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Let the fast passengers wait: Boarding an airplane takes shorter time when passengers with the most bin luggage enter first

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ABSTRACT

Airlines usually organize the passenger queue by letting certain groups of passengers enter the airplane in a specific order. The total boarding time of such airplane boarding policies can be estimated and compared by a Lorentzian metric based on the one used in Einstein's theory of relativity. The metric accounts for aisle-clearing times that depend on the passengers' queue positions, in particular when passengers in the back of the queue with increasing probability will have to wait for already seated aisle or middle seat passengers to rise up and let the others pass to a seat closer to the window. We provide closed-form expressions for the asymptotic total boarding time when the number of passengers is large, and prove that the best queue ordering with low congestion is according to decreasing luggage-handling time. The effect of seat interference amplifies the previously shown superiority of slow first vs. random boarding and fast first. That this ranking of policies also holds for realistic congestion is illustrated by both analytical methods and simulations, and parameters are taken from empirical data. However, the result is non-trivial, as the ranking shifts for unrealistically high congestion. Based on the analytical results, we demonstrate that the slow-first policy can be improved by dividing the passengers into more than two groups based on their number of bin luggage items, and let the slowest groups with the most luggage items enter the queue first.

1. Introduction

The airplane boarding process has been shown to be on the critical path of the turnaround time that airplanes spend at the gate (Neumann, 2019). While other time-critical processes, like fueling, are hard to shorten, there is a significant potential for reducing the boarding time, which is the time from the first passenger entering the airplane until the last passenger sits down. The boarding time is influenced by several factors (Hutter et al., 2019), and since cost reductions could be significant for every minute saved (Cook & Tanner, 2015; Nyquist & McFadden, 2008), airlines address this by implementing boarding policies.

A boarding policy describes how the queue should be arranged prior to boarding, and typically, the passengers are divided into groups based on, e.g., designated row number, and then the groups enter the queue in a prescribed order. With the back-to-front policy, which is widely used in the US, the passengers who have designated seats in the back of the airplane are allowed to enter first, followed by groups heading for seats closer to the front rows (Delcea et al., 2018). In Europe many airlines apply the random boarding policy, without organizing the queue, except from usually letting passengers with special needs, like families with children and disabled persons, enter first. This can, in fact, be considered as a slow-first policy, and that this courtesy to certain groups that tend to be slow actually has associated benefits, will be further elucidated in this paper.

In the last 20 years, there has been a large and growing body of work on the analysis of airplane boarding procedures and policies (see, e.g., Jaehn & Neumann, 2015; Willamowski & Tillmann, 2022). Much of the analysis of the boarding process is carried out by simulations with varying degrees of parameters and randomness, which provide the flexibility needed to model different boarding scenarios and to examine different aspects of the boarding procedures (Audenaert et al., 2009; Delcea et al., 2018; Ferrari & Nagel, 2005; Schultz, 2018; Steffen, 2008; Van Landeghem & Beuselinck, 2002). Another approach is mathematical programming, where the

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boarding time can be written as a non-linear (van den Briel et al., 2005) or linear (Bazargan, 2007; Willamowski & Tillmann, 2022) program. The corresponding optimization introduces optimal solutions in some space of allowable policies.

These approaches identified two different types of interferences (interactions) between passengers, which make the boarding process less efficient and should therefore be minimized. The first is aisle interference in which one passenger directly or indirectly prevents another passenger from getting to their designated row, by blocking the aisle while getting organized to take their seat in the row. The other type, seat interference, refers to the situation where upon arrival at the designated row, some passengers are blocking the passenger from getting to their designated seat, for example, these other passengers are already seated in the aisle and middle seats of the row, and have to get up to allow the incoming passenger to reach the window seat. Generally speaking, these studies show a trade-off between the minimization of boarding time, the amount of control that the policies exert on the passengers, and the difficulty of implementing them.

Another approach can be applied to boarding policies that divide passengers into a limited number of groups and where the sequence of passengers within each group is random (Bachmat et al., 2006; 2009; 2013; Erland et al., 2019; 2021; Frette & Hemmer, 2012). In this approach, the boarding process is described by a model with several parameters. It is possible to simulate the process, but instead an average boarding time estimate can in many cases be expressed by formulas that allow us to compare different policies analytically. The formulas describe the boarding time of the model with an accuracy that improves as the number of passengers N becomes large. The analysis of the model is geometrical and more specifically provides an interpretation of the airplane boarding process in terms of space-time (Lorentzian) geometry — the type of geometry that is also used for describing relativity theory. More specifically, airplane boarding is described via a wave propagation in this geometry, and the boarding time estimate corresponds to the relativistic (proper) time needed for the waves to completely cover a domain, which in the airplane boarding case is taken to be the unit square. The advantage of the approach is that it provides insights into what makes good policies and connects to several areas of mathematics and physics, which allow us to apply tools from those fields. The asymptotic formulas, which are exact when $N \to \infty$, allow us to consider large (theoretically infinite) families of policies all at once. Moreover, the computability and accuracy do not decline when the number of passengers grows — unlike the case of mathematical programming, which in many cases cannot solve to optimality when N is large.

