Velocity and cluster distributions in a bottleneck system

Vidar Frette[1,*] and Per C. Hemmer[2,†]
[1]*Department of Engineering, Stord/Haugesund College, Bjørnsonsgt. 45, N-5528 Haugesund, Norway*
[2]*Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway*
(Received 4 June 2009; revised manuscript received 9 September 2009; published 13 November 2009)

Velocity and cluster distributions for particles with unidirectional motion in one dimension are studied. The particles never pass each other, like cars on a narrow road that does not allow overtaking. As a result, particles cluster behind slow particles (queues are formed behind slow cars). Thus, the *actual* velocity of each particle is to a large extent determined by slow particles further ahead. Considering all possible permutations of N particles with *initial* velocities $\{v_i\}$, the average number of particles with *actual* velocity v_i is $(N+1)/[i(i+1)]$ (in the sequence $\{v_i\}$, the initial velocities are listed with monotonically increasing values). For i large and $v_i \propto i$ the average number of actual velocities is thus a power law in v_i, even though the average cluster density is found to be independent of cluster size, L. On the other hand, the cluster density varies significantly with cluster velocity; we obtain $[(N-i)!(N-L)!]/[N\cdot N!(N-L-i+1)!]$. The average velocity at a given position in the sequence of N particles, and the average global velocity are determined. Explicit results for several distributions of the initial velocities show that the global velocity depends sensitively on the form of this distribution.

DOI: 10.1103/PhysRevE.80.051115 PACS number(s): 02.50.−r, 05.40.−a

I. INTRODUCTION

Particle systems have a widespread use as models of transport processes; from hydrodynamics to emergency evacuation of buildings. In all cases, configurations of particles restricted in a certain geometry are updated according to given rules. With properly chosen updating schemes, particle systems are able to mimic large classes of transport phenomena realistically.

When restricted to one dimension, these systems are susceptible to single particles with irregular behavior. Strong effects may arise, since one particle potentially may influence the motion of a large number of others. This is highlighted in the asymmetric exclusion process, a simple model system for nonequilibrium transport in one dimension [1–3]. Here, each lattice site may be empty or contain one particle. The particle configuration is updated by shifting a randomly chosen particle one step in a preferred direction, when possible (when the neighboring site is empty). The global particle current in the system has been found to be very sensitive to boundary conditions [4] as well as sites [5–8] and particles [9–12] with anomal properties, since bottlenecks are formed. Experimental situations to which this type of models are related range from highway traffic [13] to ant motion on trails [14].

In this contribution, we analyze a one-dimensional particle system in which any particle may form a bottleneck. The particles all have different potential as to formation of bottlenecks. However, there are no correlations between particle position and its bottleneck potential. Such a system may be said to possess *hierarchical disorder*. We will show that this seemingly weak property has several striking consequences.

*vidar.frette@hsh.no
†per.hemmer@ntnu.no

This paper is organized as follows. In Sec. II the particle system is described and previous results summarized. The number of clusters formed, as function of cluster size and cluster velocity, is computed in Sec. III. Based on this result, we find in Sec. IV the number of particles moving with a given velocity. The average particle velocity is calculated in Sec. V for several distributions of the initial velocities. A discussion of the results and some concluding remarks are given in Secs. VI and VII, respectively.

II. PARTICLE SYSTEM

Consider particles that move along a line (cars that drive along a narrow road), as shown in Fig. 1. All particles have the same direction of motion. Each particle has an initial velocity at which it will move if alone, as indicated in the upper line of the figure. All these *initial* (preferred) velocities are different. The particles are not allowed to pass each other, as with cars on a narrow road that does not allow overtaking.

The order of the particles along the line is random, as when a large number of cars disembark from a ferry onto a

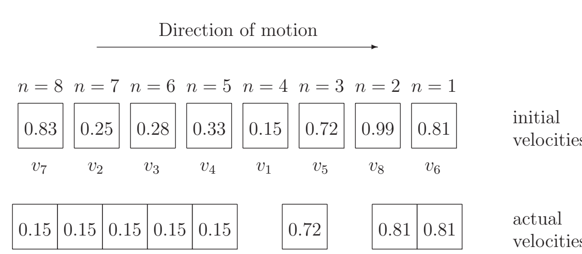

FIG. 1. An example of clusters formed with $N=8$ particles in unidirectional motion to the right. The upper line shows the initial velocity of each particle, with values between 0 and 1. The lower line indicates the clusters formed due to slow particles: cluster sizes are 2, 1, and 5. The actual velocity of each particle is given in this line. The numbering of the N sites is indicated: $n=1$ corresponds to the foremost particle.

single-lane road. Thus, there are no correlations between initial velocity and position. We list the velocities with monotonically increasing values, $v_1 < v_2 < \ldots < v_N$, which will be referred to as the sequence $\{v_i\}$.

Due to clustering behind slow particles (queues behind slow cars) the *actual* velocities will differ from the initial ones. There will be more low velocities and less high velocities after clustering. This tendency is also seen for the case in Fig. 1, where the lower line shows the clusters (queues) and the actual velocities.

Any realization may be represented as a certain permutation of the initial velocities v_i. For the case in Fig. 1 one has $v_7 v_2 v_3 v_4 v_1 v_5 v_8 v_6$. We will consider the ensemble of all $N!$ permutations of velocities v_i. For each new realization, new numerical values are drawn for initial velocities v_i. Note that several of our quantitative results below are independent of the initial velocity distribution and depend only on the velocity rank number i.

A similar model, with clustering behind slow particles, was studied by Ben-Naim and co-workers [15]. However, they focused on the kinetics of the clustering process, where as our interest is in the statistics of the fully clustered state. In the literature on exclusion models, these designs belong to the class of particle-wise disorder [16,17].

