Proyecciones Journal of Mathematics
Vol. 31, N^o 4, pp. 73-90, March 2013.
Universidad Católica del Norte
Antofagasta - Chile

A new convergence analysis for the two-step Newton method of order three

I. K. Argyros
Cameron University, U. S. A.
and
S. K. Khattri
Stord Haugesund University College, Norway
Received : September 2012. Accepted : March 2013

Abstract

We present a tighter than before semilocal convergence analysis for the two-step Newton method of order three using recurrent functions. Numerical examples are also provided to show that our convergence criteria are satisfied but earlier studies such as in nine,thirteen,fifteen are not satisfied.

AMS Subject Classification : *65H10; 65G99; 65J15; 47H17; 49M15*

Key Words : *Two-step Newton method, Newton's method, Banach space, Kantorovich hypothesis, majorizing sequence, Lipschitz/center-Lipschitz conditions.*

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution $x^{\star}$ of equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where, $\mathcal{F}$ is Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Many problems in computational mathematics can brought in the form (1.1). The solutions of these equations are rarely found in closed form. Therefore most solution methods for these equations are iterative. Newton's method

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \tag{1.2}$$

is undoubtedly the most popular method for generating a sequence $\{x_n\}$ converging quadratically to $x^{\star}$. Two-step Newton method (TSNM)

$$y_n = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}),$$

$$\mathrm{x}_{n+1} = y_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n)$$

(1.3)

has also been used to generate a cubically convergent sequence $x^{\star}$ five,nine. Note that (1.3) requires one more evaluation of $\mathcal{F}$ per step than Newton's method (1.2)

In particular Ezquerro, Hernández and Salanova nine used the following conditions (in non-affine invariant form) ($\mathbf{C}_{\mathrm{K}}$)

$\mathrm{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ for some $x_0 \in \mathcal{D}$;

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \nu$$

$$\left\|\mathcal{F}'(x_0)^{-1}\Big[\mathcal{F}'(x) - \mathcal{F}'(x_0)\Big]\right\| \leq L_0 \left\|x - x_0\right\| \quad forall x \in \mathcal{D};$$

$$\left\|\mathcal{F}'(x_0)^{-1}\Big[\mathcal{F}'(x) - \mathcal{F}'(y)\Big]\right\| \leq L \left\|x - y\right\| \quad forall x, y \in \mathcal{D};$$

$$h_k = L\,\eta \leq \frac{1}{2}$$

(1.4)

and

$$U(x_0,\lambda)=\Big\{x\in\nabla\rceil\lceil\mathcal{D}\Big|\,\|x-x_0\|\le\lambda\Big\}\subseteq\mathcal{D},$$

for specified $\lambda\ge 0$.

The same ($\mathbf{C}_\mathrm{k}$) conditions have been used to show the semilocal convergence for the Newton's method (1.2). Note that (1.4) is the, famous for its simplicity and clarity, Kantorovich sufficient convergence hypothesis for the Newton's method (1.2). A current survey on Newton-type methods can be found in [][and the references therein]five (see also thirteen,fifteen). We have shown five the quadratic convergence of the Newton's method (1.2). Using the set of conditions ($\mathbf{C}_\mathrm{AH}$)

$\mathrm{F}'(x_0)^{-1}\in L(\mathcal{Y},\mathcal{X})\quad forsome\ \ x_0\in\mathcal{D};$
$\Big\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\Big\|\le red\eta$
$\Big\|\mathcal{F}'(x_0)^{-1}\Big[\mathcal{F}'(x)-\mathcal{F}'(x_0)\Big]\Big\|\le L_0\,\|x-x_0\|\quad forall\ x\in\mathcal{D};$
$\Big\|\mathcal{F}'(x_0)^{-1}\Big[\mathcal{F}'(x)-\mathcal{F}'(y)\Big]\Big\|\le L\,\|x-y\|\quad forall\ x,y\in\mathcal{D};$
$h_{AH}=L\,\eta\le\dfrac{1}{2}$
(1.5)