One of the insights provided by the approach is that the congestion k is an important parameter. Congestion is the ratio of total queue length when all passengers are standing one after the other, to the total length of all aisles in the airplane. Some boarding methods such as the back-to-front policy, are very performance sensitive to the value of k. Certain simulation approaches do not have k as a parameter, but rather as an implicitly defined constant, and the use of different constants can be used to explain discrepancies between results of different simulations. Other major results from this approach include a complete analysis of back-to-front boarding policies, which shows that the usual implementations of such policies are not effective in realistic scenarios (Bachmat et al., 2013). More recently it has been shown how the total boarding time is affected by dividing the passengers into two groups with different aisle-clearing time distributions. If there is a fast group of passengers without luggage and a slower group of passengers with luggage, it is universally better (i.e., independent of model parameters) to board the slow group first and then the fast group, compared to the other way around or mixing the two populations as in random boarding (Erland et al., 2019; 2021). On the other hand, it is universally better to board the fast group first in order to minimize the average boarding time of individual passengers (Bachmat et al., 2021).

The drawback of this geometrical approach is that in order to explicitly solve the model by formulas instead of simulations, the model needs to be simplistic so it can be described geometrically, and in addition the geometric description has to be analytically tractable. For example, in order to set up the geometric interpretation, it must be assumed that when passengers are not blocked (experience aisle interference), they proceed to walk down the aisle towards their designated row at infinite speed, i.e., they immediately reach a blocking passenger or arrive at their row. This simplification means that differences in boarding times tend to be larger in the model than in simulations that take walking speed into account. Similarly, the models did not take into account seat interferences since it previously was not clear how to handle them in an analytically tractable fashion. Consequently, all the results above were indicative of behavior in models that lacked/ignored various realistic features, and it was not clear whether they hold and to what extent when the features are added.

The purpose of this paper is to show that we can to a large extent take seat interference into account in an analytically tractable way with varying (in the boarding scenarios) degrees of explicitness. This allows us to examine the extent to which earlier model-based insights remain true when seat interferences are supplemented into the model.

Our main findings are the following:

1. When the passengers are divided into a limited number of groups, taking seat interferences into account still yields an analytically tractable model.
2. When we have two passenger groups, and add seat interferences into the model, boarding slow passengers first is better in realistic cases and in all cases below a certain congestion threshold. However, the result is not universal, in the sense that for some very high and unrealistic congestion values, boarding the fast passengers first is better. The results are also observed in model simulations with a realistic number of passengers.
3. If there are more than two groups, ranking them in the order from slowest to fastest can be shown analytically to be optimal for low congestion.

The structure of the paper is as follows. In Section 2 we give a high-level presentation of the boarding process model. This is followed by analysis of the boarding time in Section 3 that allows for aisle-clearing times that depend continuously on the queue location. Expressions for the effective aisle-clearing time that include seat interferences, with parameters derived from empirical data are presented in Section 4. The main result is presented in Section 5 where we compare and rank the three main policies under consideration — slow first, fast first, and random boarding. In Section 6 we suggest a new improved policy that aims to narrow the gap to what we show is an optimal (but infeasible) policy. We end the paper with a discussion in Section 7.

2. General high level model description

We explain our modeling approach at a high level. In particular we point out the challenges at the various stages of the boarding time estimation and what makes it sometimes analytically tractable. A more detailed description of the model is found in Bachmat (2014) and Erland et al. (2019).

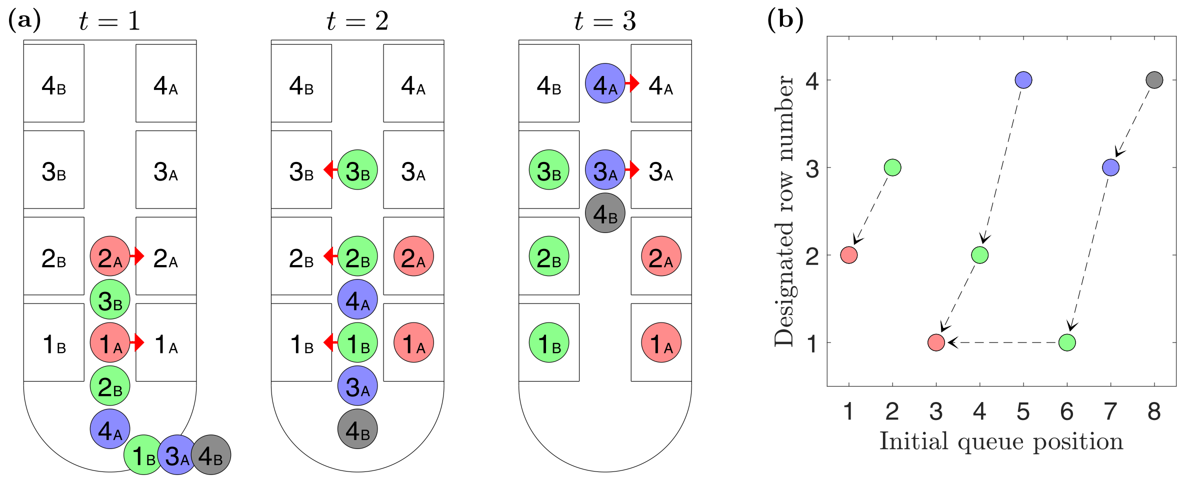


Fig. 1. Geometric representation of the boarding process with $N = 8$ passengers and two seats per row. (a) The queue advances step-wise, and each passenger is represented by a circle with designated row number. At each time step, the queue moves forward, and passengers arriving at their designated rows sit down simultaneously. The wave front of passengers who take their seat at that time step is marked by red arrows and color coded. (b) Each point in the queue-row-diagram represents a passenger, indicated by initial queue position and designated row number. Passengers who do not sit down in the first time step have been blocked by at least one passenger in a preceding wave front, and such blocking relations are indicated by arrows. Passengers who consecutively block each other form blocking paths, and the boarding time $T = 4$ is determined by the sum of aisle-clearing times of the passengers in the maximal path (the connected dashed lines). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2.1. Geometric representation of the boarding process

We consider the boarding process from the time the first passenger enters the airplane, until the last passenger is seated. We assume that a passenger cannot proceed before any other passengers in the aisle has cleared the aisle and is seated, as illustrated by the example in Fig. 1(a). The time a passenger needs to move before halting at the next blocking passenger is considered negligible compared to the aisle-clearing time of the blocking passenger.