The present particle system was first described by Hemmer [18], who found that on average, that is, per realization or case, there is 1 cluster of length 1. The argument is reviewed in Sec. III A. An extension of this result is given in Sec. III B.

In Ref. [19] the distribution of *actual* velocities was found, as a function of the velocity rank number i. The total number of occurrences of v_i as *actual* velocity, when considering all $N!$ permutations, is

$$\mathcal{A}_i = \frac{1}{i \cdot (i+1)} \cdot (N+1)\,! \,, \tag{1}$$

which falls off rapidly with increasing i. We will give a more concise and much shorter derivation of this result in Sec. IV below.

For any realization (case) one may compute the average over actual velocities. For the case in Fig. 1 one finds 0.39. Considering initial velocities uniformly distributed on [0,1] and averaging over all realizations, the average cluster (queue) velocity was found in Ref. [19] to be

$$\langle v \rangle = \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{1}{i+1}, \tag{2}$$

which falls off with increasing N. Based on our main result in Eq. (16), we rederive this expression in Sec. V.

III. CLUSTER NUMBER DENSITIES

Both the number of clusters and their sizes vary much from realization to realization. For the case in Fig. 1 there are three queues, of sizes 2, 1, and 5. However, for $N=8$, one may also have only one cluster of size 8 (when the particle in front has velocity v_1), eight clusters of size 1 (when initial velocities happen to be arranged with monotonically increasing values along the line), as well as several other configurations.

The average number of clusters of a given size L is computed in Secs. III A and III B. This quantity, however, only give partial information on the cluster structures, since actual velocities are not specified. In Sec. III C we calculate the average number of clusters, as a function of both size and actual velocity. The expression obtained in Eq. (16) is the main result of this paper.

A. Cluster size $L=1$

In Ref. [18] the average number of clusters of size 1 was calculated. The average is over realizations, i.e., over the $N!$ permutations of initial velocities v_i. The calculation is reviewed here.

Consider N particles with some cluster structure. Let $B_1(N)$ be the average number of clusters of size 1. To calculate this quantity, we remove the particle with the highest initial velocity (v_N), leaving the remaining structure unchanged. The average number of clusters of size 1 in the remaining structure is $B_1(N-1)$. We will now calculate the average change in the number of clusters of size 1 as the removed particle is again added to the cluster structure. The particle is, however, not necessarily added at the position it had before, but to any of N possible positions in the $N-1$-particle structure, with equal probability $1/N$. Thereby, the *average* change in the number of cluster of size 1 is obtained.

The addition of the particle with highest initial velocity may affect the number of clusters of size 1 in two ways. If added in front, it will increase the number of clusters of size 1 by 1. At any of the remaining $N-1$ possible positions, the particle will be part of a larger cluster, since there will be slower particles further ahead. In particular, when the particle is added at the position behind a cluster of size 1, the number of such clusters will be reduced by 1. This will happen in $B_1(N-1)$ cases. Thus,

$$B_1(N) - B_1(N-1) = \frac{1}{N} - B_1(N-1) \cdot \frac{1}{N}, \tag{3}$$

which may be rewritten as

$$N \cdot [B_1(N) - 1] = (N-1) \cdot [B_1(N-1) - 1]. \tag{4}$$

Here, if $B_1(N-1)=1$ one has $B_1(N)=1$. Since $B_1(1)=1$ one obtains

$$B_1(N) = 1, \tag{5}$$

for all N. On average, there is 1 cluster of size 1, for any N.

B. Cluster size $L \geq 1$

In Sec. III A above, the average number of clusters of size 1 was shown to be $B_1(N)=1$ for all N. The average number of clusters of size L, $B_L(N)$, will now be calculated.

The number of clusters of size L, $\mathcal{B}_L(N)$, considering jointly all permutations of velocities v_i, will be found first. Obviously, $B_L(N)=\mathcal{B}_L(N)/N!$. Consider the changes in clusters that occurs when all cases for N is generated from all cases for $N-1$ by inserting a new particle, with velocity v_N, at all N possible positions. When v_N is located in front, it forms a new cluster of size 1. Thus, this happens once for each of the $(N-1)!$ permutations of the $N-1$ particles.

At all other positions, the new particle will be part of a cluster of size $L\geq 2$, increasing its length by 1. Hence a new cluster of length L is generated when the particle is added to an existing cluster of size $L-1$, in $L-1$ possible positions. On the other hand, L-clusters are left unchanged, and repeated, as the particle is added in $N-L$ different positions. Thus,

$$\mathcal{B}_L(N) = (L-1)\cdot\mathcal{B}_{L-1}(N-1) + (N-L)\cdot\mathcal{B}_L(N-1). \quad (6)$$

We will prove by induction that the solution is

$$\mathcal{B}_L(N) = \frac{N!}{L}. \quad (7)$$

From Eq. (5) one has $\mathcal{B}_1(N)=B_1(N)\cdot N!=N!/1$, which has the form of Eq. (7). For $L=2$, Eq. (6) becomes

$$\mathcal{B}_2(N) = \mathcal{B}_1(N-1) + (N-2)\cdot\mathcal{B}_2(N-1). \quad (8)$$

Assuming that $\mathcal{B}_2(N-1)=(N-1)!/2$ one has

$$\mathcal{B}_2(N) = \frac{(N-1)!}{1} + (N-2)\cdot\frac{(N-1)!}{2} = \frac{N!}{2}. \quad (9)$$

Since one easily finds that $\mathcal{B}_2(2)=1=2!/2$, the induction is complete for $L=2$. Successive inductions for higher values of L establish the expression in Eq. (7) for all values of N and all $L\leq N$. It follows that

$$B_L(N) = \frac{\mathcal{B}_L(N)}{N!} = \frac{1}{L}. \quad (10)$$

The average number of *particles* that belong to a cluster of size L is thus $L\cdot\frac{1}{L}=1$, for all L and N. Thus, on average there is 1 particle that is part of a cluster of size L, for each L. In this sense the cluster structures may be said to be *homogeneous*.