and

$$U(x_0,\lambda_0)\subseteq\mathcal{D},$$

for some specified $\lambda_0\ge 0$, where

$$L=\frac{1}{8}\left(L+4L_0+\sqrt{L^2+8L_0L}\right). \tag{1.6}$$

Note that

$$L_0\le L \tag{1.7}$$

holds in general, and L/L_0 can be arbitrarily large four,five. Moreover, redL_0 the Center-Lipschitz is not an additional condition, since L_0 is a special case of red$L$. Furthermore, we have by (1.4)-eq:17

$$h_K \leq \frac{1}{2} \quad \Longrightarrow \quad h_{AH} \leq \frac{1}{2} \tag{1.8}$$

but not necessarily vise versa unless if $L_0 = \text{red}L$. The error analysis under eq:15 is also tighter than eq:14. Hence, the applicability of Newton's method (1.2) has been extended.

In this study, we provide the sufficient convergence conditions for (TSNM) corresponding to (1.4). The paper is organized as follows: §2 contains the semilocal convergence analysis for (TSNM), whereas the numerical examples are given in §3.

2. Semilocal Convergence Analysis for (TSNM)

We need the following result on majorizing sequence for (TSNM).

Lemma 2.1. *Let L_0, L, η be positive constants. Assume: there exist parameters α and ϕ such that*

$$\frac{L\eta}{2} \leq \alpha \leq \frac{L}{2L_2}, \tag{2.1}$$

$$\frac{L_1\eta}{2(1 - L_2\eta)} \leq \phi \leq \phi_0 \tag{2.2}$$

and

$$\eta \leq \min\left\{\frac{2}{L_1 + 2L_2(1+\phi)}, \frac{1}{L_2}\right\} \tag{2.3}$$

where,

$$L_1 = \alpha(2+\alpha)L, \qquad L_2 = (1+\alpha)L_0, \tag{2.4}$$

$$\phi_0 = \min\left\{\frac{2L_1}{L_1 + \sqrt{L_1^2 + 8L_1L_2}}, \frac{L - 2\alpha L_2}{L}, \frac{2\alpha(1 - L_2\eta)}{L\eta}\right\}. \tag{2.5}$$

Then, sequences $\{s_n\}$, $\{t_n\}$ generated by

$$\begin{aligned} &t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0t_n)}, \\ &s_{n+1} = t_{n+1} + \frac{L\left[2(s_n - t_n) + t_{n+1} - reds_n\right](t_{n+1} - reds_n)}{2(1 - L_0t_{n+1})} \end{aligned} \tag{2.6}$$

are non-decreasing, bounded from above by

$$t^{\star\star} = \left(\frac{1+\alpha}{1-\phi}\right)\eta, \tag{2.7}$$

and converge to their common least upper bound $t^\star \in [0, t^{\star\star}]$*. Moreover, the following estimates hold*

$$0 \le t_{n+1} - s_n \le \alpha(s_n - t_n), \tag{2.8}$$

and

$$0 \le s_{n+1} - t_{n+1} \le \phi(s_n - t_n). \tag{2.9}$$

Proof. We shall show using induction on k:

$$0 \le \frac{L(s_k - t_k)}{2(1 - L_0 t_k)} \le \alpha, \tag{2.10}$$

and

$$0 \le \frac{L_1(s_k - t_k)}{2(1 - L_0 t_{k+1})} \le \phi. \tag{2.11}$$

Note that estimates (2.8) and (2.9) will then follow from (2.10) and (2.11), respectively. Estimates (2.10) and (2.11) hold by the left hand side hypotheses in (2.1),(2.2), respectively. It follows from (2.6), (2.10) and (2.11) that estimates (2.8) and (2.9) hold for $redn = 0$. Let us assume estimates (2.10) and (2.11) hold for all $k \le redn$. It then follows that estimates (2.8) and (2.9) hold for $n = redk$. We then have:

$$0 \le s_k - t_k \le \phi(s_{k-1} - t_{k-1}) \le \phi \cdot \phi(s_{k-2} - t_{k-2}) \le \cdots \le \phi^k \eta, \tag{2.12}$$

$$0 \le t_{k+1} - s_k \le \alpha(s_k - t_k) \le \alpha\phi^k\eta, \tag{2.13}$$

and

$$t_{k+1} \le s_k + \alpha\phi^k\eta \le t_k + \alpha\phi^k\eta + \phi^k\eta$$
$$\le s_{k-1} + \alpha\phi^{k-1}\eta + \alpha\phi^k\eta + \phi^k\eta$$

$$\leq t_{k-1} + \phi^{k-1}\eta + \alpha\phi^{k-1}\eta + \alpha\phi^k\eta + \phi^k\eta$$

$$= \mathrm{t}_{k-1} + (\phi^{k-1} + \phi^k)\eta + \alpha(\phi^{k-1} + \phi^k)\eta \leq \cdots$$

$$\leq s_0 + \alpha(\eta + \phi\eta + \cdots + \phi^k\eta) + \alpha(\phi\eta + \cdots + \phi^k\eta)$$

$$= (1{+}\alpha)(1 + \phi + \cdots + \phi^k\eta) \leq t^{\star\star}. \tag{2.14}$$

In view of (2.12) and (2.14), estimate (2.10) certainly holds, if

$$0 \leq \frac{L\phi^k\eta}{2\left[1 - L_2(1 + \phi + \cdots + \phi^{k-1})\eta\right]} \leq \alpha, \tag{2.15}$$

or

$$L\phi^k\eta + 2\alpha L_2(1 + \phi + \cdots + \phi^{k-1})\eta - 2\alpha \leq 0. \tag{2.16}$$

Estimate (2.16) motivates us to introduce recurrent functions f_k on $[0, 1)$ by

$$f_k(t) = L\eta t^k + 2\alpha L_2(1 + t + \cdots + t^{k-1})\eta - 2\alpha. \tag{2.17}$$

We need a relationship between two consecutive functions f_k:

$$f_{k+1}(t) = Lt^{k+1}\eta + 2\alpha L_2(1 + t + \cdots + t^k)\eta - 2\alpha - Lt^k\eta -$$

$$2\alpha L_2(1 + t + \cdots + t^{k-1})\eta + 2\alpha + f_k(t)$$

$$= \mathrm{f}_k(t) + Lt^{k+1}\eta - Lt^k\eta + 2\alpha L_2 t^k\eta$$

$$= \mathrm{f}_k(t) + g(t)t^k\eta, \tag{2.18}$$

where

$$g(t) = Lt - L + 2\alpha L_2. \tag{2.19}$$

Note that $g(\phi) \leq 0$ by (2.2). Using (2.17) we see that (2.16) holds

$$\begin{aligned} redif \quad f_k(\phi) &\leq 0 \\ redor \quad redf_1(\phi) &\leq 0, \end{aligned} \tag{2.20}$$

(2.21) *since,* $\quad g(\phi) \leq 0 \quad$ *and* $\quad f_{k+1}(\phi) = f_k(\phi) + g(\phi)\phi^k\eta \leq f_k(\phi),$

where ϕ is chosen as in the right hand side inequality of (2.1). But (2.20) also holds by (2.2). Moreover, define function f_∞ on $[0, 1)$ by

$$f_\infty(t) = \lim_{k\to\infty} f(t). \tag{2.22}$$

Then, we have by (2.19) that

$$f_\infty(\phi) \leq 0. \tag{2.23}$$

Hence, (2.8) and (2.10) hold for all k. Similarly, (2.11) holds, if

$$L_1\phi^k\eta \leq 2\phi\left[1 - L_2(1 + \phi + \cdots + \phi^k)\eta\right] \tag{2.24}$$

or

$$L_1\phi^k\eta + 2\phi L_2(1 + \phi + \cdots + \phi^k)\eta - 2\phi \leq 0. \tag{2.25}$$

