained in the generalized mass matrix [M]:

$$\left[M\right]\equiv\begin{bmatrix} m^{(1)}I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & J_C^{(1)} & 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & m^{(2)}I_3 & 0_{3\times3} & \cdots & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & J_C^{(2)} & \cdots & 0_{3\times3} & 0_{3\times3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & \cdots & m^{(\alpha)}I_3 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & J_C^{(\alpha)} \end{bmatrix} \tag{66}$$

We now focus on the work done. Hamilton's Principle does not account for non-conservative forces such as applied loads or damping. For that, we extend Hamilton's Principle as the engineering Principle of Virtual Work. Thus, we must formulate the work done. Wittenburg [8] used a weighted virtual angular velocity, to extract Euler's equation. However, moment and angular velocity define the power, not the work. This was the weakest point in the classical multibody dynamics. The MFM has rectified this [9], and independently so, by Holm [10].

In the MFM, moment vs. virtual rotation represent a natural pair—they are conjugate to each other, and expressed in the moving body frame. To represent the virtual rotation, the MFM presents a *restriction* on the angular velocity for use in the extension of Hamilton's Principle. The equations of motion are then easier to obtain. This restriction reduces as a simple matrix based equation:

To continue, define virtual rotational displacement $\delta\pi^{(\alpha)}(t)$ as the un-skewed form of $\overleftrightarrow{\delta\pi^{(\alpha)}(t)}$, which is defined as the product of the transpose and the variation of the rotation matrix:

$$\overleftrightarrow{\delta\pi^{(\alpha)}(t)} = \left(R^{(\alpha)}(t)\right)^T \delta R^{(\alpha)}(t) \tag{67}$$

With the following equation, we structure the virtual Cartesian displacements $\{\delta \tilde{X}(t)\}$:

$$\left\{\delta \tilde{X}(t)\right\} = \begin{pmatrix} \delta x^{(1)}(t) \\ \delta\pi^{(1)}(t) \\ \delta x^{(2)}(t) \\ \delta\pi^{(2)}(t) \\ \vdots \\ \delta x^{(\alpha)}(t) \\ \delta\pi^{(\alpha)}(t) \end{pmatrix} \tag{68}$$

Next, the variation of the velocities is called the virtual Cartesian velocities $\{\delta \dot{X}(t)\}$:

$$\left\{\delta \dot{X}(t)\right\} = \begin{Bmatrix} \delta \dot{x}_C^{(1)}(t) \\ \delta\omega^{(1)}(t) \\ \delta \dot{x}_C^{(2)}(t) \\ \delta\omega^{(2)}(t) \\ \vdots \\ \delta \dot{x}_C^{(\alpha)}(t) \\ \delta\omega^{(\alpha)}(t) \end{Bmatrix} \tag{69}$$

For the linear displacement, the variation of the derivative is equal to the derivative of the variation:

$$\delta \dot{x}_C^{(\alpha)}(t) = \frac{d}{dt} \delta x_C^{(\alpha)}(t) \tag{70}$$

However, the restriction on the variation of the angular velocity is:

$$\delta\omega^{(\alpha)}(t) = \frac{d}{dt} \delta\pi^{(\alpha)}(t) + \overleftrightarrow{\omega^{(\alpha)}(t)}\, \delta\pi^{(\alpha)}(t) \tag{71}$$

The last two equations are written in compact form as:

$$\left\{\delta \dot{X}(t)\right\} = \left\{\delta \dot{X}(t)\right\} + [D]\left\{\delta \tilde{X}(t)\right\} \tag{72}$$

Where $[D]$ is a skew symmetric matrix that contains the angular velocity matrices for each frame:

$$[D(t)] \equiv \begin{bmatrix} 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & \cdots & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & \overleftrightarrow{\omega^{(1)}(t)} & 0_{3x3} & 0_{3x3} & \cdots & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & \cdots & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & \overleftrightarrow{\omega^{(2)}(t)} & \cdots & 0_{3x3} & 0_{3x3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & \cdots & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & \cdots & 0_{3x3} & \overleftrightarrow{\omega^{(\alpha)}(t)} \end{bmatrix} \tag{73}$$

The variation of the kinetic energy is re-expressed as:

$$\delta K(t) = \left\{\delta \dot{X}(t)\right\}^T [M]\left\{\dot{X}(t)\right\} \tag{74}$$

Next, we need the forces and moments acting on the different bodies of the system. They are expressed in a single column matrix $\{Q(t)\}$.

