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The average velocity in a queue

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Abstract
A number of cars drive along a narrow road that does not allow overtaking. Each driver has a certain maximum speed at which he or she will drive if alone on the road. As a result of slower cars ahead, many cars are forced to drive at speeds lower than their maximum ones. The average velocity in the queue offers a non-trivial example of a mean value calculation. Approximate and exact results are obtained using sampling, enumeration and calculations. The geometrical nature of the problem as well as the separate levels of averaging involved are emphasized. Further problems suitable for exploration in student projects are outlined.

1. Introduction

The mean or average values are extensively used as the simplest characterization of data sets and distributions. One encounters them every day, in science and in the news. Yet, neither the calculation nor the interpretation of the average value is in all situations as obvious as the definition may indicate. This should be conveyed to students.

This paper outlines an average-value calculation in a non-trivial situation. In undergraduate courses, it should follow simpler examples. In his *Statistical Mechanics*, Ma [1] uses the distribution function formed from the last digits of the telephone numbers in portions of a directory to illustrate the average value. A number of other simple examples are easily found or constructed. The material could also be used in more extensive student projects, at an undergraduate or a graduate level. See section 7 for suggestions on how to use this paper in teaching and projects. The paper is intended for lecturers, and should also be directly accessible to students, at both undergraduate and graduate levels.

This paper may also serve as an introduction to the broad and very active research field called complex systems. The system considered is complex in the sense that it has *a large number of states, which are not equivalent.* Even for relatively small systems, it is practically impossible to list or work through all cases (states). Furthermore, there is a *large variability in the geometrical structure* of these states, and the geometrical structures have *very different statistical weights*. The calculation of an average value can involve *several distinct averaging*

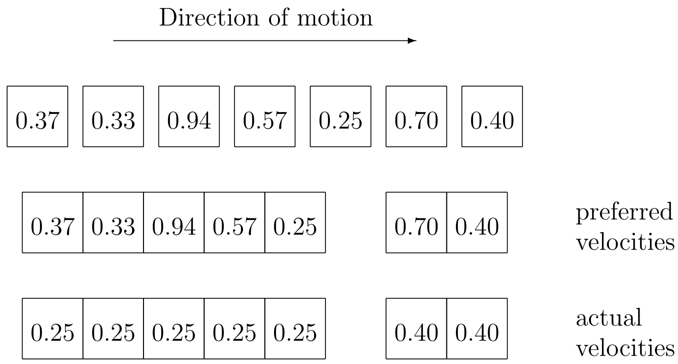

Figure 1. Seven cars, all driving to the right. The upper line shows the preferred velocity of each car. All cars behind the car with preferred velocity 0.25 have higher preferred velocities. They are slowed down and form a queue together with the 0.25 car. By the same token, the 0.70 and 0.40 cars also form a queue. These two queues are shown in the middle line. The bottom line shows the actual velocity of all cars in this queue pattern.

procedures; in the case described below, there are three. The reader will find these and related aspects explained and discussed in depth in [2], a graduate-level textbook that to a large extent is also accessible to undergraduate students. This book has three main topics: percolation, Ising model and self-organized criticality.

2. The model system

The situation, or a model system, to be considered here consists of cars driving along a narrow road. Each driver has a certain maximum speed at which he or she will drive if alone on the road. All these preferred speeds (velocities) are different. The preferred velocities are numbers between 0 and 1, uniformly distributed.

It is assumed that the narrow road does not allow overtaking. It is further assumed that the cars are distributed along the road in a random sequence. In other words, there are no correlations between the preferred velocities of the drivers and their order. One may imagine that the cars have arrived on the narrow road from a ferry boat. This situation (model system) was first described by Hemmer [3].

A driver will only very rarely be able to keep his or her preferred velocity. In most cases, many drivers will have to drive at lower velocities due to slower cars (drivers with lower preferred velocities) further ahead. As a result, queues are formed behind slow cars.

An example is shown in figure 1. Here, there are seven cars, with preferred velocities 0.37, 0.33, 0.94, 0.57, 0.25, 0.70, 0.40 (for simplicity, only two digits are used in this example). The direction of motion is to the right, as it will be in subsequent figures. Thus, the car with preferred velocity 0.40 is in front, as shown in the first (topmost) line. The second line in the figure illustrates the queues that are formed in this particular case. There is a queue consisting of five cars from the slowest car (0.25) and backwards and a queue consisting of two cars associated with the slowest car among the remaining two (0.40). Each car is represented by a box. Connected boxes indicate cars in the same queue. An interstice indicates a new queue (moving at a different velocity). We will refer to a series of queues like those in figure 1 as a *queue pattern*.

In the third line of the figure the same queues (the same queue pattern) are redrawn, with the velocities the cars will actually have. All five cars in the hindmost queue have actual velocity 0.25 and the two cars in the front queue have actual velocity 0.40. In any queue, the actual velocity of all cars is determined by the preferred velocity of the car in front.

The interstices between queues in figure 1 and in figures later in this paper serve as a visual representation of the subdivision into queues. The representation does not convey correctly the spatial structure of the queue pattern on a more detailed level. The actual distances between queues will increase with time, with a rate that depends on the queue velocities.

The average value of the actual velocities in queue patterns like that in figure 1 will be used to characterize cases (states). In the present case, the average velocity is

$$\langle v \rangle_c = \tfrac{1}{7}(5 \times 0.25 + 2 \times 0.40) \simeq 0.29.$$

In this particular case, the average velocity is markedly lower than 0.50. We shall soon see that this is true in most cases.

This is the first level of averaging: given a set of preferred velocities and given their specific ordering, find the average velocity. This is indicated by index c (for cars) in $\langle .. \rangle_c$. We have averaged over cars.

One soon realizes that a variety of queue patterns are possible, as a result of both different ways to order (different sequences) and different choices for the preferred velocities. Two further levels of averaging will appear, as discussed below.

In most cases, the transport is not optimal, since cars with low preferred velocities will slow down cars with higher preferred velocities. It turns out that as the number of cars increases, optimal or near-optimal transport becomes very rare.

Hemmer found that on average (per queue pattern), there is one car in each queue pattern that drives alone; in other words, there is on average one queue of length 1 [3]. The present paper focuses on the average velocity as a function of the number of cars.

3. Guessing

It is always useful to start by guessing. The students should first have opportunity to play with the model. One way is to construct more cases similar to that in figure 1. Another way is to first consider only one car (the average velocity must equal 0.5), then two cars (the fastest car will in some cases be slowed down by the slowest one; thus, the average velocity must be below 0.5), etc.

Small exercises like these will indicate that the average velocity decreases as the number of cars increases. The more cars, the higher the probability that further ahead there is a car with low preferred velocity. One may then ask for the average velocity $\langle v(N) \rangle$ as the number of cars N becomes very large. Will it approach some non-zero limiting value or will it approach 0?

An argument for 0 as the limiting value is as follows. Suppose that there is a non-zero limiting value for $\langle v(N) \rangle$ as $N \to \infty$. Then this non-zero value should be given by some combination of (tunable) parameters of the model. However, this model does not have any such parameters. Thus, the limiting value must be 0.

4. Sampling

A simple method for obtaining a numerical estimate is sampling: a number of samples (cases) are generated in a random way and one then averages the results. In the present case one generates a number of cases like that shown in figure 1, computes the average velocity in each case and then averages over cases.