As in (2.17) we define functions p_k on $[0, 1)$ by

$$p_k(t) = L_1t^k\eta + 2tL_2(1 + t + \cdots + t^k)\eta - 2\phi. \tag{2.26}$$

We need a relationship between two consecutive functions $redh_k$:

$$p_{k+1}(t) = [t]L_1t^{k+1}\eta + 2tL_2(1 + t + \cdots + t^{k+1})\eta - 2\phi - L_1t^k\eta$$

-2tL$_2(1 + t + \cdots + t^k)\eta + 2\phi + p_k(t)$
= p$_k(t) + L_1t^{k+1}\eta - L_1t^k\eta + 2L_2t^{k+2}\eta$

= p$_k(t) + g_1(t)t^k\eta$
(2.27)

where

$$g_1(t) = 2L_2t^2 + L_1t - L_1. \tag{2.28}$$

Note that $g_1(\phi) \leq 0$ by (2.2) and that

$$redr = red\frac{2L_1}{L_1 + \sqrt{L_1^2 + 8L_1L_2}} \tag{2.29}$$

redis the positive root of g_1. In view of (2.26), estimate (2.25) holds

$$if \quad p_k(\phi) \leq 0 \quad or \quad p_1(\phi) \leq 0 \tag{2.30}$$

since, $\quad \mathrm{g}_1(\phi) \leq 0 \quad and \quad p_{k+1}(\phi) = p_k(\phi) + g_1(\phi)\phi^k\eta \leq p_k,$
where ϕ is chosen as in the right hand side of (2.2). Note now that (2.30) holds by (2.3). Furthermore, define functions p_∞ on $[0, 1)$ by

$$p_\infty(t) = \lim_{k\to\infty} p_k(t). \tag{2.31}$$

We then have

$$p_\infty(\phi) \leq 0. \tag{2.32}$$

That completes the induction for (2.9) and (2.11). Finally, in view of (2.8), (2.9) and (2.14), sequences $\{t_n\}$, $\{s_n\}$ converge to $t^\star$. That completes the proof of the Lemma. □

We need an Ostrowski-type relationship between iterates $\{x_n\}$ and $\{y_n\}$ fourteen.

Lemma 2.2. *Let us assume iterates $\{x_n\}$ and $\{y_n\}$ in (TSNM) are well defined for all $n \geq 0$. Then, the following identities hold:*

$$\mathcal{F}(x_{n+1}) = \int_0^1 \Big[\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(x_n)\Big](x_{n+1} - y_n)d\theta,$$

(2.33)

and

$$\mathcal{F}(y_n) = \int_0^1 \Big[\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)\Big](y_n - x_n)d\theta. \tag{2.34}$$

Proof. Identity (2.34) follows from the Taylor's theorem and the first iteration in (TSNM), whereas (2.35) follows from Taylor's theorem and the second iteration in (TSNM). That completes the proof of the Lemma. □

We can show the following semilocal convergence result for (TSNM).

Lemma 2.3. *Let $\mathcal{F} : \mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ be Fréchet-differentiable operator. Assume: there exist $x_0 \in \mathcal{D}$, $L_0 > 0$, $L > 0$ and $\eta > 0$ such that for all $x, y \in \mathcal{D}$:*

$$\mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}), \tag{2.35}$$

$$\left\| \mathcal{F}'(x_0)^{-1} \mathcal{F}(x_0) \right\| \leq \eta, \tag{2.36}$$

$$\left\| \mathcal{F}'(x_0)^{-1} \left(\mathcal{F}'(x) - \mathcal{F}'(x_0) \right) \right\| \leq L_0 \left\| x - x_0 \right\|, \tag{2.37}$$