However, this simple result does not allow us to calculate the average velocity [see Eq. (2) above] or related quantities, since clusters of the same size will not all move at the same actual velocity. The cluster number density, as function of both cluster size and actual velocity, will be calculated in Sec. III C below. A beautiful structure will emerge when the homogeneous distribution in Eq. (10) is decomposed.

C. Velocity dependency

We want to determine the average number $C_{i,L}$ of clusters (queues) with velocity v_i and length L. To determine $C_{i,L}$, we enumerate how many of the altogether $N!$ possible permutations of the disembarking cars (particles) produce queues as specified.

Clusters with velocity v_i must have particle number i as the leading one, and all the $i-1$ slower particles behind this cluster. For the moment we assume $i>1$. Moreover, the particle just behind our cluster must be one of these slower particles, otherwise no gap will arise. Let there be altogether be $k-1$ particles behind the particle with velocity v_i. Of these L belongs to the cluster we consider, with $k-L$ particles behind the cluster. Of these $k-L$ particles, the one in front must be one of the $i-1$ slower particles, with initial (preferred) velocities lower that v_i, while the remaining $i-2$ slower particles can be distributed freely among $k-L-1$ positions. Thus, there are

$$(i-1)\frac{(k-L-1)!}{(k-L+1-i)!} \quad (11)$$

different allowed configurations of the $i-1$ slowest particles.

When the i (initially) slowest particles are distributed, the remaining $N-i$ faster particles can be distributed in $(N-i)!$ different ways among the remaining $N-i$ positions. These $N-i$ positions include positions inside the cluster considered, as well as behind it and in front of it. For a fixed position k of the particle with initial velocity v_i there are thus

$$(i-1)\frac{(k-L-1)!}{(k-L+1-i)!}(N-i)! \quad (12)$$

allowed permutations of the remaining particles. Summing over the possible positions for particle i, and dividing by the total number $N!$ of realizations, we obtain

$$C_{i,L} = \frac{1}{N!}\sum_{k=L+i-1}^{N}(i-1)\frac{(k-L-1)!}{(k-L+1-i)!}(N-i)! = \frac{(i-1)(N-i)!}{N!}\sum_{s=i-2}^{N-L-1}\frac{s!}{(s-i-2)!}, \quad (13)$$

which may be rewritten as

$$C_{i,L} = \frac{(i-1)!\,(N-i)!}{N!}\sum_{s=i-2}^{N-L-1}\binom{s}{i-2}. \quad (14)$$

Using the identity

$$\sum_{s=0}^{M-1}\binom{s}{j-1} = \binom{M}{j}, \quad (15)$$

which is proved in the appendix, we find

$$C_{i,L} = \frac{(N-i)!\,(N-L)!}{N!\,(N-L-i+1)!}. \quad (16)$$

Above we assumed $i>1$. But the formula in Eq. (16) is also valid for the slowest particle, with $i=1$: each of the N possible positions of the slowest particle occurs with probability $1/N$. For all these positions, a cluster moving with velocity v_1 and extending from the chosen position and backward throughout the system is generated. Thus, $C_{1,L}=1/N$ for any L, in accordance with Eq. (16). Numerical examples are shown in Table I.

Note the symmetry $C_{i,L}=C_{L,i}$: on average, there are as many clusters of length L and velocity v_i as clusters of length i with velocity v_L. In particular, $C_{1,k}=C_{k,1}$, and since $C_{1,L}=1/N$ for all L (see above), all these values are identical, for each N value (first row and first column as displayed in Table I). For fixed $i>1$, $C_{i,L}$ falls off with increasing L, the higher

TABLE I. Examples of $C_{i,L}$, the average number of clusters with velocity v_i and size L, as obtained from Eq. (16).

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
			$N=2$			
v_1	1/2	1/2				
v_2	1/2					
			$N=4$			
v_1	3/12	3/12	3/12	3/12		
v_2	3/12	2/12	1/12			
v_3	3/12	1/12				
v_4	3/12					
			$N=6$			
v_1	10/60	10/60	10/60	10/60	10/60	10/60
v_2	10/60	8/60	6/60	4/60	2/60	
v_3	10/60	6/60	3/60	1/60		
v_4	10/60	4/60	1/60			
v_5	10/60	2/60				
v_6	10/60					

i value the faster. The expression in Eq. (16) can also be obtained using methods similar to those in Sec. III B, but the derivation is more lengthy.

From our main result in Eq. (16), several additional formulas may be derived. We will start by calculating the total number of clusters with velocity v_i, averaged over realizations, D_i. This corresponds to summing the values in a given row in Table I. One obtains

$$D_i = \sum_{L=1}^{N} C_{i,L} = \frac{(N-i)!}{N!}\sum_{L=1}^{N} \frac{(N-L)!}{(N-L-i+1)!}$$
$$= \frac{(N-i)\,!\,(i-1)!}{N!}\sum_{L=1}^{N}\binom{N-L}{i-1}$$
$$= \frac{(N-i)\,!\,(i-1)!}{N!}\sum_{s=0}^{N-1}\binom{s}{i-1}.$$

By means of the identity (15) we obtain

$$D_i = \frac{(N-i)\,!\,(i-1)!}{N!}\binom{N}{i} = \frac{1}{i}. \tag{17}$$

For $i=1$ we find $D_1=1$, as it should be, since the slowest particle causes precisely one cluster for any realization. For a large number of particles the average total number of clusters grows logarithmically,

$$D_{\rm tot} = \sum_{i=1}^{N} D_i \simeq \ln N + \mathcal{C}, \tag{18}$$

where $\mathcal{C}=0.57722\ldots$ is Euler's constant.