Table 1. Average velocities for N cars in queues. Results in the first column (few samples) were obtained using a hand calculator to generate random numbers and ten samples for each N-value. The numerical estimates in the second column (many samples) were obtained with 250 000 samples for each N-value. The third column (lists) gives exact values as obtained from equation (5). The last column gives approximate values based on equation (6).

N	Few samples	Many samples	Lists—exact	$\langle v_1 \rangle$
1	0.549	0.498 991	$\frac{1}{2} = 0.5$	0.500
2	0.440	0.417 032	$\frac{5}{12} \simeq 0.4167$	0.333
3	0.370	0.361 270	$\frac{13}{36} \simeq 0.3611$	0.250
4	0.279	0.320 589	$\frac{77}{240} \simeq 0.3208$	0.200
5	0.320	0.290 028	$\frac{29}{100} = 0.29$	0.167
6	0.313	0.265 995	$\frac{223}{840} \simeq 0.2655$	0.143
7	0.191	0.245 339	$\frac{481}{1960} \simeq 0.2454$	0.125
8	0.201	0.228 300	$\frac{4609}{20160} \simeq 0.2286$	0.111

When writing a program for sampling the average velocity, one needs an algorithm for finding the actual velocities. Referring to the case shown in figure 1, the values in the third line must be found from the values in the first line. A simple algorithm is as follows. Find the car with the lowest preferred velocity. In figure 1, it has velocity 0.25. All cars behind (and including) the car have to keep this velocity. Then, find the slowest of the remaining cars (0.40 in the example); all cars between this car and the previously found car have the velocity of the second slowest car. Continue till the car found is the car in front.

Results from sampling for $N = 1$ to $N = 8$ are shown in the first two columns of table 1. The values should be compared to results obtained by other methods and approximations, as given in the other columns. The first column contains results from sampling using a hand calculator and few samples. This method is fast and gives an idea of the variability in the data, but the results are necessarily inaccurate.

5. List all alternatives

5.1. List all permutations

In this model, all preferred velocities are different. One may get a survey of all possible queue patterns by considering all permutations of velocities $v_1, v_2, \ldots, v_N$. Here, v_1 denotes the lowest preferred velocity, v_2 the lowest-but-one, etc, in a sample. Thus, in the sequence v_i the preferred velocities are arranged with monotonically increasing values. Consider the case in figure 1 as an example. Here one finds that $v_1 = 0.25$, $v_2 = 0.33$, etc.

In figure 2, all $3! = 6$ permutations of velocities v_1, v_2 and v_3 are shown (to the left), together with the queue patterns they lead to (in the middle). To the right, expressions for the average velocity of each queue pattern are given. Only actual velocities contribute to the average velocity. Consider the second line in figure 2 as an example. Here, the hindmost and foremost cars move at their preferred velocities (v_1 and v_2, respectively), while the car in the middle has actual velocity v_2, which is lower than the preferred velocity v_3. Thus, in this case, v_1 occurs once and v_2 twice as actual velocities.

In figure 2, some queue patterns arise twice (for two different permutations) while others come only once (for only one permutation). The pattern where all cars drive at their preferred velocity (upper line) occurs only for one permutation. This queue pattern has the highest

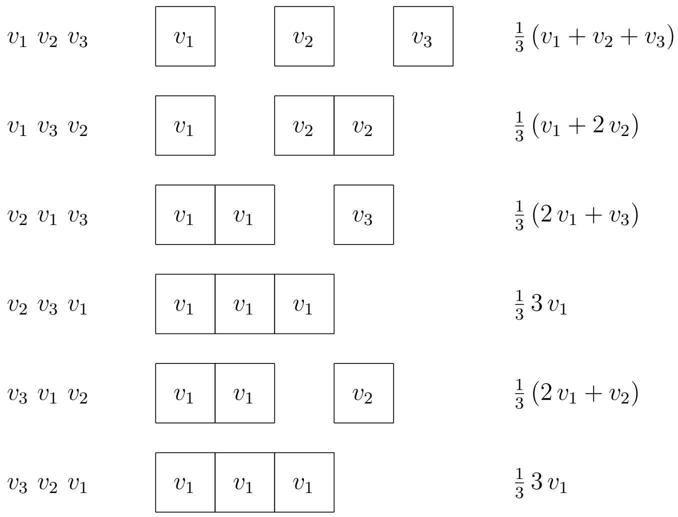

Figure 2. All possible queue patterns for $N = 3$ (3 cars). All possible permutations of the lowest (v_1), the lowest-but-one (v_2) and the highest (v_3) velocity are listed to the left. In the middle, the queue pattern is given for each case (each permutation), and to the right the expressions for the average velocity are given.

possible average velocity among the cases listed in figure 2. As N increases, still, only one among the $N!$ possible permutations leads to the highest possible average velocity, with all cars driving at their preferred speed. Thus, optimal transport becomes increasingly rare. There are four *distinct* queue patterns in figure 2: line 1, line 2, lines 3 and 5, and lines 4 and 6.

All permutations are equally probable. Thus, in the $N = 3$ case shown in figure 2, each queue pattern occurs with frequency $\frac{1}{6}$. Averaging over queue patterns, the average velocity becomes

$$\begin{aligned}\langle v(N=3)\rangle_{c,p} &= \tfrac{1}{6} \times \tfrac{1}{3}(v_1 + v_2 + v_3) + \tfrac{1}{6} \times \tfrac{1}{3}(v_1 + 2v_2) + \tfrac{1}{6} \times \tfrac{1}{3}(2v_1 + v_3) \\ &\quad + \tfrac{1}{6} \times \tfrac{1}{3}3v_1 + \tfrac{1}{6} \times \tfrac{1}{3}(2v_1 + v_2) + \tfrac{1}{6} \times \tfrac{1}{3}3v_1 \\ &= \tfrac{1}{3} \times \tfrac{1}{6}(12v_1 + 4v_2 + 2v_3).\end{aligned}$$

Here, we have averaged twice: first over actual car velocities for each permutation, leading to the expressions to the right in figure 2, and then over permutations, as indicated by c and p, respectively, in $\langle .. \rangle_{c,p}$. The above expression shows that $\langle v \rangle$ depends on all values v_1, v_2 and v_3, but v_1 dominates and v_2 is more important than v_3.

This manual method can be extended to higher N-values. For $N = 4$, the result is

$$\langle v(N=4)\rangle_{c,p} = \tfrac{1}{4} \times \tfrac{1}{24}(60v_1 + 20v_2 + 10v_3 + 6v_4).$$

However, since the number of permutations grows like $N!$, this approach soon becomes impracticable.

5.2. List all queue patterns

One can proceed further by considering instead the distinct queue patterns for each N-value. For $N = 3$, as shown in figure 2, there are four distinct patterns. For arbitrary N, the number of distinct queue patterns is 2^{N-1}. There are at least two ways to get this result, as follows.