$$\left\| \mathcal{F}'(x_0)^{-1} \left(\mathcal{F}'(x) - \mathcal{F}'(y) \right) \right\| \leq L \left\| x - y \right\|, \tag{2.38}$$

$$U(x_0, t^\star) \subseteq \mathcal{D}. \tag{2.39}$$

Hypotheses of Lemma 2.1 hold, where $t^\star$ is given in Lemma 2.1. Then, sequences $\{x_n\}$ and $\{y_n\}$ generated by (TSNM) are well defined, remain in $U(x_0, t^\star)$ for all $n \geq 0$ and converge to a solution $x^\star \in U(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$.

Moreover, the following estimates hold

$$\|y_n - x_n\| \leq s_n - t_n, \tag{2.40}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{2.41}$$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{2.42}$$

$$\|y_{n+1} - y_n\| \leq s_{n+1} - s_n, \tag{2.43}$$

$$\|x_n - x^\star\| \leq t^\star - t_n, \tag{2.44}$$

$$\|y_n - x^\star\| \leq t^\star - s_n. \tag{2.45}$$

Furthermore, if there exists $R \geq t^\star$ *such that*

$$U(x_0, R) \subseteq \mathcal{D} \tag{2.46}$$

and

$$L_0(t^\star + R) < 2, \tag{2.47}$$

then, $x^\star$ *is the only solution of* $\mathcal{F}(x) = 0$ *in* $U(x_0, R)$.

Proof. We shall show using induction on k that (TSNM) is well defined, the iterates remain in $U(x_0, t^\star)$ for all $n \geq 0$ and estimates (2.41) and (2.42) hold for all $n \geq 0$. Iterate y_0 is well defined by the first equation in (TSNM) for $n = 0$ and (2.36). We also have by (2.6) and (2.37)

$$\|y_0 - x_0\| = \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta = reds_0 = s_0 - t_0 \leq t^\star.$$

That is (2.41) holds for $n = 0$ and $y_0 \in U(x_0, t^\star)$. Using (TSNM) for $n = 0$, we see that x_1 is well defined. Moreover, in view of (2.35) for $n = 0$, (TSNM), (2.6) and (2.37)-(2.39), we get

$$\|x_1 - y_0\| = \left\|\int_0^1 \mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(x_0 + \theta(y_0 - x_0)) - \mathcal{F}'(x_0)\right]d\theta(y_0 - x_0)\right\|$$

$$\leq L_0 \textstyle\int_0^1 \theta\|y_0 - x_0\|^2 d\theta = \dfrac{L_0}{2}\|y_0 - x_0\|^2$$
$$\leq \frac{L_0}{2}(s_0 - t_0)^2 = t_1 - s_0$$

which shows (2.42) for $n = 0$. We also have

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 - t_0 \leq t^\star,$$

which implies (2.43) holds for $n = 0$ and $x_1 \in U(x_0, t^\star)$.

Let $w \in U(x_0, t^\star)$. Then, we have by Lemma 2.1 and (2.38) that

$$\left\|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(w) - \mathcal{F}'(x_0)\right]\right\| \leq L_0\|w - x_0\| \leq L_0 t^\star < 1. \tag{2.48}$$

It follows from (2.49) and the Banach lemma on invertible operators five,thirteen,fifteen that $\mathcal{F}'(w)^{-1}$ exists and

$$\left\|\mathcal{F}'(w)^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1 - L_0\|w - x_0\|}. \tag{2.49}$$

In particular, for $x_1 \in U(x_0, t^\star)$, we have

$$\left\|\mathcal{F}'(x_1)^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1 - L_0\|x_1 - x_0\|} \leq \frac{1}{1 - L_0(t_1 - t_0)} = \frac{1}{1 - L_0 t_1}. \tag{2.50}$$