Due to the symmetry $C_{i,L}=C_{L,i}$, summing values in a row in Table I is equivalent to summing values in the column with the same value for the velocity rank number i. Thus, from Eq. (17) B_L, the average number of clusters of size L, irrespectively of velocity, must be $B_L=1/L$, a result already obtained in a different way above [see Eq. (10)].

IV. OCCUPATION NUMBERS

In the previous section, expressions for the number of clusters of different types, averaged over the $N!$ realizations, were found. For cluster velocity v_1 the number of clusters is constant as a function of L, whereas it falls off with L for any v_i with $i>1$, see Table I. On the other hand, the larger L, the more particles in the cluster.

The average total number of *particles* with velocity v_i is

$$A_i = \sum_{L=1}^{N} LC_{i,L} = \frac{(N-i)!}{N!}\sum_{L=1}^{N}\frac{L(N-L)!}{(N-L-i+1)!}$$
$$= \frac{(N-i)\,!\,(i-1)!}{N!}\sum_{L=1}^{N} L\binom{N-L}{i-1}. \tag{19}$$

Using the identity

$$\sum_{j=1}^{M} j\binom{M-j}{k} = \binom{M+1}{k+2}, \tag{20}$$

proved in the appendix, we obtain

$$A_i = \frac{(i-1)\,!\,(N-i)!}{N!}\binom{N+1}{i+1} = \frac{N+1}{i(i+1)}. \tag{21}$$

This is the same result as in Eq. (1), since $A_i=\mathcal{A}_i/N!$. Note that the expression in Eq. (21) does not depend on the probability distribution function for the initial velocities v_i. One easily finds $\Sigma_i A_i=N$, as it should be: for each realization, there are N particles.

V. AVERAGE VELOCITIES

The results in Secs. III and IV give the average number of clusters with certain properties [see Eqs. (5), (10), and (16)] and the distribution of actual particle velocities [see Eq. (21)]. However, these quantities alone do not allow us to compute the average velocity $\langle v\rangle$, which results from averaging all *actual* velocities in all clusters under all $N!$ configurations and all choices for *initial* velocities (see Ref. [19]). In addition, it is necessary to specify the probability distribution function from which the initial velocities are drawn.

A. Initial velocities

As a simple example we assume that the preferred (initial) velocities are uniformly distributed in an interval $[0,V_{\max}]$. The *average* value of each initial velocity v_i is then given by

$$\langle v_i\rangle = V_{\max}\frac{i}{N+1} \qquad i=1,2,\ldots,N, \tag{22}$$

since the N values for $\langle v_i\rangle$ divides $[0,V_{\max}]$ in $N+1$ equal parts (see also Ref. [19]). The expression in Eq. (22) will be used in calculations below.

More realistically, the preferred velocity could be distributed uniformly between a minimum and a maximum velocity. This case is also covered by Eq. (22) by simply adding a constant.

In Sec. V D we consider other distributions of the initial velocities, with power-law velocity dependence. While all results in Secs. V B and V C for the uniform distribution are exact for any value of N, for the power-law distributions in Sec. V D we merely give asymptotic results valid for large N.

B. Global average velocity

Using Eqs. (22) and (21) the average velocity may be computed as

$$\langle v\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle v_i\rangle A_i = \frac{1}{N}\sum_{i=1}^{N} V_{\max}\frac{i}{N+1}\frac{N+1}{i(i+1)} = \frac{V_{\max}}{N}\sum_{i=1}^{N}\frac{1}{i+1}. \tag{23}$$

This is the same result as in Eq. (2), with $V_{\max}=1$.

For large N, $\langle v\rangle$ is to dominating order

$$\langle v\rangle \simeq \frac{\ln N}{N}V_{\max}, \tag{24}$$

which is much smaller than the averaged initial velocity $V_{\max}/2$.

C. Site average velocity

The average value found in Eq. (23) results from averaging also over the N sites available. However, these sites are not equivalent. At the front site, any particle will move with its preferred velocity. Thus, at this site, the average velocity is $V_{\max}/2$. On the other hand, the velocity at the last site is always v_1, irrespectively of where the particle with the lowest preferred velocity is placed. Thus, at this site the average velocity is $\langle v_1\rangle=V_{\max}/(N+1)$. We will show that the average velocity increases monotonically from the last to the first site. To obtain these N site average velocities, we compute first the full probability distribution function for having initial velocity v_i as actual velocity at a given site.

Let us number the cars (particles) by n, starting from the first car in the train of queues; in other words, according to the order of disembarking, see Fig. 1. The actual velocity $v(n)$ of car (particle) number n may be any of the preferred velocities v_i, and we ask for the probability $P_n(v_i)$ that $v(n)$ equals v_i. The probability is zero if $i>N-n+1$, because in these cases it is impossible that all of the $n-1$ particles in front of particle number n have larger velocities than v_i.

For particle number n to have velocity v_i, one of the n first particles must have velocity v_i, with probability n/N. Moreover, all particles with velocity lower than v_i must be behind, i.e., at positions $n+1, n+2, \ldots, N$, which occurs with probability

$$\frac{N-n}{N-1}\cdot\frac{N-n-1}{N-2}\cdot\ \ldots\ \cdot\frac{N-n-i+2}{N-i+1} = \frac{(N-n)\,!\,(N-i)!}{(N-n-i+1)\,!\,(N-1)!}. \tag{25}$$

We find the probability

$$P_n(v_i) = \frac{n(N-n)\,!\,(N-i)!}{(N-n-i+1)\,!\,N!} = \frac{\binom{N-i}{n-1}}{\binom{N}{n}}. \tag{26}$$

By means of Eq. (15) one checks that the probabilities for each value of n sum to unity.