In Fig. 1(b) the passengers are represented as points in a square. The first coordinate is the location of the passenger in the boarding queue, while the second coordinate represents the designated row number of the passenger. In the remainder of the paper we will refer to the normalized queue and row location, and in terms of these coordinates, all passengers will be represented as points in the unit square.

Note that the distribution of points representing passengers in the unit square will vary according to the airline boarding policy employed. Policies that do not couple the row location and queue location will lead to a uniformly random distribution (when the number of passengers $N \to \infty$), while others such as back-to-front will lead to non-uniform distributions. The policies considered in this paper will lead to a uniform distribution.

The main parameters in the airplane boarding model are listed below:

- N — the number of passengers in the airplane, R is the number of rows.
- $\tilde{q}$ — the initial queue position of a passenger, $q = \tilde{q}/N$ is the normalized queue position.
- $\tilde{r}$ — the designated row number of a passenger, $r = \tilde{r}/R$ is the normalized row position.
- k — the congestion, i.e., the ratio of the total length the queue of all passengers as they stand in the aisle and the total aisle length of the airplane. Let ρ be the occupancy of the plane, s the number of seats per row, n_a the number of aisles, d the distance between consecutive rows and w the distance between passengers in the aisle(s). Then $k = \rho s w/(n_a d)$. The congestion k reflects both how densely the passengers stand in the aisle and the interior design of the airplane.[1]
- h — the number of seats per row segment. E.g., if there are $n_a = 1$ aisle in the airplane and $s = 6$ seats per row, $h = 3$.
- Y_q — the aisle-clearing time needed for a passenger to organize bin luggage and take a seat after reaching one's row. Y_q depends on the queue position q and is a sum of the luggage-handling time X, and the waiting time W for already seated passengers to give way. In Fig. 1 all passengers have an aisle-clearing time of $Y_q \equiv 1$ time steps. The effective aisle-clearing time $\tau_Y(q)$ is an intrinsic parameter in the asymptotic estimate of the total boarding time and is determined by the distribution of Y_q only. Explicit expressions are generally not available, except when $Y_q \equiv c$ is constant, in which case $\tau_Y = c$.
- p — the proportion of passengers that are designated as slow passengers with long luggage-handling time. The remaining fraction, $1 - p$, are considered fast.

We assume that only the front door of the airplane is used, and for convenience we use a single aisle, i.e., $n_a = 1$ and a realistic number of $N = 180$ passengers with full occupancy $\rho = 1$ as the default.

2.2. Wave fronts, maximal paths and space-time geometry

Once the passengers are represented geometrically in the unit square as outlined above, we can describe the boarding process as a “wave propagation” in the unit square, where the wave fronts represent passengers who sit down at roughly the same time (color-coded in Fig. 1(b)). The process ends when the last wave front covers the last remaining passengers, and when the aisle-clearing time is one time unit as in Fig. 1, the boarding time is given by the number of fronts. However, when passengers have different aisle-clearing times, the computation of boarding time is not so straightforward.

A dual way to consider the boarding time, is to consider for every passenger the passenger who last blocked them from get-

[1] In Fig. 1, $k = 1$ since $\rho = 1$, $d = 2w$, $n_a = 1$, and $s = 2$.

ting to their row, and then recursively, the passenger who blocked the passenger who blocked and so forth until reaching a passenger who was not blocked by anyone and proceeded unobstructed to their designated row when the boarding started. This creates for each passenger, a path or sequence of passengers each being blocked by the next one. Plotting these paths in the unit square provides a system of paths, starting at every point representing a passenger and ending usually very near the origin. In Fig. 1(b), the path of the last passenger sitting down is shown as connected dashed lines ending up in a passenger in the first wave front.

The next steps are intended to relate these paths that have a discrete number of segments to a continuous geometry, so we can use the rich structure and tools associated to the continuous geometry. We let the number of passengers N go to infinity and establish a geometric law of large numbers that leads to a continuous geometry description. As $N \to \infty$, we just scale the length of the airplane (i.e., the number of rows) and keep the other parameters fixed. These parameters describe the airplane boarding process and consist of the congestion parameter k, a given airline boarding policy, and a description of the characteristics of the passenger population. When $N \to \infty$, the boarding process can be described as a wave propagation in a continuous geometry that is given by a boarding scenario dependent space-time structure (Lorentzian metric) on the unit square (Bachmat, 2014).