Induction. All queue patterns for N can be obtained from the queue patterns for $N - 1$ by constructing two N patterns from each $N - 1$ pattern: by adding a car in front belonging to,

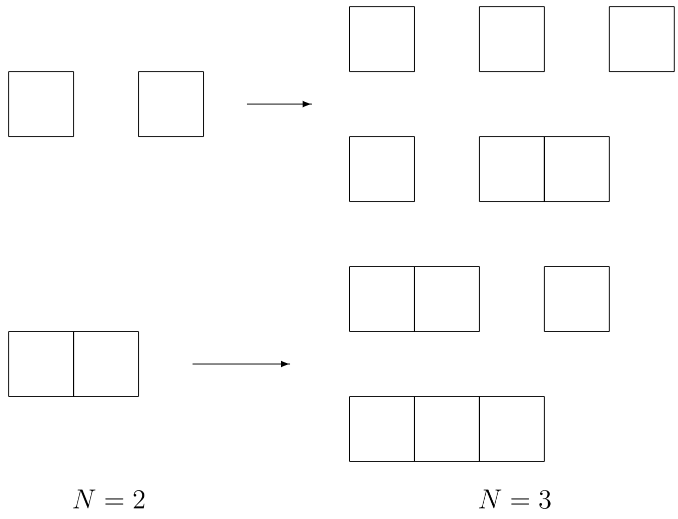

Figure 3. The generation of all possible queue patterns for three cars ($N = 3$) based on all possible queue patterns for two cars ($N = 2$). For each $N = 2$ pattern, two $N = 3$ patterns are constructed, by adding a car in front that is either disconnected (upper cases) or connected (lower cases) to the car (queue) behind it. The four queue patterns that are generated for $N = 3$ are identical to the four *distinct* patterns in figure 2.

and not belonging to, the foremost queue. This is illustrated for the case $N = 2 \longrightarrow N = 3$ in figure 3. Thus, the number of patterns is doubled when N increases by 1. For $N = 1$, there is only one queue pattern. Therefore, the number of queue patterns is 2^{N-1}.

All possible interstices. One may instead consider all possible ways to locate interstices between queues to form a queue pattern. The lowest possible number of interstices is 0, with all cars in a single queue (bottom cases in figure 3). The highest possible number of interstices is $N - 1$, with all cars separated (topmost cases in figure 3). There is only one pattern with 0 interstices, whereas 1 interstice can be at $N - 1$ different positions, each leading to a separate queue pattern. With 2 interstices there are $(N-1)(N-2)/(1 \times 2)$ queue patterns, since the interstices cannot be distinguished. Therefore, the number of queue patterns is

$$1 + \frac{N-1}{1!} + \frac{(N-1)(N-2)}{2!} + \cdots + \frac{(N-1)(N-2)(N-3)\cdot \ldots \cdot 1}{(N-1)!}$$
$$= \sum_{k=0}^{N-1} \binom{N-1}{k} = 2^{N-1}.$$

A list of all queue patterns, like in figure 3, is a more compressed representation of the alternatives than a list of all permutations, like in figure 2. Two challenges remain. One needs to find the number of permutations that leads to a certain queue pattern, *without* listing all the permutations. One needs to determine if all these sub-cases of one particular queue pattern have the same average velocity or not.

When seeking for the number of permutations that leads to a certain queue pattern, one needs to consider all possible exchanges of velocities v_i that will not change the pattern.

A simple example is the case of all cars in a single queue, as shown in the bottom cases in figure 3. This queue pattern is only possible as long as the slowest car is (locked) in the front. However, all other cars may have any position behind the front one. Thus, there are $(N-1)!$ permutations that generate this particular queue pattern. All cars have actual velocity v_1 in this case. Therefore, the average velocity is v_1 for all the $(N-1)!$ cases (for $N=3$ there are $(3-1)! = 2$ cases or permutations, as is also shown in figure 2).

As a further example, consider the case with a single car driving alone in front and the rest forming a queue behind, as in the lowest-but-one cases in figure 3. As in the previous example, only the slowest car has a locked position (as number 2); all others may be freely exchanged. Again, there are $(N-1)!$ permutations that generate the queue pattern. However, the permutations do not have the same average velocity, since in each case one car is driving alone in front at its preferred velocity. For $N = 4$ there are three sub-cases, each with average velocity

$$\bar{v}_j = \tfrac{1}{4}(3v_1 + v_j), \qquad j = 2, 3, 4.$$

Each of these sub-cases is generated by two permutations, corresponding to exchanges between the trailing two cars. Thus, there are indeed $3 \times 2 = (4-1)!$ permutations that lead to this queue pattern.

A still more involved example is given and explained in figure 4. The method is to start from the rear end of the queue pattern and work forwards. One needs to distinguish between cars that must be locked at certain positions for the queue pattern to be maintained and cars that may be freely exchanged, thereby generating new cases (permutations). Note that not all the 180 permutations found for this case have the same velocity.

A list of all queue patterns offers a more compact and faster way of working through all alternatives than a full list of permutations. The number of permutations is $N!$, which grows much faster than the number of queue patterns, 2^{N-1}. However, also treating the list of queue patterns manually soon becomes impracticable, and one needs to turn to computer listing.

5.3. Computer listing

The average velocity can be computed from lists of all permutations of velocities $v_1, v_2, \ldots, v_N$. All permutations for $N = 3$ are shown to the left in figure 2. Algorithms for generating all permutations for arbitrary N are given in [4].

Alternatively, permutations may be generated successively for increasing N by inserting v_N at all possible positions, including foremost and hindmost, in all $N-1$ permutations. Thus, from v_1v_2 (for $N = 2$) one obtains $v_3v_1v_2$, $v_1v_3v_2$ and $v_1v_2v_3$ (for $N = 3$), and from v_2v_1 (for $N = 2$) one obtains $v_3v_2v_1$, $v_2v_3v_1$ and $v_2v_1v_3$ (for $N = 3$).

For each permutation, the actual velocities may be found as explained in section 4. One then counts the occurrences of each v_i as actual velocity. In figure 2, these numbers are given to the right for each permutation (after excluding the $\frac{1}{3}$-factors). Let $c_{i,N}$ be the number of times velocity v_i occurs as actual velocity when considering *all* permutations for N cars. Consider the occurrences of v_1 for $N = 3$ as an example. Again referring to figure 2, one finds that $c_{1,3} = 1 + 1 + 2 + 3 + 2 + 3 = 12$.

In general, when all $c_{i,N}$ (counts) are determined, the average velocity is found as

$$\langle v \rangle_{c,p} = \frac{1}{N} \cdot \frac{1}{N!} \sum_{i=1}^{N} c_{i,N} \cdot v_i. \tag{1}$$

This equation should be compared to the expressions at the end of section 5.1. The prefactor $\frac{1}{N}$ corresponds to averaging over cars for one permutation, while $\frac{1}{N!}$ corresponds to averaging over permutations. A third level of averaging will be discussed in section 6.