For the first particle in particular $P_1(v_i)=1/N$ for all v_i, as it should be. Thus, the average velocity of the foremost particle equals the average of the preferred velocities, as stated above. As n increases, there is a shift of probability toward low i values. As a function of n, the probability for v_1 increases linearly, the probability for v_2 is symmetric, while the cases for higher i values becomes more and more asymmetric.

The average velocity $\langle v(n)\rangle$ for site number n is obtained from Eqs. (26) and (22),

$$\langle v(n)\rangle = \sum_{i=1}^{N} P_n(v_i)\cdot\langle v_i\rangle = \frac{V_{\max}}{N+1}\binom{N}{n}^{-1}\sum_{i=1}^{N} i\binom{N-i}{n-1}. \tag{27}$$

By means of the identity (20) we obtain the simple result

$$\langle v(n)\rangle = \frac{V_{\max}}{N+1}\binom{N}{n}^{-1}\binom{N+1}{n+1} = \frac{V_{\max}}{n+1}. \tag{28}$$

The average velocities decrease monotonically with n, from $\langle v(1)\rangle=V_{\max}/2$ for the first car to the lowest preferred velocity for the last car, $\langle v(N)\rangle=V_{\max}/(N+1)=\langle v_1\rangle$. Note that the final expression in Eq. (28) is independent of N, the number of particles. We will comment further on this point in Sec. VI below.

The average velocity $\langle v\rangle$ for the whole train of queues (clusters) is now obtained as

$$\langle v\rangle = \frac{1}{N}\sum_{n=1}^{N}\langle v(n)\rangle = \frac{V_{\max}}{N}\sum_{n=1}^{N}\frac{1}{n+1}, \tag{29}$$

in agreement with Eq. (23).

D. Power-law distribution of initial velocities

We consider now two power-law probability densities $p(v)$ for the initial velocities, one increasing as $p(v)\propto v^{a}$, the other decreasing as $p(v)\propto v^{-b}$ for large v, with $a>0$ and $b>1$. The corresponding cumulative distributions are

$$P(v) = \begin{cases} (v/V_{\max})^{a+1} & \text{for } v \le V_{\max} \\ 1 & \text{for } v > V_{\max} \end{cases}, \tag{30}$$

and

$$P(v) = 1 - \left(\frac{v + v_0}{v_0}\right)^{1-b}. \tag{31}$$

In the calculations below we will need $\langle v_i \rangle$, the average values of the initial velocities. For large N we have asymptotically [20]

$$P(\langle v_i \rangle) \simeq \frac{i}{N+1}. \tag{32}$$

The form (32) is exact for the uniform distribution $P(v) = v/V_{\max}$ [see Eq. (22)].

1. Increasing power law

Using Eq. (32), we obtain for the increasing power law (30) the following asymptotic values for the average initial velocities:

$$\langle v_i \rangle \simeq V_{\max}\left(\frac{i}{N+1}\right)^{1/(a+1)}. \tag{33}$$

The most interesting quantity is the global average velocity $\langle v \rangle$. We have shown in Sec. V B that for any distribution of the initial velocities $\langle v \rangle$ is given by

$$\langle v \rangle = \frac{1}{N}\sum_{i=1}^{N} \langle v_i \rangle \frac{N+1}{i(i+1)}. \tag{34}$$

In the present case we obtain

$$\langle v \rangle \simeq \frac{V_{\max}}{N}\sum_{i=1}^{N}\left(\frac{N+1}{i}\right)^{a/(a+1)} \cdot \frac{1}{i+1}. \tag{35}$$

Compared to the result for a uniform distribution of initial velocities in Eq. (23), there is an additional factor $[(N+1)/i]^{a/(a+1)}$. Since this factor is always larger then 1, every term in the sum, and therefore the global average velocity, is larger than for the uniform case. The difference increases with a. In other words, when the values for $\langle v_i \rangle$ for low i (in particular) are displaced upward in the interval, the global velocity increases significantly. This is expected, since the low initial velocities create bottlenecks and thereby obtain large weight in the average value. When N increases the global velocity decreases, as for the uniform distribution, but it decreases less fast,

$$\langle v \rangle \propto N^{-1/(1+a)}. \tag{36}$$

Strong effects from reducing the number of low initial velocities are also seen for the site average velocities $\langle v(n) \rangle$. For the front particle we find the size-independent value $\langle v(1) \rangle \simeq V_{\max}(a+1)/(a+2)$, in agreement with the uniform distribution value $V_{\max}/2$ when $a=0$. The last and slowest particle, however, has a velocity

$$\langle v(N) \rangle = \langle v_1 \rangle = V_{\max}/(N+1)^{1/(a+1)}, \tag{37}$$

much larger than the last-particle velocity $V_{\max}/(N+1)$ for the uniform distribution.

2. Decreasing power law

Using Eq. (32) for the decreasing power law (31) we find the asymptotic values

$$\langle v_i \rangle \simeq \frac{v_0}{\left(1 - \dfrac{i}{N+1}\right)^{1/(b-1)}} - v_0. \tag{38}$$

By inserting (38) into Eq. (34) we obtain the global average velocity

$$\langle v \rangle \simeq \frac{N+1}{N} v_0 \sum_{i=1}^{N} \frac{1}{i(i+1)}\left[\left(1 - \frac{i}{N+1}\right)^{-1/(b-1)} - 1\right]. \tag{39}$$

For the special value $b=2$ this equals

$$\langle v \rangle \simeq \frac{N+1}{N} v_0 \sum_{i=1}^{N} \frac{1}{(i+1)(N+1-i)}, \tag{40}$$

which is easily shown to give

$$\langle v \rangle \simeq 2v_0 \frac{\ln(N)}{N} \tag{41}$$

to leading order. This is a low global velocity, with the same dependence upon N as for the uniform distribution, Eq. (24).