For each of the bodies, a force vector consisting of the forces and moments are listed in a set. This force will include the forces and torques exerted by translational motors and internal forces.

$$\{F(t)\} = \begin{Bmatrix} F_{3x1}^{(1)}(t) \\ M_{3x1}^{(1)}(t) \\ F_{3x1}^{(2)}(t) \\ M_{3x1}^{(2)}(t) \\ F_{3x1}^{(3)}(t) \\ M_{3x1}^{(3)}(t) \\ \vdots \\ F_{3x1}^{(\alpha-1)}(t) \\ M_{3x1}^{(\alpha-1)}(t) \\ F_{3x1}^{(\alpha)}(t) \\ M_{3x1}^{(\alpha)}(t) \end{Bmatrix} = \begin{Bmatrix} N^{(1)} - m^{(1)}ge_3 \\ -T^{(1)}(t) \\ N^{(2)} - m^{(2)}ge_1 \\ T^{(1)}(t) - T^{(2)}(t) \\ N^{(3)} - m^{(3)}ge_3 \\ T^{(2)}(t) - T^{(3)}(t) \\ \vdots \\ N^{(\alpha-1)} - m^{(\alpha-1)}ge^{(\alpha-1)} \\ T^{(\alpha-1)}(t) - T^{(\alpha)}(t) \\ N^{(\alpha)} - m^{(\alpha)}ge^{(\alpha)} \\ T^{(\alpha)}(t) \end{Bmatrix} \tag{75}$$

The terms are:

N = Normal force for the respectively body
$-mg$ = Force from gravity on the respectively body
T = Torque from the motor
$-T$ = Reverse torque from the previous motor

The Equation of motion will be numerically integrated. It will produce a $3n\,x1$ row vector with the next time step for the essential generalized velocity $\dot{q}(t)_{n+1}$.

The virtual work done by the generalized forces can then be expressed as:

$$\delta W = \left\{\delta \tilde{X}(t)\right\}^T \{Q(t)\} \tag{76}$$

The B-matrix that relates the Cartesian velocities $\{\dot{X}(t)\}$ to the essential generalized velocities $\{\dot{q}(t)\}$, also relates the virtual generalized displacements $\{\delta \tilde{X}(t)\}$ to the essential virtual displacements $\{\delta q(t)\}$

$$\{\delta \tilde{X}(t)\} = [B(t)]\{\delta q(t)\} \tag{77}$$

The transpose of the above is used to rewrite equation (76):

$$\delta W(t) = \{\delta q(t)\}^T \{F^*(t)\} \tag{78}$$

Where the essential generalized forces $\{F^*(t)\}$ are defined as:

$$\{F^*(t)\} = [B(t)]^T \{Q(t)\} \tag{79}$$

By inserting the expressions obtained for the variation of the kinetic energy and the virtual work into equation (61), we obtain the basis for the equation of motion:

$$\int_{t_0}^{t_1} \left(\{\delta \dot{X}(t)\}^T [M]\{\dot{X}(t)\} + \{\delta q(t)\}^T \{F^*(t)\} \right) dt = 0 \tag{80}$$

Equation of motion

After performing integration by parts on (80), and accounting for zero virtual displacement at the endpoints, we obtain a second order coupled differential equation:

$$[M^*(t)]\{\ddot{q}(t)\} + [N^*(t)]\{\dot{q}(t)\} = \{F^*(t)\} \tag{81}$$

Where the following terms are defined:

$$[M^*(t)] \equiv [B(t)]^T [M][B(t)] \tag{82}$$

$$[N^*(t)] \equiv [B(t)]^T \left([M][\dot{B}(t)] + [D(t)][M][B(t)] \right) \tag{83}$$

Solving (81) with respect to the list of generalized accelerations $\{\ddot{q}(t)\}$, yields:

$$\{\ddot{q}(t)\} = [M^*(t)]^{-1} \left(\{F^*(t)\} - [N^*(t)]\{\dot{q}(t)\} \right) \tag{84}$$

This list of five equations, one for each generalized coordinate, will shortly be integrated numerically using the method of Runge-Kutta.

SIMPLIFICATION OF MODEL

An accurate simulation should account for gravity, normal reaction force and friction of motion. This would require, for the sake of simulation, a module to assess if the robot snake has lifted off the table; and that, then determines the application of gravity and normal reaction forces and friction. However, it is emphasized that this is an undergraduate student project. Thus, in this simulation, one may consider the snake moving in a zero-gravity field with no friction.