Values for $c_{i,N}$ obtained from computer listing are given in table 2. For each value of N, $c_{i,N}$ decreases when i increases. The lower preferred velocities dominate when calculating the average value. This is a reflection of the fact that cars with low preferred velocities tend to

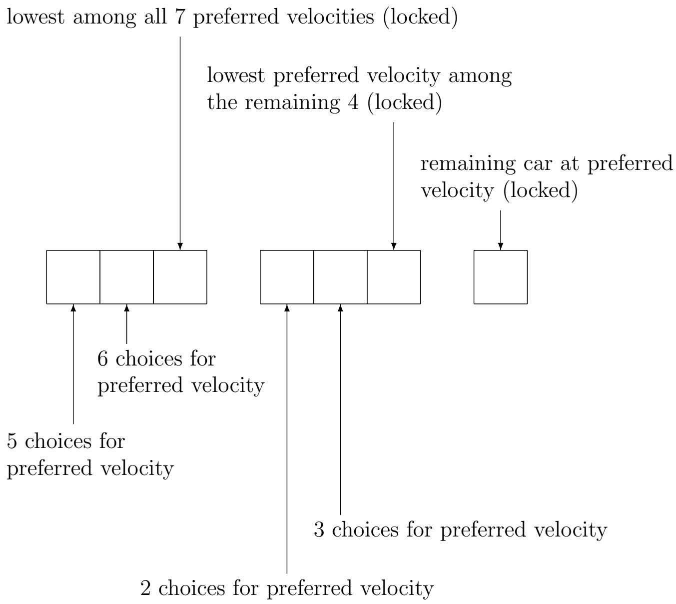

Figure 4. The figure shows an example of how to find the number of permutations that leads to a given (fixed) queue pattern. In the present pattern, there are three queues: a queue consisting of three cars hindmost, a queue consisting of three cars in the middle and a queue consisting of one car in front. The upper part of the figure contains explanations on positions where specific cars (that is, drivers with their preferred velocities) need to be placed for the queue pattern to be maintained. These positions are at the front of each queue. Since these cars are fixed at the specified positions, they do not generate new alternatives (permutations). At the positions explained in the lower part of the figure, on the other hand, several cars may be placed, leading to a number of alternatives (permutations). The main rule is to start with the hindmost queue and work forwards. The slowest car, among all seven, must be in front of the hindmost queue, else a different queue pattern will arise. However, at the position just behind, any of the remaining six cars may be placed. For each of these six choices, any of the remaining five cars may be placed at the last position of the queue. Then there are four cars left, but they may not be distributed freely at the remaining positions: the slowest among these four cars must be in front of the middle queue (the hindmost of the remaining queues) for the given queue pattern to be maintained. This car (which need not be the slowest-but-one among all seven cars) is locked at that position. Behind it, however, any of the remaining three cars may be placed, leaving two choices for the last position in the middle queue. The number of permutations that leads to this particular queue pattern is, therefore, $6 \times 5 \times 3 \times 2 = 180$.

slow down (impose low actual velocities on) several other cars further behind. Note that the values for $N = 3$ and $N = 4$ in table 2 were already given in section 5.1.

5.4. An integer triangle

The values in table 2 may seem arbitrary. However, there is a hidden structure, indicated by numbers A_N and B_N. Using these numbers, or multiplicators, all values for $c_{i,N}$ can be obtained from values for $c_{i,N-1}$.

Any $c_{i,N}$, for $i < N$, equals $c_{i,N-1}$ times the value in column A_N, row N. Formally, one may write $c_{i,N} = c_{i,N-1} \cdot A_N$. Taking $c_{2,5}$ as an example, one finds $c_{2,5} = c_{2,4} \cdot A_5 = 20 \times 6 = 120$, as also obtained from the computer listing.

Moreover, any $c_{N,N}$ may be obtained from $c_{N-1,N-1}$ by multiplying with the value in column B_N, row N. Formally, one may write $c_{N,N} = c_{N-1,N-1} \cdot B_N$. Using $c_{5,5}$ as an example, one gets $c_{5,5} = c_{4,4} \cdot B_5 = 6 \times 4 = 24$, which is the value from the computer listing.

Table 2. The table gives values for all $c_{i,N}$ for N (the number of cars) up to 8. $c_{i,N}$ is the total number of times v_i occurs as *actual* velocity when all permutations of *preferred* velocities are considered. v_1 is the lowest preferred velocity, v_2 the lowest-but-one preferred velocity and so on. It is necessary that $i \leqslant N$. All values were obtained by complete enumeration: all permutations were generated, actual velocities computed and the number of occurrences of $v_1, v_2, \ldots, v_N$ as actual velocities counted.

N	$c_{1,N}$	$c_{2,N}$	$c_{3,N}$	$c_{4,N}$	$c_{5,N}$	$c_{6,N}$	$c_{7,N}$	$c_{8,N}$	A_N	B_N
1	1									
2	3	1							3	1
3	12	4	2						4	2
4	60	20	10	6					5	3
5	360	120	60	36	24				6	4
6	2 520	840	420	252	168	120			7	5
7	20 160	6 720	3 360	2 016	1 344	960	720		8	6
8	181 440	60 480	30 240	18 144	12 096	8 640	6 480	5 040	9	7

The values for A_N and B_N increase by 1 when N increases by 1. Can this simple pattern be extended to an arbitrary value of N, thus allowing us to build an integer triangle without any computer listing? Formally, do expressions $A_N = N + 1$ and $B_N = N - 1$ hold for any N? The answer is yes, as shown in section 5.5.

Also, note that all $c_{i,N}$ values for a given N sum to $N \cdot N!$. Since there are $N!$ permutations of the velocities, and N actual velocities for each of these alternatives, there are altogether $N \cdot N!$ actual velocities. With $N = 5$ as an example, one finds $360 + 120 + 60 + 36 + 24 = 600$, which equals $5 \times 5! = 5 \times 120 = 600$. Formally, we write

$$\sum_{i=1}^{N} c_{i,N} = N \cdot N!,$$

which should be compared to equation (1). The weights $\frac{1}{N} \cdot \frac{1}{N!} \cdot c_{i,N}$ in that expression must sum to 1 (normalization), which is the same statement as made above.

5.5. Construction of the triangle

To explain the structure of the integer triangle in table 2, we use the method described in section 5.3 for obtaining all permutations for N from all permutations for $N - 1$: inserting the new car at all possible positions. This is illustrated in figure 5, where all alternatives for $N = 3$ are obtained from all alternatives for $N = 2$ by this method.

First, consider $c_{N,N}$, the diagonal entries in table 2. When the new velocity v_3 is inserted at all possible positions of an $N = 2$ alternative in figure 5, it will in almost all cases end up behind a slower car and thus have an actual velocity lower than v_3. Only in the last case for each $N = 2$ alternative, when the v_3 car is placed in front (cases in lines 3 and 6), the actual velocity will be v_3 and $c_{3,3}$ is increased by 1 (count added). This will be the case for any value of N. Thus, $c_{N,N}$ must be equal to the number of permutations for $N - 1$, that is,

$$c_{N,N} = (N - 1)! = (N - 1) \cdot (N - 2)! = B_N \cdot c_{N-1,N-1},$$

which is consistent with the values in table 2. We have found that $B_N = N - 1$, for all N.

Figure 5 also indicates how the values for $c_{i,N}$, with $i < N$, arise. There are two contributions to $c_{i,3}$. First, no occurrence of v_1 or v_2 as actual velocity for $N = 2$ is affected when v_3 is inserted. However, they are repeated three (in general, N) times. Consider the first $N = 2$ permutation, $v_1 v_2$ (preferred velocities), as an example. As shown in line 1 of figure 5, there is one occurrence of v_1 and one occurrence of v_2 as actual velocity for $N = 2$.