The summand in Eq. (39) decreases with increasing b, and therefore (41) is an upper limit for the global velocity for all $b > 2$. To find a lower bound we use the inequality

$$f(x) = (1-x)^{-c} \geq 1 + cx, \tag{42}$$

a consequence of $f(0)=1$ and $f'(x) > c$ for $0 < x < 1$. Using Eq. (42) in Eq. (39) we obtain the lower bound

$$\langle v \rangle > \frac{v_0}{N(b-1)} \sum_{i=1}^{N} \frac{1}{i+1}. \tag{43}$$

The dominant contribution from the sum is $\ln(N)$. Thus we obtain

$$\frac{v_0}{b-1}\frac{\ln(N)}{N} < \langle v \rangle < 2v_0 \frac{\ln(N)}{N} \tag{44}$$

for large values of N. Hence, for all $b \geq 2$ we have a low global velocity, with the same size dependence as we found for the uniform distribution of the initial velocities.

The remaining possibility $1 < b < 2$ leads to very different behavior. That in general $\langle v \rangle$ will be considerable larger than the values for $b \geq 2$ is not surprising since for $b < 2$ the tail of the probability density $p(v)$ decays so slowly that the average initial velocity, $\int v p(v) dv$, is not finite. However, we show below that there are striking effects on the average velocity when the particle velocities are modified through clustering (i.e., when initial velocities are replaced by actual velocities).

We find that there are two subranges, $3/2 \leq b < 2$ and $1 < b < 3/2$. In the first range the global velocity will decrease with increasing N, but more slowly than the previous size dependence $\ln(N)/N$. In the second range the global velocity will *increase* with increasing N. The reason for this is that with increasing N more and more particles will have a high initial velocity corresponding to the probability distribution tail. We show below that for both subranges, that is for $1 < b < 2$, the following relation holds

$$\langle v\rangle \propto N^{-2+1/(b-1)}. \tag{45}$$

For the special value $b=3/2$ we can evaluate (39) analytically, with the asymptotic result

$$\langle v\rangle \simeq v_0\pi^2/6 \quad \text{for } b=3/2. \tag{46}$$

That the result in this case is *independent* of N is in agreement with Eq. (45). More generally, for small values of $b-1$ the last terms in the sum (39) are the dominating ones. Introducing $j=N+1-i$ we have to dominating order

$$\frac{\langle v\rangle}{v_0} \simeq \sum_{j=1}^{N}\left(\frac{N+1}{j}\right)^{1/(b-1)}\frac{1}{(N+1-j)(N+2-j)}. \tag{47}$$

The first factor shows that small values of j give the dominating terms in the sum. Assuming that in this range of j values we may put $N-j\simeq N$, we find asymptotically

$$\frac{\langle v\rangle}{v_0} \simeq N^{-2+1/(b-1)}\sum_{j=1}^{\infty} j^{-1/(b-1)} = N^{-2+1/(b-1)}\zeta[1/(b-1)], \tag{48}$$

where $\zeta(z)$ denotes the Riemann zeta function. For the special value $b=3/2$ the right-hand side of Eq. (48) equals $\zeta(2)=\pi^2/6$, in complete agreement with Eq. (46). Since small j values in Eq. (48) are more dominant the smaller b is, we conclude that the asymptotic result (48) is valid for all $1<b\le 3/2$.

To sum up, for the decreasing power-law distribution the size dependence of the global average velocity is as follows:

$$\frac{\langle v\rangle}{v_0} \propto \begin{cases} \ln(N)/N, \text{ decreasing toward zero with increasing } N & \text{for } b\ge 2 \\ N^{-2+1/(b-1)}, \text{ decreasing toward zero with increasing } N & \text{for } 3/2<b<2 \\ \text{constant} & \text{for } b=3/2 \\ N^{-2+1/(b-1)}, \text{ increasing with increasing } N & \text{for } 1<b<3/2 \end{cases}. \tag{49}$$

A characteristic site average velocity in the slow end of the queues is

$$\langle v(N)\rangle = \langle v_1\rangle \simeq v_0\left(\frac{N+1}{N}\right)^{1/(b-1)} \simeq \frac{v_0}{(b-1)N}, \tag{50}$$

of the same order as for the uniform distribution. In the fast end of the queues, however, the significance of the different ranges of b shows up. For the front particle we have

$$\langle v(1)\rangle = N^{-1}\sum_{i=1}^{N}\langle v_i\rangle. \tag{51}$$

The largest contribution in the sum is

$$\frac{\langle v_N\rangle}{N} \simeq v_0\frac{(N+1)^{1/(b-1)}-1}{N} \simeq v_0 N^{(2-b)/(b-1)}. \tag{52}$$

For $b<2$ this contribution increases with N, and therefore the average velocity of the front particle will necessarily increase with N. Since the average velocity for the whole train is $\langle v\rangle=\Sigma_i\langle v(n)\rangle/N$, the contribution from the term (52) alone is of the order

$$N^{(3-2b)/(b-1)}, \tag{53}$$

which implies that for $1<b<3/2$ the global average velocity increases without bound with increasing N, in agreement with Eq. (49).