3D VISUALIZATION AND WEBGL

WebGL (Web Graphics Library) is a JavaScript interface for rendering interactive 2D and 3D computer graphics.

http://home.hib.no/prosjekter/dynamics/2019/robot/

VALIDATE THE EQUATION OF MOTION

To validate and test the MFM, the following was done:

We provided time-dependent polynomial functions for the angle motion and used that to obtain the associated moments from Eqn. (84). Then, we applied those moments to confirm the angles.

We did this using RK4 numerical integration, to obtain the predicted motion of the robot. The accordion and cobra movement (ref.: Website) were given polynomial functions to simulate and validate 2D motion. The simulation is visually compared to the motion wanted, which will confirm the MFM and its application. To simulate 3D movement a spiral function was given as an input on the website. Figure 2 presents a snapshot of the animation.

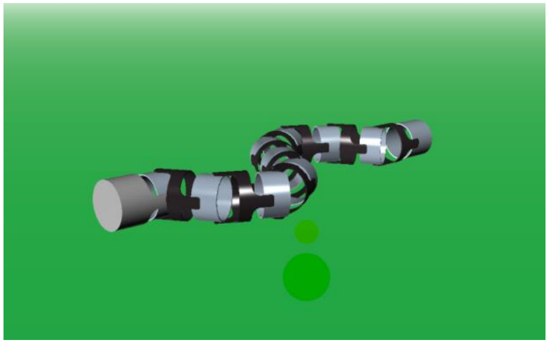

Figure 2. One snapshot of the Virtual Snake Robot

BUILDING THE REAL MODEL

Figure 3. Two snapshots of the Physical Snake Robot

Robotic Integration

To validate and test the MFM, we built a multi-link robot as shown in Figure 3. The robot provides a real physical system for comparison.

Design method and criteria

The multi-linked robotic system consists of multiple joints which are driven by electrical servo motors. The robot provides data during run time providing degrees on each joint. This is used to compare the robotic model to the theoretical simulation.

Mechanical design

The design of the robot consists of eight joints excluding the head and tail. The joints are connected with servo motors, which determines the angle of each joint in regard to the previous. This gives the robot a horizontal movement. In order to reduce weight and more efficient prototyping, the bodies where 3D-printed using polylactic acid (PLA). Finite element method simulations show that a thickness of 5 mm and a few strengthening features in the design, makes it sufficient to withstand the maximum load that the motors can exert on the frame of the robot. Overall, this structural design results in a robust, lightweight cylindrical frame construction.

Hardware components

The robot is controlled by a microcontroller sending pulse width modulation (PWM) signals to the servo motors. The width of the pulse determines the angle of the motor, which in turn determines the angle of the joint. The microcontroller contains the code necessary to control the motion of the robot. As power supply the robot has a battery pack containing rechargeable NiMH battery cells, with a total capacity of 8Ah at 7V. The microcontroller has a separate power supply to produce a stable 5V required to operate.

Software design

All software is written in the C language. The microcontroller contains the program controlling the motion of the robot. The different PWM signals given to the servo motors is the recipe for the sinusoidal formation. For each time-step a constant input of angle is given, resulting in a linear response of motion. The program writes a log file of data from the servo motors in a text document.

Validation of the MFM

The data from the servo motors contains the position and duration corresponding to each time-step of the movement. This way, the robot's actual movement can be compared to the simulated results. A list of data and a corresponding graph for each joint of the robot and the simulation is put side by side for visual inspection.

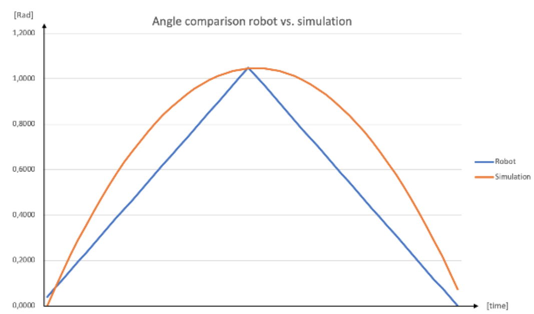

Figure 4. Graph showing the comparison between the robot and the simulation (MFM).