A space-time structure defines a class of permissible curves, which in physics describes the motions of objects at or under the speed of light (causal trajectories). It also provides a weight to each of these curves. In physics, this corresponds to the time passing on a clock that is attached to an object whose trajectory is traced by the curve. In the case of airplane boarding, the maximal weight of permissible curves starting in a particular point in the unit square, provides a normalized estimate of the amount of time it will take a passenger with those coordinates to sit down in the given boarding scenario. In physics, such maximal curves correspond to trajectories of objects who are only influenced by gravity, i.e., free falling objects. Such curves are called geodesics. In airplane boarding there is an added complication, since such maximal curves may sometimes have segments that lie on the boundary of the unit square domain. Thus, maximal curves are composed of boundary segments and geodesic segments that lie in the interior of the square. The boarding time is the maximal time of any passenger to sit down. Hence, it is given as the maximal curve weight in the space-time structure. Consequently, computing the (normalized version of the) boarding time amounts to finding the curve of maximal weight and computing its weight.

2.3. Curvature and explicit solutions of maximal curve weights

Given the geometric structure, the next steps in the analysis attempt to explicitly solve for the boarding time. As noted above, the curve of maximal weight can have portions on the boundary of the unit square and will also have portions in the interior of the square. The mixture of boundary and interior curve segments, depends on the parameters of the boarding scenario. This breaks up the parameter space into different pieces, which depend on the combinatorics of this mix. The weight of the boundary pieces is given by some integrals, which often can be solved explicitly. For the interior segments we have to find the maximizing curve, which amounts to solving the geodesic equation with the appropriate boundary conditions for the given Lorentzian metric (space-time structure). In general, the geodesic equation is given by a second order ODE, which is an Euler-Lagrange equation. However, there is no reason to expect that it is possible to explicitly solve the ODE. The main exception is a flat space-time structure, which corresponds to special (rather than general) relativity. In the case of a flat space-time in standard coordinates (corresponding to the point of view of an object at rest), the solutions to the geodesic equation are simply straight lines. However, sometimes the flat space is given in non-standard coordinates, and we need to change coordinates to standard ones.

Given a space-time structure (in any coordinate system) there is a point dependent (local) quantity that we can compute, called the curvature, which provides a criterion to determine whether the space-time is flat. The criterion is that the curvature has to vanish at every point. By an amazing stroke of luck, when seat interferences were ignored, the space-time structures that correspond to reasonable boarding scenarios, correspond to locally flat space-time structures. That is, the unit square can be broken into sub-domains, and on each sub-domain the curvature vanishes, and we can find the coordinate change to standard coordinates (Bachmat, 2019; Bachmat et al., 2005).

From a technical standpoint, the main results of this paper show that even after taking seat interference into account, the corresponding structures are still flat. In addition, we can in many cases go further and explicitly compute the maximal curve weight and compare boarding policies in terms of total boarding time.

2.4. Curve weight integral

The total boarding time with a finite number of passengers is determined by the *heaviest path*, that is the path that has a maximal sum of aisle-clearing times. When the number of passengers $N \to \infty$, the heaviest path approaches a *heaviest causal curve*. If we assume that the aisle-clearing time of each passenger is deterministically given by the passengers' normalized queue and row position, $Y = \tau(q, r)$, the *weight* of a curve $r(q)$ when the passengers are uniformly distributed in the unit square, can be defined by (Bachmat, 2014)

$$\mathcal{W}(r) = \int_{q_0}^{q_1} \tau(q, r(q))\sqrt{r'(q) + k[1 - r(q)]}dq. \tag{1}$$

A curve is called *causal* when the square root in the integral is real, and this indicates the ability of one passenger on the curve to block the next passenger on the curve. The square root measures the number of passengers along the causal curve, and this is weighted by the respective aisle-clearing time $\tau(q, r(q))$ along the same part of the curve.

The *heaviest* causal curve $r^*(q)$ is the one maximizing the integral. Since the normalized queue and row number (q, r) of all passengers are within the unit square, the heaviest causal curve must obey the same restriction. In addition, r^* must be continuous and connect the points (0,0) and (1,1) (Bachmat, 2014).

When the aisle-clearing times Y are stochastic, $\tau(q, r)$ in Eq. (1) can be replaced by an effective aisle-clearing time $\tau_Y(q, r)$, which is independent of k (Bachmat, 2019). In this paper, we consider boarding policies where the aisle-clearing time distribution depends on q only, such that $\tau_Y(q, r) = \tau_Y(q)$ (for brevity we will usually write $\tau(q)$ in the following).

For the three main policies analyzed in this paper, we assume that the passengers can be divided into two groups according to their anticipated aisle-clearing time when seat interferences are not taken into account. The slow group with larger effective aisle-clearing time τ_S could be those with overhead-bin luggage, while the fast ones without luggage are characterized by τ_F. Without seat interference, the resulting effective aisle-clearing time function $\tau(q)$ is constant within the respective q-intervals of each group,

$$\tau(q) = \begin{cases} \tau_S \text{ when } q \in [0, p],\ \tau_F \text{ when } q \in (p, 1], & \text{Slow first (SF)} \\ \tau_F \text{ when } q \in [0, 1-p], & \\ \quad \tau_S \text{ when } q \in (1-p, 1], & \text{Fast first (FF)} \\ \tau_A \gtrapprox \sqrt{\tau_S p + \tau_F(1-p)} \text{ when } q \in [0, 1], & \text{Random boarding.} \end{cases} \tag{2}$$