Using (TSNM), (2.6), (2.34) (for $n = 0$) and (2.51), we get

$$\|y_1 - x_1\| = \left\|\left[\mathcal{F}'(x_1)^{-1}\mathcal{F}'(x_0)\right]\left[\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_1)\right]\right\|$$

$$\leq \|\mathcal{F}'(x_1)^{-1}\mathcal{F}'(x_0)\|\,\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_1)\|$$

$$\leq \frac{1}{1 - L_0 t_1}\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(y_0 + \theta(x_1 - y_0)) - \mathcal{F}'(x_0)\right] d\theta (x_1 - y_0)\right\|$$

$$\leq \frac{L_0}{1 - L_0 t_1}\int_0^1 \left[\|y_0 - x_0\| + \theta\|x_1 - y_0\|\right] d\theta \|x_1 - y_0\|$$

$$\leq \frac{redL}{1 - L_0 t_1}\left[(s_0 - t_0) + \frac{1}{2}(t_1 - s_0)\right](t_1 - s_0) = s_1 - t_1,$$

which implies (2.41) for $n = 1$. We then have that

$$\|y_1 - y_0\| \leq \|y_1 - x_1\| + \|x_1 - y_0\| \leq s_1 - t_1 + t_1 - s_0 = s_1 - s_0,$$

$$\|y_1 - x_0\| \leq \|y_1 - y_0\| + \|y_0 - x_0\| \leq s_1 - s_0 + s_0 - t_0 = s_1 \leq t^\star,$$

which imply (2.44) for $n = 0$ and $y_1 \in U(x_0, t^\star)$. Let us now assume (2.41)-(2.44), $y_n, x_k \in U(x_0, t^\star)$ for all $n \leq k$. Using (TSNM), (2.6), (2.34), (2.35), (2.39) and the induction hypotheses, we have in turn that

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\|$$

$$\leq t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 = t_{k+1} \leq t^\star, \tag{2.51}$$

$$\|y_{k+1} - x_{k+1}\| = \left\| \left[\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0) \right] \left[\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1}) \right] \right\|$$

$$\leq \|\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\| \, \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\|$$

$$\leq \frac{1}{1 - L_0\|x_{k+1} - x_0\|} \int_0^1 \left\| \mathcal{F}'(x_0)^{-1} \left[\mathcal{F}'(y_k + \theta(x_{k+1} - y_k)) \right.\right.$$

$$\left.\left. - \mathrm{F}'(x_k) d\theta (x_{k+1} - y_k) \right\|$$

$$\leq \frac{L}{1 - L_0 t_{k+1}} \int_0^1 \left[\|y_k - x_k\| + \theta\|x_{k+1} - y_k\| \right] d\theta \left\| x_{k+1} - y_k \right\|$$

$$\leq \frac{L}{1 - L_0 t_{k+1}} \left[s_k - t_k + \frac{1}{2}(t_{k+1} - s_k) \right] (t_{k+1} - s_k)$$

$$= \mathrm{s}_{k+1} - t_{k+1},$$

(2.52)

$$\|x_{k+2} - y_{k+1}\| = \left\| \left[\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0) \right] \left[\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_{k+1}) \right] \right\|$$

$$\leq \frac{1}{1 - L_0 t_{k+1}} \int_0^1 \left\| \mathcal{F}'(x_0)^{-1} \left[\mathcal{F}'(x_{k+1} + \theta(y_{k+1} - x_{k+1})) \right.\right.$$

$$\left.\left. - \mathrm{F}'(x_{k+1}) d\theta (y_{k+1} - x_{k+1}) \right\|$$

$$\leq \frac{L}{1 - L_0 t_{k+1}} \int_0^1 \theta \|y_{k+1} - x_{k+1}\|^2 d\theta$$

$$\leq \frac{L}{2(1 - L_0 t_{k+1})} (s_{k+1} - t_{k+1})^2 = t_{k+2} - s_{k+1},$$

(2.53)