CONCLUSION AND FUTURE WORK

The linear curve from the output of the motors and the polynomial curve from the simulation intersects proximally at the same maxima. Because the two curves are of a different characteristic, the path will deviate. Thus, the initial conditions and the maxima which are the crucial intersection points can be compared using the two different estimations. Future work may contain a polynomial function for the motors to get a better assessment.

To validate the 3D movement, one may observe that the RK4 numerical analysis for the equation of motion gives a correct output as the intended path. Hence, the equation of motion holds for a multi-linked robotic system. For future work one may use the real-life forces from the robot to get a visual comparison to measure the deviation from the desired movement. Due to internal friction in motors in combination with environmental effects the desired path is assumed to deviate.

Further, to obtain a more accurate motion of the model, it is possible to implement an artificial intelligence engine. The engine predicts the motion and adjusts the real-time feedback position.

The time rate of the B-matrix algorithm may be computed, as shown in the B-matrix. This will improve how to obtain the equation of motion for a generalized alpha-body multi-linked system.

The MFM and the algorithms hold for 2D and 3D movement. For future work the control system may be improved, allowing the robot to move in three dimensions as simulated using JavaScript.

This shows that a complex time-dependent 3D problem can be solved by undergraduates, which would be difficult using classic multi-body dynamics approach.

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APPENDIX

The time-dependent is removed from the rotation matrix, due to the comprehensiveness of the equation.

Nomenclature for the B-matrix

α = body
m = row
n = columns
$i, j, k \in \mathbb{N}_1$:

$$B_{1,1} = -R^{(1)} \overleftrightarrow{l_1} e_1 \quad (85)$$

$$B_{1,2} = e_1 \quad (86)$$

$$B_{3,1} = B_{1,1} - \left(R^{(1)} (\overleftrightarrow{R^{(2/1)} l_2} + \overleftrightarrow{l_1}) \right) e_1 \quad (87)$$

$$B_{4,1} = R^{(2/1)^T} e_1 \quad (88)$$

$$B_{5,1} = B_{3,1} - \left(R^{(1)} (\overleftrightarrow{R^{(2/1)} R^{(3/2)} l_3} + \overleftrightarrow{R^{(2/1)} l_2}) \right) e_1 \quad (89)$$

$$B_{6,1} = R^{(3/2)^T} R^{(2/1)^T} e_1 \quad (90)$$

$$B_{7,1} = B_{5,1} - \left(R^{(1)} (\overleftrightarrow{R^{(2/1)} R^{(3/2)} R^{(4/3)} l_4} + \overleftrightarrow{R^{(2/1)} R^{(3/2)} l_3}) \right) e_1 \quad (91)$$

$$B_{8,1} = R^{(4/3)^T} R^{(3/2)^T} R^{(2/1)^T} e_1 \quad (92)$$

$$B_{(m-1),1} = \left(\begin{array}{l} -R^{(1)} \overleftrightarrow{l_1} - R^{(1)} (\overleftrightarrow{R^{(2/1)} l_2} + \overleftrightarrow{l_1}) \\ -R^{(1)} \sum_{i=3}^{\alpha} \left(\overleftrightarrow{\left(\prod_{j=2}^{i} R^{(j/j-1)} \right) l_i} + \overleftrightarrow{\left(\prod_{k=3}^{i} R^{(k-1/k-2)} \right) l_{i-1}} \right) \end{array} \right) e_1 \quad (93)$$

$$B_{m,1} = \left(\prod_{i=2}^{\alpha} R^{(i/i-1)} \right)^T e_1 \quad (94)$$

$$B_{3,2} = -R^{(2)} \overleftrightarrow{l_2} e_3 \quad (95)$$

$$B_{4,2} = e_3 \quad (96)$$

$$B_{5,2} = B_{3,2} - \left(R^{(2)} (\overleftrightarrow{R^{(3/2)} l_3} + \overleftrightarrow{l_2}) \right) e_3 \quad (97)$$

$$B_{6,2} = R^{(3/2)^T} e_3 \tag{98}$$

$$B_{7,2} = B_{5,2} - \left(R^{(2)} (\overleftrightarrow{R^{(3/2)} R^{(4/3)} l_4} + \overleftrightarrow{R^{(3/2)} l_3}) \right) e_3 \tag{99}$$