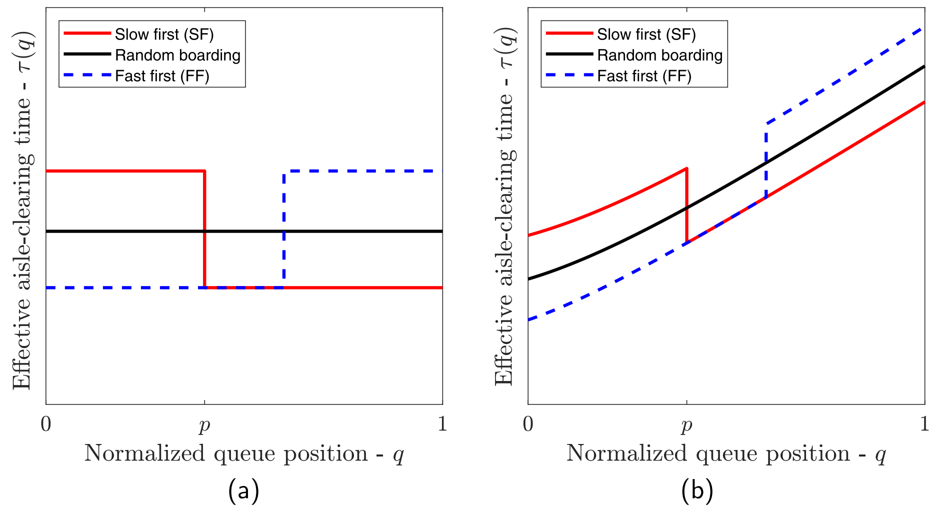


Fig. 2. Illustration of effective aisle-clearing time as a function of normalized queue position q for the three studied policies. There are $p = 40\%$ slow passengers, and the effective luggage-handling time is $\tau_F = 1$ and $\tau_S = 2$ for the fast and the slow group, respectively. (a) $\tau(q)$ without seat interference as in Eq. (2). (b) Seat interference and the time it takes to wait for already sitting passengers to give way often has a non-linear effect on $\tau(q)$. Here we have used that $\tau^2(q) \approx \langle Y^2 \rangle$ as in Eq. (21) with $h = 3$.

This $\tau(q)$ has been applied previously in the literature for the three mentioned policies (Bachmat et al., 2021; Erland et al., 2019; 2021). In Section 4 we derive approximate expressions for $\tau(q)$ that take seat interference into account — and hence depend continuously on q. Both cases — with and without seat interference — are exemplified in Fig. 2.

3. Asymptotic total boarding time

While estimates of the asymptotic total boarding time previously have been obtained for piecewise constant $\tau(q)$, the major contribution of this section is to provide estimates also when $\tau(q)$ varies continuously. Let r^* be the heaviest causal curve from (0,0) to (1,1) within the unit square that maximizes $\mathcal{W}(r)$ in Eq. (1). The total boarding time converges to a multiple of the weight of r^* (Bachmat, 2014),

$$\frac{T}{\sqrt{N}} \overset{\text{a.s.}}{\to} 2\mathcal{W}(r^*).$$

The result is based on a general formula by Myrheim (1978). The average total boarding time is asymptotically given by

$$\langle T \rangle \sim 2\sqrt{N}\mathcal{W}(r^*) \equiv \hat{T}. \tag{3}$$

The asymptotic boarding time $\hat{T}$ is a leading term. For finite N and for realistic values of k, the average boarding time $\langle T \rangle$ is overestimated by a relative error of order $o(N^{-\frac{1}{4}})$ (Bachmat et al., 2013).

The result in Eq. (3) is used below to derive analytical expressions for the asymptotic boarding time for the three main policies under consideration in this paper: random boarding, slow first and fast first. We exploit that the heaviest causal curve r^* that maximizes Eq. (1) with integrand $L(q, r, r')$ can be found by solving the Euler-Lagrange equation

$$\frac{\partial L}{\partial r} - \frac{d}{dq}\frac{\partial L}{\partial r'} = 0. \tag{4}$$

The estimate for the total boarding time is found by multiplying the weight of the solution $\mathcal{W}(r^*)$ with $2\sqrt{N}$.

3.1. Total boarding time when $k = 0$

When $k = 0$, passengers are paper thin, and the solution of Eq. (4) when L is the integrand of Eq. (1), is given by

$$r^{*\prime}(q) = a\tau^2(q)$$
$$\Downarrow$$
$$r^*(q) = a\int_0^q \tau^2(\rho)d\rho + b \quad = \quad \frac{\int_0^q \tau^2(\rho)d\rho}{\int_0^1 \tau^2(\rho)d\rho}, \tag{5}$$

where the constants a, b in the last line are determined by the boundary conditions $r^*(0) = 0$ and $r^*(1) = 1$.

The corresponding total boarding time is from Eq. (3)

$$\langle T \rangle \sim 2\sqrt{N}\sqrt{\int_0^1 \tau^2(q)dq}. \tag{6}$$

If seat interference is not taken into account and there is, e.g., two groups, each having a fixed effective aisle-clearing time τ_S, τ_F as in Fig. 2(a), then the ordering of the groups in the queue will not affect the total boarding time in Eq. (6). This argument easily extends to situations with more groups, but as we show in Section 5, the situation is more complex when seat interferences are taken into account. However, if two groups are mixed (as in random boarding), the effective aisle-clearing time is slightly larger than the second moment approximation in Eq. (2), and the boarding time in Eq. (6) will be somewhat larger than for policies where the groups are separated (Erland et al., 2021).