$$\|y_{k+2} - y_{k+1}\| \leq \|y_{k+2} - x_{k+2}\| + \|x_{k+2} - y_{k+1}\|$$

$$\leq s_{k+2} - t_{k+2} + t_{k+2} - s_{k+1} = s_{k+2} - s_{k+1},$$

(2.54)

$$\|x_{k+2} - x_{k+1}\| \leq \|x_{k+2} - y_{k+1}\| + \|y_{k+1} - x_{k+1}\|$$
$$\leq t_{k+2} - s_{k+1} + s_{k+1} - t_{k+1} = redt_{k+2} - t_{k+1}$$
(2.55)

which show (2.41)-(2.44) hold for all $n \geq 0$. Estimates (2.45) and (2.46) follow from (2.43) and (2.44), respectively by using standard majorization technique five,thirteen,fifteen. Moreover, from Lemma 2.1 and (2.41)-(2.44) we deduce that (TSNM) is Cauchy in a Banach space $\mathcal{X}$ and as such it converges to some $x^\star \in U(x_0, t^\star)$ (since $U(x_0, t^\star)$ is a closed set).

Moreover, we have by (2.53)

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_{k+1})\right\| \leq L\left[\|y_k - x_k\| + \frac{1}{2}\|x_{k+1} - y_k\|\right]\|x_{k+1} - y_k\|$$
$$\to 0, \quad as \quad k \to \infty.$$
(2.56)

That is $\mathcal{F}(x^\star) = 0$. Finally to show uniqueness, let $y^\star \in U(x_0, R)$ be a solution of equation $\mathcal{F}(x) = 0$. Let us define linear operator M by

$$M = \int_0^1 \mathcal{F}'(y^\star + \theta(x^\star - y^\star))\theta. \tag{2.57}$$

Then, using (2.38), (2.47) and (2.48), we get in turn that

$$\|\mathcal{F}'(x_0)\left[M - \mathcal{F}'(x_0)\right]\| \leq L_0 \int_0^1 \|y^\star + \theta(x^\star - y^\star) - x_0\|\,\theta$$
$$\leq L_0 \textstyle\int_0^1 \left[(1-\theta)\|y^\star - x_0\| + \theta\|x^\star - x_0\|\right]\theta$$
$$\leq \frac{L_0}{2}(R + t^\star) < 1.$$
(2.58)

It follows from (2.59) and the Banach Lemma on invertible operators that M^{-1} exists. Then, in view of the identity

$$0 = \mathcal{F}(x^\star) - \mathcal{F}(y^\star) = M(x^\star - y^\star), \tag{2.59}$$

we conclude that $x^\star = y^\star$. That completes the proof of the Theorem. □

Remarks 2.4.

Limit point $t^\star$ can be replaced by $t^{\star\star}$, given in closed form by (2.7), in hypotheses (2.40) and (2.48).

The verification of conditions (2.1)-(2.3) require simple algebra (see also Example 3.1).

If $L_0 = L$, then scalar sequences $\{s_n\}$, $\{t_n\}$ given by (2.6) reduce essentially to the ones used in nine. In particular, we have in this case

$$\begin{aligned} redt_0 = 0, \quad reds_0 = \eta, \quad t_{n+1} = s_n + \frac{L(s_n - t_n)^2}{2(1 - Lt_n)}, \\ s_{n+1} = t_{n+1} + \frac{L[2(s_n - t_n) + t_{n+1} - s_n](t_{n+1} - s_n)}{2(1 - Lt_{n+1})} \end{aligned} \tag{2.60}$$

If $L_0 < L$ iteration (2.6) is tighter than eq:261. Moreover, in view of the proof of the Theorem 2.3, we note that sequence

$$\begin{aligned} t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{L^\star(s_n - t_n)^2}{2(1 - L_0 t_n)}, \\ s_{n+1} = t_{n+1} + \frac{L^\star[2(s_n - t_n) + t_{n+1} - s_n](t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}, \end{aligned} \tag{2.61}$$

is also majorizing for (TSNM), where

$$L^\star = \begin{cases} L_0, & if \quad n = 0 \\ L, & if \quad n > 0. \end{cases}$$

In case $L_0 < L$, (2.26) is even a tighter majorizing sequence than (2.61). Furthermore, L, L_1 can be replaced by $L_0, L_1^\star = \alpha(\alpha + 2)L_0$ red at the left hand sides of (2.1) and (2.2), respectively.