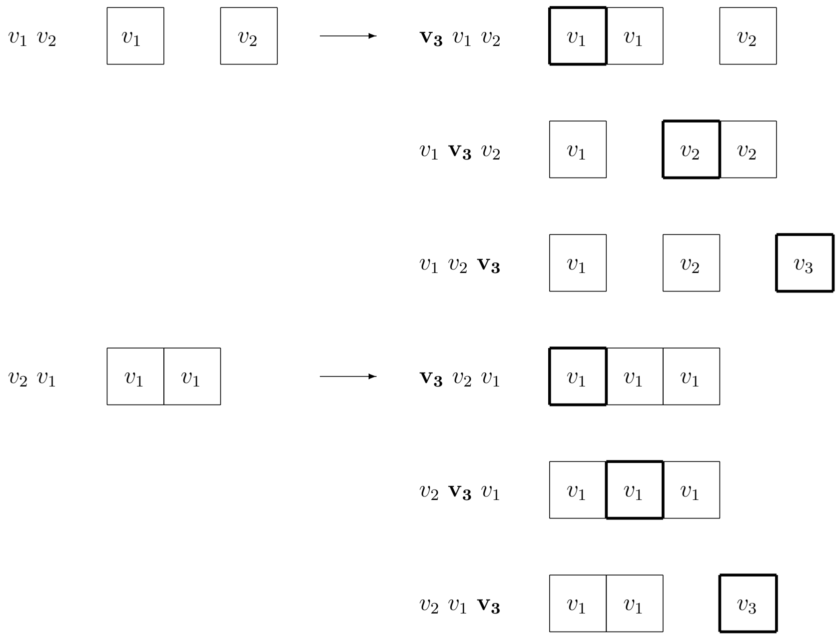

Figure 5. Generation of all permutations for $N = 3$, with queue patterns and actual velocities (to the right) from the two permutations and queue patterns for $N = 2$ (to the left). An additional car, with preferred velocity v_3, is inserted at one of the three possible positions. The inserted car is indicated by thick lines in the $N = 3$ queue patterns and shown with the actual velocity it gets. Unless in front (cases in lines 3 and 6), the inserted (added) car has an actual velocity set by one of the other cars (cases in lines 1, 2, 4 and 5).

From this $N = 2$ permutation three $N = 3$ permutations are generated, as shown. For each of these three cases, there is one occurrence of v_1 and one occurrence of v_2 as actual velocity (by repetition). This corresponds to boxes drawn with thin lines.

Second, the inserted car, with preferred velocity v_3, will have actual velocities that are lower than v_3 at all possible positions except the last (front) one (the front position, lines 3 and 6, was treated above). More precisely, the inserted car will have the actual velocity of the car directly in front of it in a queue. This car may, in turn, have had its actual velocity set by a car further on. In this way, one occurrence is added for N for each occurrence of an actual velocity for $N - 1$, since the inserted car will end up behind each of the cars in the $N = 2$ pattern exactly once. Again, consider the first $N = 2$ permutation, $v_1 v_2$ (preferred velocities), as an example. For this case, there is one occurrence of v_1 and one occurrence of v_2 as actual velocity for $N = 2$. The figure shows that the inserted car has actual velocity v_1 in line 1 and v_2 in line 2. This corresponds to boxes with thick lines.

Summing these two contributions, we find

$$c_{i,N} = N \cdot c_{i,N-1} + c_{i,N-1} = (N + 1) \cdot c_{i,N-1} = A_N \cdot c_{i,N-1},$$

which is again consistent with the values in table 2. Thus, $A_N = N + 1$, for all N.

6. Average over drivers

An expression for the average velocity was given in equation (1). There, we have averaged over cars (actual velocities for one particular sequence of cars) and over permutations (all possible sequences) of the preferred velocities $v_1, v_2, \ldots, v_N$.

Since all count values $c_{i,N}$ can be obtained from table 2, we can compute the average velocity for a given set of v_i values. For the case shown in figure 1, $v_1 = 0.25$, $v_2 = 0.33$, $v_3 = 0.37$, $v_4 = 0.40$, $v_5 = 0.57$, $v_6 = 0.70$ and $v_7 = 0.94$. One has

$$\langle v\rangle_{c,p} = \frac{1}{7} \times \frac{1}{7!}(c_{1,7} \cdot v_1 + c_{2,7} \cdot v_2 + c_{3,7} \cdot v_3 + c_{4,7} \cdot v_4 + c_{5,7} \cdot v_5 + c_{6,7} \cdot v_6 + c_{7,7} \cdot v_7).$$

By inserting numbers, one finds

$$\langle v\rangle_{c,p} \simeq 0.324.$$

This is the result when one averages over all permutations (all possible sequences) of the particular preferred velocities in figure 1. This value should be compared to the value 0.29 for the particular sequence in figure 1; see section 2.

However, we have still not obtained a general answer for the average velocity for a given number of cars, N. Only an answer for a given set of preferred velocities has been found. We now need to consider all possible choices for preferred velocities, that is, we need to average over drivers. Formally, we need to go from

$$\langle v\rangle_{c,p} = \frac{1}{N} \cdot \frac{1}{N!} \sum_{i=1}^{N} c_{i,N} \cdot v_i$$

(which is identical to equation (1)) to

$$\langle v\rangle_{c,p,d} = \frac{1}{N} \cdot \frac{1}{N!} \sum_{i=1}^{N} c_{i,N} \cdot \langle v_i\rangle. \tag{2}$$

On the right-hand side, only v_i depends on the preferred velocities (drivers). d in $\langle v\rangle_{c,p,d}$ indicates that an additional average over drivers has been taken.

To elaborate on equation (2), we will first consider $\langle v_i\rangle$ (see sections 6.1 and 6.2), then $c_{i,N}$ (see section 6.3) and finally combine the results (see section 6.4).

6.1. Sampling $\langle v_i\rangle$

The preferred velocities are random numbers between 0 and 1. The average value of random numbers between 0 and 1 is 0.5000. However, this does not mean that all average values of v_i, which will be denoted by $\langle v_i\rangle$, are equal to 0.5000. For N random numbers, the values $v_1, v_2, \ldots$ constitute an ordered sequence. When computing, e.g., $\langle v_1\rangle$, one averages the lowest values from sets that consist of N random numbers evenly distributed between 0 and 1. Therefore, one expects $\langle v_1\rangle$ to be lower than 0.5000.

Consider the following five sets of random numbers (preferred velocities) for $N = 4$:

0.05	0.10	0.59	0.54
0.55	0.37	0.47	0.36
0.03	0.09	0.11	0.84
0.33	0.12	0.78	0.27
0.63	0.92	0.93	0.30

Based on these sets of preferred velocities, one finds

$$\langle v_1\rangle = \tfrac{1}{5}(0.05 + 0.36 + 0.03 + 0.12 + 0.30) = 0.17.$$

In general, the value found for some $\langle v_i\rangle$ will depend on N. Again consider v_1 as an example. The larger a set of random numbers one has, the lower one expects the lowest of them to be. This is precisely what one finds. Results from sampling are shown in table 3, and for each column ($\langle v_1\rangle, \langle v_2\rangle, \ldots$) the values decrease when N increases.

Table 3. Numerical estimates for the average values of the lowest ($\langle v_1 \rangle$), the lowest-but-one ($\langle v_2 \rangle$), ..., the highest ($\langle v_N \rangle$) preferred velocity. For each N-value, a number of samples, each consisting of N random numbers between 0 and 1 (preferred velocities), were generated. The lowest (the lowest-but-one,..., the highest) value from each sample was then averaged. The number of samples for each N-value was around 1.5×10^6.