3.2. Total boarding time for small $k > 0$

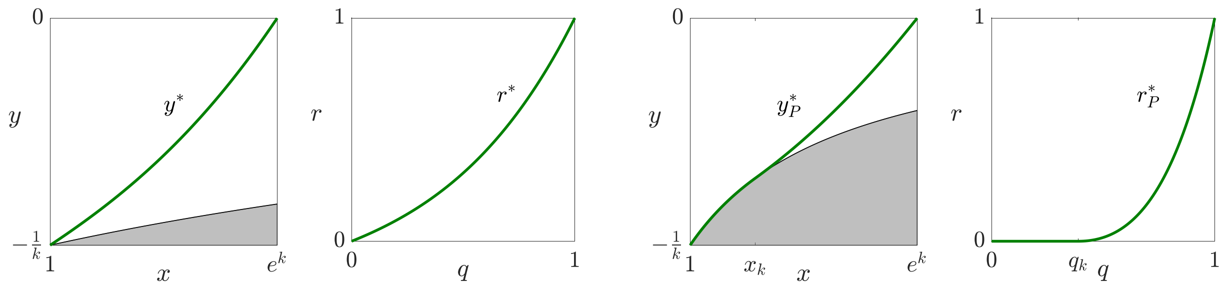
In order to solve the Euler-Lagrange equation in Eq. (4) that maximizes Eq. (1) when $k > 0$, we first make a transformation of variables (Bachmat, 2014),

$$q = \frac{\ln(x)}{k} \qquad x = e^{kq}$$
$$r = 1 + kxy \qquad y = \frac{r-1}{ke^{kq}}. \tag{7}$$

Then (by a slight abuse of notation) Eq. (1) reads

$$\mathcal{W}(y) = \int_{x_0}^{x_1} \tau\left(\frac{\ln(x)}{k}\right)\sqrt{y'(x)}dx. \tag{8}$$

That the corresponding metric is flat is shown in Appendix A.



(a) Low congestion: $0 < k < k_c$ (b) High congestion: $k_c < k$

Fig. 3. Illustration of heaviest curves for random boarding in both sets of coordinates in Eq. (7). We have used $\tau^2(q) = 1 + Bq + Cq^2$ as in Fig. 2(b). In the xy-coordinates, the curve is constrained to the region above the gray-shaded area defined by $y = -1/(kx)$ (mapped from the baseline $r \equiv 0$). (a) With small congestion $k = 0.2$, y^* and r^* are defined by Eq. (9). (b) For larger congestion $k = 0.9$, the heaviest curves are piecewise, and y_P^* and r_P^* leave the baseline in $x = x_k$ and $q = q_k$, respectively (see Eq. (12)).

With a flat metric, it is possible to find the heaviest curve y^* that maximizes Eq. (8),

$$y^*(x) = a\int_1^x \tau^2\left(\frac{\ln(z)}{k}\right)dz + b \quad = \quad \frac{1}{k}\frac{\int_1^x \tau^2\left(\frac{\ln(z)}{k}\right)dz}{\int_1^{e^k} \tau^2\left(\frac{\ln(x)}{k}\right)dx} - \frac{1}{k}. \tag{9}$$

The required start and end points of the transformed curve, $y^*(1) = -1/k$ and $y^*(e^k) = 0$, determine the constants a, b. An example of such a curve, in both xy- and qr-space, is shown in Fig. 3(a).

The corresponding maximal curve weight in Eq. (8) is

$$\mathcal{W}(y^*) = \sqrt{\int_0^1 \tau^2(q)e^{kq}dq}. \tag{10}$$

Note that the total boarding time $\langle T\rangle \sim 2\sqrt{N}\mathcal{W}(y^*)$ increases substantially if k is large and $\tau^2(q)$ is increasing. If seat interference is ignored, it immediately follows that the slow-first policy with decreasing effective aisle-clearing time is superior to the fast-first policy. In Section 5 we show that this also holds when seat interference is taken into account.

The curve y^* in Eq. (9) is only valid for the calculation of the asymptotic total boarding time if the corresponding curve r^* (obtained by the transformation in Eq. (7)) is contained in the unit square. This requires that $y^{*\prime}(1) > 1/k$, and from Eq. (9) k must be less than a critical value k_c, which is implicitly defined by

$$k_c: \quad k_c\int_0^1 \tau^2(q)e^{k_c q}dq = \tau^2(0). \tag{11}$$

3.3. Total boarding time for large $k > k_c$ and non-decreasing $\tau(q)$

When the curve y^* in Eq. (9) is not contained in the unit square, the heaviest *valid* curve $y^*(x)$ — with respect to airplane boarding — turns out to be a piecewise curve $y^* = y_P^*$ (Bachmat, 2014). For simplicity we here also assume that $\tau(q)$ is non-decreasing (as in the fast first and random boarding policies in Fig. 2).