VI. DISCUSSION

The one-dimensional system studied in this paper has a hierarchy of potential bottlenecks, which spatially are randomly distributed. This leads to a systematic reduction in *actual* velocities, as initially fast particles cluster behind slow ones. The cluster statistics is very different from the statistics of systems in which local effects or rules drive clustering. In such systems, the number of clusters with given properties are expected to be proportional to the system size. This is not the case for our system. As an example, the total number of clusters grows much slower than linearly with system size N, as demonstrated in Eq. (18).

Our main result (16) displays the same nonextensivity. The number of clusters of size L and velocity v_i per site is

$$c_{i,L} = \frac{C_{i,L}}{N} = \frac{(N-i)\,!\,(N-L)!}{N\cdot N\,!\,(N-L-i+1)!}, \tag{54}$$

which definitely is not independent of N. Rather, the cluster number density depends on N in a complicated way.

Note that the expression for the average number of clusters of size L and velocity v_i in Eq. (16) and the expression for the probability to have v_i as actual velocity at site n in Eq. (26) are close to identical, when n and L are exchanged.

Any *actual* velocity must be one of the *preferred (initial)* velocities of that particular realization. Due to the shadowing from particles with low initial velocities, high velocities are systematically underrepresented among actual velocities, as shown by Eq. (21). Per particle, that is, per position, the fraction of sites with v_i as actual velocity is, on average

$$a_i = \frac{A_i}{N} = \frac{N+1}{N}\cdot\frac{1}{i(i+1)} \overset{N\to\infty}{\to} \frac{1}{i(i+1)}. \tag{55}$$

Thus, the limiting values are $a_1=\frac{1}{2}$, $a_2=\frac{1}{6}$, $a_3=\frac{1}{12},\ldots$. Incrementing from N to $N+1$, all values a_i with $i\le N$ decrease toward their limiting values with a total amount equal to a_{N+1}.

The limiting value $a_1=\frac{1}{2}$ can be readily understood, since the particle with v_1 as initial velocity may be placed at any of the N positions, with equal probability. Since in all cases a cluster will extend from this position throughout the system backward, the average cluster size is $L/2$. On the other hand, for $i>1$; a_i will in principle depend on where the particle with initial velocity v_i is placed relatively to the ones with lower velocity. Still, remarkably, a_i follows the same simple formula (55) over the entire range, apparently without any influence from the increasing complexity in where, as i increases, it is possible to form a cluster with actual velocity v_i. However, the placement problem for different values is intervowen, or correlated. In this respect, the expression in Eq. (55) reflects a highly correlated system.

Considering the uniform distribution of initial velocities in particular, this may be displayed in a different manner. If v_i is proportional to i, as in Eq. (22), the probability distribution function for actual velocities is

$$P_{\mathrm{act}}(v)=\frac{1}{K_1 v(K_1 v+1)}\simeq K_2 v^{-2} \tag{56}$$

for large N and i. Here, K_1 and K_2 are constants. Thus, the actual velocities are power-law distributed. The bottleneck hierarchy leads to scale invariance.

The calculations in Sec. V C allow an interpretation of the expression for the (global) average velocity $\langle v\rangle$ in Eqs. (2) and (23), which apply to the uniform distribution of initial velocities. The average velocity $\langle v(n)\rangle$ at a given *site* was found in Eq. (28). From this expression, which does not contain N, $\langle v\rangle$ was found by averaging over sites, see Eq. (29). Thus, in the expressions given in Eqs. (2), (23), and (29), the first term in the sum corresponds to the average velocity at the foremost site, the second term to the site behind it, and so on. The values that enter the average, $V_{\max}/2$, $V_{\max}/3,\ldots$, do not depend on N. The reason that the (global) average velocity $\langle v\rangle$ decreases when N increases is therefore not that there are changes in any (site) average velocities $\langle v(n)\rangle$, but that new and low values from the rearmost sites enter the average.

Alternatively, the terms in the expression for the (global) average velocity $\langle v\rangle$ in Eqs. (2) and (23) may be connected to cluster size L. Suppose one computes the average velocity for each cluster size separately, that is,

$$\langle v_L\rangle=\sum_{i=1}^{N}\langle v_i\rangle\frac{C_{i,L}}{B_L}=\frac{V_{\max}}{N+1}\cdot\frac{1}{B_L}\sum_{i=1}^{N} i\cdot C_{i,L}, \tag{57}$$

where $\langle v_i\rangle$ is given by Eq. (22) and the total number of clusters of size L, B_L, as given by Eq. (10), does not depend on i. The calculation of the sum is identical to the calculation in Sec. IV, with i and L exchanged. Therefore, one finds that

$$\langle v_L\rangle=\frac{V_{\max}}{N+1}\cdot\frac{1}{\frac{1}{L}}\cdot\frac{N+1}{L(L+1)}=\frac{V_{\max}}{L+1}. \tag{58}$$

Note that this expression is independent of N. The total number of clusters of length L, B_L, decreases when L increases. However, the average number of *particles* that belong to clusters of size L is the same for any value of L, see Sec. III B. Thus, one may obtain directly the (global) average velocity $\langle v\rangle$ by averaging over the N values L can have, using the expression above and equal weight for each L value. This leads to the expression for the (global) average velocity $\langle v\rangle$ in Eqs. (2) and (23). Therefore, in the expression for $\langle v\rangle$, the first term may be interpreted as the result of averaging over all clusters of size 1, the second as the result of averaging over size 2,.... From this perspective, the reason why $\langle v\rangle$ decreases when N increases is that larger clusters, with low actual velocities, enter the average. The parameters i, the velocity rank, and L, the cluster size, play a strikingly similar role.

Our results for clusters and velocities are general and may be applied to any probability distribution of the initial velocities. As we have seen, for the uniform distribution exact results, valid for any N, were obtained. For more general distributions of the initial velocities, for example the power-law distributions in Sec. V D, we have to be satisfied with asymptotic large-N results.