N	$\langle v_1 \rangle$	$\langle v_2 \rangle$	$\langle v_3 \rangle$	$\langle v_4 \rangle$	$\langle v_5 \rangle$	$\langle v_6 \rangle$	$\langle v_7 \rangle$	$\langle v_8 \rangle$
1	0.500 047							
2	0.333 246	0.666 824						
3	0.249 808	0.500 026	0.750 084					
4	0.200 163	0.400 459	0.600 376	0.800 349				
5	0.166 581	0.333 422	0.500 206	0.666 811	0.833 389			
6	0.142 645	0.285 536	0.428 455	0.571 426	0.714 371	0.857 219		
7	0.125 032	0.250 074	0.375 004	0.499 973	0.625 036	0.750 180	0.875 047	
8	0.111 198	0.222 296	0.333 342	0.444 772	0.555 434	0.666 604	0.777 650	0.888 842

By using table 3, the average velocities can be computed. Consider, as an example,

$$\langle v(N=2)\rangle_{c,p,d} = \frac{1}{2} \times \frac{1}{2!}(3\langle v_1\rangle + \langle v_2\rangle) = \frac{1}{2} \times \frac{1}{2!}(3 \times 0.333\,246 + 0.666\,824) = 0.416\,641.$$

This is the result after averaging over all cars in a given configuration (indicated by c in $\langle ..\rangle_{c,p,d}$), over all permutations (sequences) of a given set of preferred velocities (indicated by p) and all possible sets of preferred velocities (indicated by d).

As a further example, consider $N = 7$. Above, we found $\langle v(N=7)\rangle_{c,p} \simeq 0.324$ for the particular choice of preferred velocities in figure 1. Considering all sets of preferred velocities, we now find $\langle v(N=7)\rangle_{c,p,d} = 0.245\,447$.

These values, obtained using table 3, should be compared to the exact values given in table 1.

6.2. Calculating $\langle v_i \rangle$

An exact expression can be found for $\langle v_i \rangle$, thus making the sampling reported in table 3 superfluous. Before proceeding to calculations, it will be useful to attempt to guess the result based on the values in table 3.

First, consider $\langle v_1 \rangle$ for $N = 2$, that is, the average value of the lowest of two numbers, each uniformly distributed on [0, 1]. If v is the lowest of two numbers, the highest of the two numbers must lie in the interval $[v, 1]$. Since the numbers are uniformly distributed, the probability for the highest number to belong to $[v, 1]$ is $(1 - v)$. Thus, the probability distribution function $f_1(v)$ for the lowest number to be v is proportional to $(1 - v)$. The probability distribution function must be normalized:

$$\int_0^1 f_1(v)\,\mathrm{d}v = \int_0^1 A(1-v)\,\mathrm{d}v \stackrel{!}{=} 1,$$

which gives for the constant A the value 2. One then finds the average value as

$$\langle v_1\rangle = \int_0^1 v \cdot f_1(v)\,\mathrm{d}v = \int_0^1 v \cdot 2(1-v)\,\mathrm{d}v = \frac{1}{3}.$$

Similarly, the probability distribution function for the highest of the two numbers to have a value v must be proportional to v (which is the probability for the lowest number to belong to $[0, v]$), and one finds $\langle v_2 \rangle = \frac{2}{3}$.

Consider the average value of the lowest-but-one number among 4 ($N = 4$). The probability distribution function for the lowest-but-one number to have value v must be proportional to $v \cdot (1 - v)^2$, since it is necessary that one of the other three numbers is lower and the other two numbers higher than v. The constant has value 12, and one finds $\langle v_2 \rangle = \frac{2}{5}$.

In the general case, the probability distribution function for a preferred velocity number i in the *ordered* sequence $v_1, v_2, \ldots, v_{i-1}, v_i, v_{N-i}, \ldots, v_N$ is $A \cdot v^{i-1}(1-v)^{N-i}$. Here, for v_i in a sample to have value v, $i - 1$ values must be lower than v and $N - i$ higher. A is a constant to be found from normalization. Thus, one needs to calculate

$$\langle v_i \rangle = \frac{\int_0^1 v \cdot A \cdot v^{i-1}(1-v)^{N-i}\,\mathrm{d}v}{\int_0^1 A \cdot v^{i-1}(1-v)^{N-i}\,\mathrm{d}v} = \frac{\int_0^1 v^{i}(1-v)^{N-i}\,\mathrm{d}v}{\int_0^1 v^{i-1}(1-v)^{N-i}\,\mathrm{d}v}.$$

Using integration by parts, one obtains the formula

$$\int_0^1 x^s(1-x)^t\,\mathrm{d}x = \frac{s}{t+1}\int_0^1 x^{s-1}(1-x)^{t+1}\,\mathrm{d}x.$$

By repeated use of this formula, that is, by repeated use of integration by parts one finds

$$\langle v_i \rangle = \frac{\frac{i}{N-i+1} \cdot \int_0^1 v^{i-1}(1-v)^{N-i+1}\,\mathrm{d}v}{\frac{i-1}{N-i+1} \cdot \int_0^1 v^{i-2}(1-v)^{N-i+1}\,\mathrm{d}v} = \frac{\frac{i}{N-i+1} \cdot \frac{i-1}{N-i+2} \cdot \ldots \cdot \frac{2}{N-1} \cdot \frac{1}{N} \cdot \int_0^1 (1-v)^{N}\,\mathrm{d}v}{\frac{i-1}{N-i+1} \cdot \frac{i-2}{N-i+2} \cdot \ldots \cdot \frac{1}{N-1} \cdot \int_0^1 (1-v)^{N-1}\,\mathrm{d}v},$$

which gives

$$\langle v_i \rangle = \frac{i}{N+1}. \qquad (3)$$

One may argue that this result could have been written down directly, without calculations, from symmetry. The values in equation (3) divide the interval [0, 1] in $N + 1$ equal parts, which is reasonable.

6.3. An expression for $c_{i,N}$

The starting point is the integer triangle in table 2. We found in section 5.5 that any value $c_{i,N}$ (the number of times v_i occurs as actual velocity when there are N cars, considering all permutations of preferred velocities $v_1, v_2, \ldots, v_N$) could be obtained from values above it in the integer triangle using the multipliers A_N and B_N. Consider as an example the counts for actual velocity v_2 when $N = 5$:

$$c_{2,5} = c_{2,4} \cdot A_5 = c_{2,3} \cdot A_4 \cdot A_5 = c_{2,2} \cdot A_3 \cdot A_4 \cdot A_5 = c_{1,1} \cdot B_2 \cdot A_3 \cdot A_4 \cdot A_5$$
$$= 1 \times 1 \times 4 \times 5 \times 6 = \frac{1}{2 \times 3} \times 6!.$$

The other values for $N = 5$ can be obtained in the same way, moving vertically and then along the diagonal back to $c_{1,1}$. The results are

$$c_{1,5} = 1 \times 3 \times 4 \times 5 \times 6 = \frac{1}{1 \times 2} \times 6!$$
$$c_{2,5} = 1 \times 1 \times 4 \times 5 \times 6 = \frac{1}{2 \times 3} \times 6!$$
$$c_{3,5} = 1 \times 1 \times 2 \times 5 \times 6 = \frac{1}{3 \times 4} \times 6!$$
$$c_{4,5} = 1 \times 1 \times 2 \times 3 \times 6 = \frac{1}{4 \times 5} \times 6!$$
$$c_{5,5} = 1 \times 1 \times 2 \times 3 \times 4 = \frac{1}{5 \times 6} \times 6!.$$