The y_P^* curve follows the baseline curve $y = -1/(kx)$ (mapped from $r(q) \equiv 0$) for $x \in [1, x_k]$, and is smoothly continued by a curve of the form in Eq. (9) for $x \in (x_k, e^k]$. The parameters x_k, a, b are determined by the smoothness and end point conditions. Continuity in $x = x_k$ requires that $y_P^*(x_k) = -1/(kx_k)$ such that

$$y_P^*(x) = a\int_{x_k}^x \tau^2\left(\frac{\ln(z)}{k}\right)dz - \frac{1}{kx_k}, \quad x \in (x_k, e^k]. \tag{12}$$

Smoothness of y_P^* at the same point, i.e., $y_P^{*\prime}(x_k) = 1/(kx_k^2)$ gives

$$a = \left[kx_k^2\tau^2\left(\frac{\ln(x_k)}{k}\right)\right]^{-1}. \tag{13}$$

Finally, $y_P^*(e^k) = 0$ determines the point $x_k \equiv e^{kq_k}$ by the implicit equation

$$q_k: \quad \tau^2(q_k)e^{kq_k} = k\int_{q_k}^1 \tau^2(q)e^{kq}dq, \tag{14}$$

where $q = q_k$ is the point where the corresponding heaviest curve $r_P^*(q)$ in qr-space departs from the baseline[2] [see Fig. 3(b)].

Adding the weights of the first and second part of $r_P^*(x)$ and applying the relation in Eq. (14) give the maximized weight

$$\mathcal{W}(r_P^*) = \sqrt{k}\int_0^{q_k} \tau(q)dq \quad + \quad \frac{\tau(q_k)}{\sqrt{k}}. \tag{15}$$

See Appendix C for how these calculations are applied to the fast-first policy and also for corresponding calculations for the slow-first policy. A detailed example with specific assumptions on the aisle-clearing time function $\tau(q)$ is calculated explicitly for random boarding with seat interference in Appendix B.

4. Effective aisle-clearing time $\tau(q)$ with seat interference

Passengers who have arrived at their designated row, use a certain aisle-clearing time to take a seat. In Erland et al. (2021) this time was defined as the time $X = X_q$ it takes to organize the overhead bin luggage before taking a seat. For the three main policies considered in this paper, the distribution of X_q is the same for all passengers within each group. In the absence of seat interference the effective aisle-clearing time $\tau_X(q)$ is given by the right-hand-side of Eq. (2).

In this paper we also include the effect of seat interference in the aisle-clearing time. That is, the additional time W a passenger could have to wait for an already seated passenger to rise up and let the fellow passenger get to a seat closer to the window. We utilize that the effective aisle-clearing time $\tau_Y(q)$ is well approximated by the square root of its second moment (Erland et al., 2021).

4.1. Probability of seat interference

Assume random boarding, i.e., both the queue positions $\tilde{q}$ and row positions $\tilde{r}$ (non-normalized) are uniformly distributed. As the passengers starts to fill the airplane, there is an increasing probability that other passengers are already seated at the same row segment when a passenger arrives at ones designated row. If the passenger is designated with a window seat, the already seated

[2] If $\tau(q)$ is discontinuous in a point $q = q_d$, there might not be a solution to Eq. (14). Then $x_k = e^{kq_d}$, and the heaviest curve still takes the form in Eq. (12). However, a is determined by $y_P^*(e^k) = 0$ and not by the smoothness condition in Eq. (13) (see Eq. (29) in Appendix C.1 for the calculation of the corresponding weight for the fast-first policy).

passengers will have to rise up and let the newly arrived passenger pass, and this causes a delay that increases the aisle-clearing time of the passenger. We here compute the probability $p_W(\tilde{q})$ that a passenger with queue position $\tilde{q}$ has to wait to pass other passengers who are already seated at the same row segment when arriving at that row.

First, the probability that a random row segment has precisely x free seats is given by the hypergeometric distribution. Parameters are N total seats (population size), a total of $N-\tilde{q}+1$ free seats, and h seats per row segment (draws) of which there are $x \in 0, 1, \ldots, h$ free seats:

$$\Pr\{\text{Random row segment has } x \text{ free seats}\} = \frac{\binom{N-\tilde{q}+1}{x}\binom{\tilde{q}-1}{h-x}}{\binom{N}{h}} \equiv p_{R,x}. \tag{16}$$

In total there are N/h row segments, and a proportion $p_{R,x}$ of those row segments has exactly x free seats. The probability that passenger $\tilde{q}$ has a designated seat at a row segment with precisely $S=x$ free seats is given by the total number of free seats at such row segments, divided by the total number of free seats,

$$\Pr\{\text{Random free seat at row segment with } S=x \text{ free seats}\} = \frac{x\frac{N}{h}p_{R,x}}{N-\tilde{q}+1} \equiv \Pr\{S=x\}. \tag{17}$$

Combining Eqs. (16) and (17) gives, depending on the number of seats per row segment h,

$$h=2: \Pr\{S=1\} = \frac{\tilde{q}-1}{N-1},$$
$$h=3: \Pr\{S=1\} = \frac{(\tilde{q}-1)(\tilde{q}-2)}{(N-1)(N-2)},$$
$$\Pr\{S=2\} = \frac{2(\tilde{q}-1)(N-\tilde{q})}{(N-1)(N-2)}.$$

When $h=2$, and the designated seat of passenger $\tilde{q}$ is at a row segment with $S=2$ free seats, passenger $\tilde{q}$ will not have to wait since no other passengers are sitting at that row segment. However, when $S=1$, the already seated passenger will have to let the other pass if sitting at the aisle seat, which occurs with probability 1/2. The total probability p_W that passenger $\tilde{q}$ will have to wait for already seated passengers to rise up when arriving at the designated row is given by

$$\begin{aligned} h=2: p_W &= \frac{1}{2}\Pr\{S=1\} + 0\cdot\Pr\{S=2\} = \frac{1}{2}\frac{\tilde{q}-1}{N-1} \\ &\overset{N\to\infty}{=} \frac{q}{2}, \\ h=3: p_W &= \frac{2}{3}\Pr\{S=1\} + \frac{1}{2}\Pr\{S=2\} + 0\cdot\Pr\{S=3\} \\ &= \frac{(\tilde{q}-1)}{(N-1)(N-2)}\left[\frac{2}{3}(\tilde{q}-2) + \frac{1}{2}2(N-\tilde{q})\right] \\ &\overset{N\to\infty}{=} \frac{2}{3}q^2 + \frac{1}{2}2q(1-q) = q - \frac{q^2}{3}. \end{aligned} \tag{18}$$