$$B_{8,2} = R^{(4/3)^T} R^{(3/2)^T} e_3 \tag{100}$$

$$B_{(m-1),2} = \begin{pmatrix} -R^{(2)} \overleftrightarrow{l_2} - R^{(2)} (\overleftrightarrow{R^{(3/2)} l_3} + \overleftrightarrow{l_2}) \\ -R^{(2)} \sum_{i=4}^{\alpha} \begin{pmatrix} \overleftrightarrow{\left(\prod_{j=3}^{i} R^{(j/j-1)} \right) l_i} + \\ \overleftrightarrow{\left(\prod_{k=4}^{i} R^{(k-1/k-2)} \right) l_{i-1}} \end{pmatrix} \end{pmatrix} e_3 \tag{101}$$

$$B_{m,2} = \left(\prod_{i=3}^{\alpha} R^{(i/i-1)} \right)^T e_3 \tag{102}$$

$$B_{5,3} = -R^{(3)} \overleftrightarrow{l_3} e_1 \tag{103}$$

$$B_{6,3} = e_1 \tag{104}$$

$$B_{7,3} = B_{5,3} - \left(R^{(3)} (\overleftrightarrow{R^{(4/3)} l_4} + \overleftrightarrow{l_3}) \right) e_1 \tag{105}$$

$$B_{8,3} = R^{(4/3)^T} e_1 \tag{106}$$

$$B_{(m-1),3} = \begin{pmatrix} -R^{(3)} \overleftrightarrow{l_3} - R^{(3)} (\overleftrightarrow{R^{(4/3)} l_4} + \overleftrightarrow{l_3}) \\ -R^{(3)} \sum_{i=5}^{\alpha} \begin{pmatrix} \overleftrightarrow{\left(\prod_{j=4}^{i} R^{(j/j-1)} \right) l_i} + \\ \overleftrightarrow{\left(\prod_{k=5}^{i} R^{(k-1/k-2)} \right) l_{i-1}} \end{pmatrix} \end{pmatrix} e_1 \tag{107}$$

$$B_{m,3} = \left(\prod_{i=4}^{\alpha} R^{(i/i-1)} \right)^T e_1 \tag{108}$$

$$B_{(m-1),(n-2)} = \begin{pmatrix} -R^{(n)} \overleftrightarrow{l_n} - R^{(n)} (\overleftrightarrow{R^{(n+1/n)} l_{n+1}} + \overleftrightarrow{l_n}) \\ -R^{(n)} \sum_{i=n+2}^{\alpha} \begin{pmatrix} \overleftrightarrow{\left(\prod_{j=n+1}^{i} R^{(j/j-1)} \right) l_i} + \\ \overleftrightarrow{\left(\prod_{k=n+2}^{i} R^{(k-1/k-2)} \right) l_{i-1}} \end{pmatrix} \end{pmatrix} e^{(\alpha)} \tag{109}$$

$$B_{m,(n-2)} = \left(\prod_{i=n+1}^{\alpha} R^{(i/i-1)} \right)^T e^{(\alpha)} \tag{110}$$

$$B_{(m-1),(n-1)} = \left(-R^{(n)} \overleftrightarrow{l_n} - R^{(n)} (\overleftrightarrow{R^{(n+1/n)} l_{(n+1)}} + \overleftrightarrow{l_n}) \right) e^{(\alpha)} \tag{111}$$

$$B_{m,(n-1)} = \left(\prod_{i=n+1}^{\alpha} R^{(i/i-1)} \right)^T e^{(\alpha)} \tag{112}$$

$$B_{(m-1),n} = -R^{(n)} \overleftrightarrow{l_n} e^{(\alpha)} \tag{113}$$

$$B_{m,n} = e^{(\alpha)} \tag{114}$$

For ease, the B-matrix is repeated.

$$[B(t)] = \begin{bmatrix} B_{1,1} & 0_{3x1} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{2,1} & 0_{3x1} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{3,1} & B_{3,2} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{4,1} & B_{4,2} & 0_{3x1} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{5,1} & B_{5,2} & B_{5,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{6,1} & B_{6,2} & B_{6,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{7,1} & B_{7,2} & B_{7,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ B_{8,1} & B_{8,2} & B_{8,3} & \cdots & 0_{3x1} & 0_{3x1} & 0_{3x1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{(m-1),1} & B_{(m-1),2} & B_{(m-1),3} & \cdots & B_{(m-1),(n-2)} & B_{(m-1),(n-1)} & B_{(m-1),n} \\ B_{m,1} & B_{m,2} & B_{m,3} & \cdots & B_{m,(n-2)} & B_{m,(n-1)} & B_{m,n} \end{bmatrix} \tag{115}$$