For each $c_{i,5}$ the integers i and $i+1$ are missing in the sequence, corresponding to the point where the path in table 2 bends from vertical to diagonal and one shifts from A factors to B factors. The pattern is similar for any value of N, and one has

$$c_{i,N} = \frac{1}{i \cdot (i+1)} \cdot (N+1)!. \tag{4}$$

6.4. An expression for $\langle v \rangle_{c,p,d}$

We are now in position to find an expression for the main quantity of interest, $\langle v \rangle_{c,p,d}$, the average velocity of a queue pattern, when averaging over cars, permutations and drivers. The expressions in equations (3) and (4) are inserted into equation (2), and one finds

$$\langle v \rangle_{c,p,d} = \frac{1}{N} \cdot \frac{1}{N!} \sum_{i=1}^{N} c_{i,N} \cdot \langle v_i \rangle = \frac{1}{N} \cdot \frac{1}{N!} \sum_{i=1}^{N} \frac{1}{i \cdot (i+1)} \cdot (N+1)! \cdot \frac{i}{N+1},$$

which gives

$$\langle v \rangle_{c,p,d} = \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{1}{i+1}. \tag{5}$$

This expression is exact, and allows us to compute $\langle v \rangle_{c,p,d}$ for any value of N. Consider, as an example, $N = 3$, where

$$\langle v \rangle_{c,p,d} = \tfrac{1}{3} \times \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) = \tfrac{13}{36}.$$

All values in the third column in table 1 were obtained from equation (5).

7. How to use this paper

This paper could be used in a first statistical physics course. Parts of the material may be taken as an example in lectures. Student groups could be assigned to work through the above calculations in more detail or explore the additional problems outlined in section 7.2. One should always keep in mind that the most able students are often not sufficiently challenged.

Mathematical formulae have been kept simple, and many examples with numbers are included. In most cases, examples and general formulae are developed in parallel. All details of the calculations are not included; still the above description is much more explicit than in a standard research paper. The intention is to make it practicable for students to work through this paper or parts of it on their own.

The reader will encounter elements of the above calculations in physics courses. These include induction based on geometry (figure 3 and section 5.2, first part), distinguishable in contrast to indistinguishable units under permutations (section 5.2—all possible interstices), permutations under restrictions (figure 4), recursive relations (section 5.5) and the importance of normalization for probability distribution functions (section 6.2).

7.1. Concepts/Ideas

7.1.1. Power laws. In many situations, an average value is a robust characterization of a data set or a distribution function. The case considered in this paper is to some extent a counterexample. To illustrate this point, consider the approximation

$$\langle v \rangle_{c,p,d} \simeq \langle v_1 \rangle = \frac{1}{N+1}. \tag{6}$$

Here, we simply use the occurrences of v_1 when computing the mean velocity and disregard all other actual velocities, since the actual velocities are often low. In particular, v_1 occurs frequently as actual velocity. A certain number of occurrences of v_1 should then be averaged, and the result is of course $\langle v_1 \rangle$. One can then use equation (3).

This approximation is poor; see table 1. Attempts to improve it turn out to be unsuccessful, as outlined in section 7.2, part 4. Furthermore, the median and mid-interval of the sets of actual velocities shown in table 2 do not give good estimates for the average velocity. Thus, neither the lower nor the central part of the distribution function ($c_{i,N}$, as a function of i) is representative, and the entire distribution must be used to obtain accurate results.

In other words, $c_{i,N}$ (as a function of i for N fixed) falls off fast, but not fast enough that the upper tail of the distribution function can be neglected in calculations. From equation (4) one has $c_{i,N} \simeq A \cdot i^{-2}$, where A is a constant. Distribution functions of this form, called power laws, are of much interest in statistical physics. They signal that there is no characteristic scale (length scale, energy scale or similar) in the problem, which is an unusual situation.

When sampling, on the other hand, the entire distribution is by construction utilized. As shown in table 1 (first column) reasonable results are obtained even from few samples.

7.1.2. Levels of averaging. The quantity of interest in this paper is the average velocity in a queue pattern (a sequence of subqueues). To obtain this quantity, we had to perform averaging in three rounds.

First, we averaged over the *actual* velocities of one particular queue pattern. An example is shown in figure 1. This is referred to as averaging over cars, and the result is $\langle v \rangle_c$. The calculation and result for the example are given in section 2.

Second, we averaged over all possible sequences of the same, particular set of *preferred* velocities. This leads to $N!$ queue patterns, and we averaged over the $\langle v \rangle_c$-values obtained from the *actual* velocities of each of them. This is referred to as averaging over permutations, and the result is $\langle v \rangle_{c,p}$. For the case (the *preferred* velocities) in figure 1, the calculation and result are given at the beginning of section 6.

Third, we averaged over all possible choices for the preferred velocities. This is referred to as averaging over drivers, and the result is $\langle v \rangle_{c,p,d}$. The result for the case used as an example above ($N = 7$) is given at the end of section 6.1 (using numerical estimates for $\langle v_i \rangle$) and in the third column of table 1 (using the exact expression in equation (5)).

This sequence of rounds (or levels) of averaging seems natural. However, one could imagine other sequences or methods. One could claim that $\langle v \rangle_{c,d} = \langle v \rangle_{c,p,d}$ and that the averaging over permutations was merely a computational aid. When one finds a numerical estimate by sampling, it is $\langle v \rangle_{c,d}$ that is computed. In our case, the final result does not depend on the method (sequence) used. There are problems in statistical physics where this is no longer true.

7.1.3. Numerical simulations. Numerical simulations are emphasized in this paper. Simulations (sampling) are even used for quantities that are later calculated exactly. One example is the average velocity in a queue, our main quantity (exact result in the third column of table 1, results from sampling in the first and second columns). Another example is $\langle v_i \rangle$, the average value of the preferred velocity number i, when arranged in an increasing order (exact result in equation (3), results from sampling in table 3). If one has struggled with numerical simulations for some time, one appreciates exact solutions more.

Results obtained using few samples and a hand calculator are given in the first column of table 1. This paper suggests using hand calculators as a fast method to get estimates for quantities of interest. The random number generators may not be of good quality. Still, in a short time one can obtain useful results, as shown in table 1. In courses, this approach may supplement more extensive sampling where one needs to write a computer program.

Sampling should be distinguished from lists of all alternatives (enumeration). Enumeration was used to generate table 2. However, it turned out that all values could be calculated from simple formulae. Thus, the computer listing of all alternatives (enumeration) was redundant. Still, in courses, listing may serve as a useful exercise and a preparation for the calculations in sections 5.4 and 5.5.

7.1.4. Related systems. The model system considered in this paper is one-dimensional; all cars follow the same path (a narrow road that does not allow overtaking). This construction leads to a bottleneck effect, where cars with low preferred velocities slow down cars with higher preferred velocities. This is the mechanism behind the decrease in queue velocity when N, the number of cars, increases.

The system and the mechanism may seem artificial. However, similar one-dimensional systems are of much interest in statistical physics. Two examples will be briefly mentioned.

In the *totally asymmetric exclusion process*, particles are moved in a stochastic manner in one (specified or imposed) direction along a line. Between moves, the particles occupy a number of specified positions (lattice points) along the line. Each of these positions can contain at most one particle. Due to the one-dimensional nature of the system, the current of particles is very sensitive to boundary conditions and disorder. For example, if one allows a finite fraction of particles to also have moves against the main direction, for specified sites or for specified particles, a state with vanishing current can be reached [5].