If $\alpha = 0$, reddefine $L_1 = L$, then it is simple algebra to show that conditions of Lemma 2.1 reduce to (1.5). Moreover, if $L_0 = L$, these conditions reduce to (1.4). That is we have Newton's method (1.2), and iteration (2.6) reduces to

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0 t_{n+1})}. \tag{2.62}$$

In the case of Newton's method for $L_0 = L$, we have the well-known Kantorovich majorizing sequence four,five,thirteen,fifteen

$$\nu_0 = 0, \quad \nu_1 = \eta, \quad \nu_{n+2} = \nu_{n+1} + \frac{L(\nu_{n+1} - \nu_n)^2}{2(1 - L\nu_{n+1})}. \tag{2.63}$$

Note that if $L_0 < L$, $\{t_n\}$ is a tighter majorizing sequence than $\{\nu_n\}$ for the Newton's method five,thirteen,fifteen.

3. Numerical Examples

Let $\mathcal{X} = \mathcal{Y} = R^2$ be equipped with the max-norm, $x_0 = (1, 1)^T$, $\mathcal{D} = U(x_0, 1 - p)$, $p \in [0, 1/2)$ and define $\mathcal{F}$ on $\mathcal{D}$ by

$$\mathcal{F}(x) = \left(\xi_1^3 - p, \xi_2^3 - p\right)^T, \quad x = (\xi_1, \xi_2)^T. \tag{3.1}$$

Using (2.35)-(2.37), we get

$$\eta = \frac{1 - p}{2}, \quad L_0 = 3 - p \quad and \quad L = 2(2 - p) > L_0. \tag{3.2}$$

The Newton-Kantorovich hypothesis (1.4) is violated, since

$$\frac{4}{3}(1 - p)(2 - p) > 1 \quad forall \quad p \in [0, 1/2).$$

Hence, there is no guarantee that (TSNM) converges to $x^\star = (\sqrt[3]{p}, \sqrt[3]{p})$. That is the results in rednine,thirteen,fifteen cannot apply to solve equation (3.1).

Using (2.1)-(2.5) and (TSNM) for $p = 0.49$, we get

$$\eta = 0.17, \quad L_0 = 2.51, \quad L = 3.02, \quad L_1 = 1.774552, \quad L_2 = 3.1626.$$

So, (2.1)-(2.3) become

$$0.2567 < 0.26 < 0.477455258,$$

$$0.326234049 < 0.33 < 0.407656274,$$

$$\eta \leq 0.196327344.$$

Moreover, we have

$$t^{\star\star} = 0.319701493 < 1 - p = 1 - 0.49 = 0.51,$$

$$t^{\star\star} \leq R < \frac{2}{L_0} - t^{\star\star} = 0.47711256 < 1 - p = 0.51.$$

Hence, the conclusions of Theorem 2.2 apply and (TSNM) converges to

$$x^{\star} = \left(\sqrt[3]{0.49}, \sqrt[3]{0.49}\right)^T = (0.788373516, 0.788373516)^T.$$

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I. K. Argyros
Department of Mathematical Sciences,
Cameron University, Lawton,
Oklahoma 73505-6377,
USA
e-mail : iargyros@cameron.edu

and

S. K. Khattri
Department of Engineering,
Stord Haugesund University College,
Norway
e-mail : sanjay.khattri@hsh.no