For an increasing power law, $p(v)\propto v^a$, the main effect is a large increase in the global velocity with increasing a, i.e., a reduction of the bottle-neck-creating low initial velocities. For a decreasing power law, $p(v)\propto v^{-b}$, $\langle v\rangle$ is crucially dependent upon the value of b. For $b>2$, the global velocity has the same qualitative size dependence, $\langle v\rangle\propto\ln(N)/N$, as the uniform distribution. For $3/2<b<2$, $\langle v\rangle$ decreases less rapidly than this, while for $1<b<3/2$ the global velocity increases with increasing N.

For both these two subranges, i.e., for $1<b<2$, averaging the *initial* values $\langle v_i\rangle$ over particles gives a divergent result when N increases. However, as a result of the mapping from initial (preferred) velocities to actual velocities [see Eq. (21)], the (global) average velocity is reduced. For one subrange, $3/2<b<2$, the average value over actual velocities turns out to decrease with increasing N; for the other subrange, $1<b<3/2$, the (global) average velocity still diverges. At the transition between these two subranges, for $b=3/2$, the (global) average velocity is constant, that is independent of N, to dominating order. The transition point divides the interval $1<b<2$ in two equal parts.

VII. CONCLUSIONS

We have analyzed a simple one-dimensional particle system with a hierarchy of bottlenecks. Many quantities could be calculated explicitly. A high degree of symmetry was found. Due to its transparency, this particle system may be a useful building block when constructing more complex models.

Our results form a general framework, and can be used to obtain actual average velocities for any distribution of the initial (preferred) velocities. We have analyzed three types of initial velocity distributions, increasing and decreasing power-law distributions, in addition to a uniform distribution. For the uniform distribution and for the decreasing power law, $p(v)\propto v^{-b}$ with $b\geq 2$, the global velocity $\langle v\rangle$ is very low for large N, we found $\langle v\rangle\propto\ln(N)/N$. This is due to the low initial velocities acting as bottlenecks. When the number of low initial velocities is reduced, as for the increasing power-law distribution $p(v) \propto v^a$, the global velocity decreases less rapidly when N is increased.

Interesting results are obtained for the decreasing power law with $b<2$, corresponding to a considerable fraction of high-velocity initial velocities. The effect of these high initial velocities is that for $3/2<b<2$ the global velocity decreases with increasing N slower than in the $b \geq 2$ case, and for $1<b<3/2$ it even *increases* with increasing system size.

APPENDIX

We shall evaluate the two sums in Eqs. (15) and (20). The first sum is

$$S_j = \sum_{s=0}^{M-1} \binom{s}{j-1}. \tag{A1}$$

Multiplication by x^{j-1} and summation over j yields

$$\sum_j S_j x^{j-1} = \sum_{s=0}^{M-1} (1+x)^s = \frac{(1+x)^M - 1}{x} = \sum_j \binom{M}{j} x^{j-1}. \tag{A2}$$

Thus,

$$S_j = \binom{M}{j}, \tag{A3}$$

which is Eq. (15).

The second sum is

$$T_k = \sum_{j=1}^{M} j \binom{M-j}{k}. \tag{A4}$$

Multiplication with x^k and summation over k yields

$$\begin{aligned}\sum_{k\geq 0} T_k x^k &= \sum_{j=1}^{M} j(1+x)^{M-j} \\ &= \frac{(1+x)^{M+1} - (M+1)(1+x) + M}{x^2} \\ &= \sum_{m\geq 2} \binom{M+1}{m} x^{m-2} = \sum_{k\geq 0} \binom{M+1}{k+2} x^k.\end{aligned}$$

Thus,

$$T_k = \binom{M+1}{k+2}, \tag{A5}$$

which is Eq. (20).

[1] T. M. Liggett, Ann. Probab. **25**, 1 (1997).
[2] B. Derrida, Phys. Rep. **301**, 65 (1998).
[3] R. Stinchcombe, Physica A **372**, 1 (2006).
[4] J. Krug, Phys. Rev. Lett. **67**, 1882 (1991).
[5] S. A. Janowsky and J. L. Lebowitz, Phys. Rev. A **45**, 618 (1992).
[6] G. Tripathy and M. Barma, Phys. Rev. E **58**, 1911 (1998).
[7] S. Goldstein and E. R. Speer, Phys. Rev. E **58**, 4226 (1998).
[8] M. C. Romano, M. Thiel, I. Stansfield, and C. Grebogi, Phys. Rev. Lett. **102**, 198104 (2009).
[9] M. R. Evans, Europhys. Lett. **36**, 13 (1996).
[10] M. R. Evans, J. Phys. A **30**, 5669 (1997).
[11] M. Bengrine, A. Benyoussef, H. Ez-Zahraouy, J. Krug, M. Loulidi, and F. Mhirech, J. Phys. A **32**, 2527 (1999).
[12] R. Juhász, L. Santen, and F. Iglói, Phys. Rev. Lett. **94**, 010601 (2005).
[13] B. S. Kerner, Phys. Rev. Lett. **81**, 3797 (1998).
[14] A. John, A. Schadschneider, D. Chowdhury, and K. Nishinari, Phys. Rev. Lett. **102**, 108001 (2009).
[15] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Phys. Rev. E **50**, 822 (1994).
[16] J. Krug, Braz. J. Phys. **30**, 97 (2000).
[17] M. Barma, Physica A **372**, 22 (2006).
[18] P. C. Hemmer, Fra fysikkens verden **53**, 51 (1991); **53**, 72 (1991).
[19] V. Frette, Eur. J. Phys. **30**, 497 (2009).
[20] E. J. Gumbel, Nat. Bur. Standards Appl. Math. Ser. **33**, 1 (1954).