Consider a porous medium where all pores are filled with a fluid. If a second fluid of lower density is injected into the porous medium from a point source, it will tend to move upwards due to buoyancy. During this upward motion, the invading fluid has to follow the pore space. Experiments show that the competition between buoyancy and the pore geometry leads to vertically aligned structures. These invasion structures are quasi-one-dimensional, with short side branches but no loops [6].

7.2. Extensions/Problems

7.2.1. Distributions. The main quantity of interest in this paper has been the queue velocity. We found in equation (5) an exact expression for its average value, after having averaged over cars, over permutations of cars and over drivers. One obtains more detailed information if one computes *probability distribution functions* for the average velocity from many realization (sampling or lists).

Each realization (sample) only involves averaging over cars (over actual velocities in the specific sample), and the result may fall anywhere in [0, 1]. Using many samples one may find the fraction of cases (samples) for which the average velocity is, e.g., between 0 and 0.05. With results for all intervals in some subdivision of [0, 1], one obtains the probability distribution function for the average velocity.

It is useful to make a guess on the shape of these probability distribution functions before computing them. The shape as obtained from computations may be surprising. Note the changes as N increases.

7.2.2. Susceptibility. For a given N (the number of cars), the average velocity is rather insensitive to changes like omitting a number of arbitrarily chosen configurations. Single configurations, like that shown in figure 1, may be more susceptible to perturbations (changes).

As an example, consider the probability that the average velocity is altered when two arbitrarily chosen cars are exchanged, that is, when two *preferred* velocities change position. In figure 1, the average velocity does not change if 0.33 and 0.57 are exchanged or if 0.94 and 0.70 are exchanged but changes if 0.25 and 0.94 are exchanged or if 0.37 and 0.70 are exchanged. There are 21 possible exchanges of this type: six where the preferred velocity 0.37 is involved, further five with 0.33 and so on—a total of $6 + 5 + 4 + 3 + 2 + 1 = 21$.

Working through all 21 exchanges of this type in figure 1, one finds 13 cases with changed average velocity. Thus, the probability for a changed average velocity is $13/21 \simeq 0.62$. In most of these cases, the queue pattern is also changed. However, there are some cases where the queue pattern is unaltered (the 5-plus-2 pattern shown in figure 1 is retained) but the average velocity is changed. One finds 10 cases with a changed queue pattern. Thus, the probability of having a changed queue pattern is $10/21 \simeq 0.48$.

One may also average the actual velocity over the set of configurations that is generated by exchanging two preferred velocities once. Including the configuration in figure 1, that is, with altogether 22 configurations, the answer is $\langle v\rangle_{c,p^*} \simeq 0.305$. Here, p^* denotes the restricted set of permutations described above.

This does not correspond to a full average over permutations, since only a subset of the $7! = 5040$ permutations of the preferred velocities is generated. The average velocity when including all permutations of the preferred velocities in figure 1 was obtained at the beginning of section 6: $\langle v\rangle_{c,p} \simeq 0.324$. This value is not far from $\langle v\rangle_{c,p^*}$. Thus, the 22 configurations that are connected through the exchanges described above sample reasonably well the full set of 5040 configurations.

However, both $\langle v\rangle_{c,p}$ and $\langle v\rangle_{c,p^*}$ deviate from $\langle v\rangle_{c,p,d} \simeq 0.2454$ (see table 1), the result obtained after averaging over drivers in addition. This can be understood from the large weight the lowest preferred velocities have in average-velocity calculations (see the expression for $\langle v\rangle_{c,p}$ at the beginning of section 6), since $c_{1,7} > c_{2,7} > c_{3,7} > \cdots$. For the specific choice of preferred velocities in figure 1, both v_1 and v_2 are higher than average values from equation (3): $v_1 = 0.25 > 1/(7+1) = 0.125$ and $v_2 = 0.33 > 2/(7+1) = 0.25$. This lifts the value above $\langle v\rangle_{c,p,d}$.

7.2.3. Correlations. Estimates obtained from numerical simulations are necessarily inaccurate, that is, they deviate more or less from the (known or unknown) exact values. One may assume that these deviations are entirely random, but this is not always the case.

In table 3 numerical estimates are given for $\langle v_i\rangle$, that is, the average value of the lowest random number (preferred velocity) of N ($\langle v_1\rangle$), the lowest-but-one ($\langle v_2\rangle$), etc. For $\langle v_i\rangle$, we obtained the expression in equation (3); thus, the deviations of the numerical estimates from the exact values may be computed. These deviations are correlated.

Consider $N = 8$ as an example. Here, the first four estimates ($i = 1, 2, 3, 4$) are slightly too high, while the last four estimates ($i = 5, 6, 7, 8$) are slightly too low. There are similar patterns for the other N-values. The deviation between the numerical estimates and the exact values does not change in a stochastic way, but only slowly when i increases. The correlations can be quantified by comparing to randomized cases, that is, several realizations for the same N value, and random exchanges of $\langle v_i\rangle$ values between the realizations.

The correlations can be understood from the random numbers used. The random numbers are supposed to be uniformly distributed on the interval [0, 1]. Any realization, however, will deviate slightly from the uniform case. Consider a realization where there are fewer random numbers below 0.5 than in the uniform case. For $N = 8$, $\langle v_1\rangle$, $\langle v_2\rangle$, $\langle v_3\rangle$ and $\langle v_4\rangle$ will then tend to be too high.

It is a useful exercise to sketch how the actual probability distribution of random numbers must have been in the sampling, based on these deviations.

7.2.4. Approximations. It was concluded in section 7.1 that $\langle v_1 \rangle$ is a poor approximation for $\langle v \rangle_{c,p,d}$. One may try to mend it by using a fixed fraction of actual velocities, extending from the lowest and upwards. As an example, consider a fraction of 0.80 for $N = 4$. From the 96 actual velocities one then uses the 76.8 lowest, that is, all 60 v_1 and 16.8 of v_2. One then has

$$\langle v \rangle_{c,p,d} \simeq \frac{1}{76.8}\left(60\langle v_1 \rangle + 16.8\langle v_2 \rangle\right) = \frac{1}{76.8}\left(60 \times \frac{1}{5} + 16.8 \times \frac{2}{5}\right) \simeq 0.244.$$

The deviations from the exact results in table 1 are large for all $N > 1$, even at high fractions. The relative error as compared to the exact values increases with N. For $N = 8$ and a fraction of 0.90, it is 22%. Thus, this is also a poor approximation.

The median actual velocity may also be tried. It turns out to be an even worse approximation. Mid-intervals of actual velocities also lead to poor estimates for the average velocity in the queue. The central 300 cases for $N = 5$ (a fraction of 0.50 of all cases) give a relative error of 25%. Other variants could be explored.

7.2.5. Queue distributions. Using a similar procedure as indicated in figure 5, one can find the number of queues as a function of length. The result is that there are $(1/L)N!$ queues of length L when considering all $N!$ permutations of the N cars. Thus, there is on average (per queue pattern) one *car* that belongs to a queue of length L, for each L. This extends the result found in [3] to all queue lengths. Derivation of this result, as well as further extensions, will be given elsewhere.

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