When $N\to\infty$, $p_W(q)$ is linear in q for $h=2$ and follows a second order polynomial for $h=3$. If we for $h=3$ distinguish between the waiting time for passengers that have to wait for either one (W_1 when $S=2$, or $S=1$ and the middle seat is free) or two (W_2 when $S=1$ and the window seat is free) already seated passengers, then the respective probabilities of waiting are given by the first two terms in Eq. (18), such that $p_{W_1}(q) = q - 2q^2/3$ and $p_{W_2}(q) = q^2/3$.

4.2. Effective aisle-clearing time — second moment approximation

We assume that the resulting aisle-clearing time $Y = Y_q$ is the sum of the time used to stow bin luggage X_q, and the time needed to wait for other passengers to rise W, such that

$$Y_q = \begin{cases} X_q & \text{with probability } 1-p_W(q), \\ X_q + W & \text{with probability } p_W(q). \end{cases} \tag{19}$$

The effective aisle-clearing time parameter τ_Y that scales the Lorentzian metric in Eq. (1) can be approximated well by $\tau_Y^2 \approx \langle Y^2\rangle$ (Erland et al., 2021). The second moment of the aisle-clearing time Y_q when the waiting time W is equal for both one and two already seated passengers to rise, is given by

$$\langle Y_q^2\rangle = \langle X_q^2\rangle\left(1 + p_W(q)\frac{2\langle X_q\rangle\langle W\rangle + \langle W^2\rangle}{\langle X_q^2\rangle}\right). \tag{20}$$

Here we have assumed that X_q is independent of W.

If we for $h=3$ distinguish between the waiting time of passengers that have to wait for either one (W_1) or two (W_2) already seated passengers, we get

$$\begin{aligned} h=2: \tau_Y^2(q) &\approx \langle X_q^2\rangle(1+Bq), \\ h=3: \tau_Y^2(q) &\approx \langle X_q^2\rangle\left(1+B_1q+Cq^2\right). \end{aligned} \tag{21}$$

The constants are determined by the distributions of W_1, W_2 and X_q,

$$\begin{aligned} B_i &= \frac{2\langle X_q\rangle\langle W_i\rangle + \langle W_i^2\rangle}{\langle X_q^2\rangle}, \qquad i=1,2 \\ B &= B_1/2 \\ C &= (B_2 - 2B_1)/3. \end{aligned} \tag{22}$$

Note that analytical expressions are readily available for the integrals that are used to calculate the asymptotic total boarding time in Section 3 when $\tau(q)$ is of the form given in Eq. (21).

4.3. Parameters derived from empirical data

Empirical data for airplane boarding has previously been presented in Steiner & Philipp (2009). The data were based on video recordings of the boarding of six different flights at Zurich airport. The effective aisle-clearing time τ was estimated accurately based on the resulting empirical distributions, for several different groupings of passengers in Erland et al. (2021). However, even though the occurrence of seat interference for each passenger was also recorded in the data, this aspect of the data was not taken into account in the latter.

Here, we have re-analyzed the data set in order to estimate how much of the aisle-clearing time Y that is due to luggage-handling (X) and waiting times (W_1, W_2), respectively. Luggage-handling includes the time needed to clear the aisle except from the waiting time for other passengers to rise, and hence passengers without luggage are also assigned a “luggage-handling” time.

The moments of the resulting empirical distributions are presented in Table 1. Note that the expected waiting time for one passenger to give way $\langle W_1\rangle$ is quite similar to the “luggage-handling” time $\langle X_F\rangle$ of a passenger without luggage. However, it takes more than three times as long to wait for both the aisle and middle passenger to rise when one is heading for a window seat, i.e., $\langle W_2\rangle \approx 3.1\langle W_1\rangle$.

The moments are used to estimate the parameters in the effective aisle-clearing time functions in Eqs. (21) and (22). This enables comparisons of the asymptotic boarding times for random boarding, fast first and slow first. The underlying empirical distributions are used in simulations with finite number of passengers.

5. Superiority of slow first with seat interference

It has previously been shown that without seat interference slow first is superior to both fast first and random boarding, while

Table 1
Moments of the empirical distributions of luggage-handling time X based on the empirical data in Steiner & Philipp (2009). The moments of the waiting time distributions are $\langle W_1 \rangle = 5.8$, $\langle W_1^2 \rangle = 39$, $\langle W_2 \rangle = 18.2$ and $\langle W_2^2 \rangle = 385$. Units of first moments are in seconds. Corresponding parameters for the aisle-clearing time functions in Eqs. (21) and (22) are also presented.

Group definition	Proportion	X	$\langle X \rangle$	$\langle X^2 \rangle$	B_1	B_2	B	C
All passengers	100%	X_A	15.2	507	0.43	1.86	0.21	0.81
0 items	45%	X_F	5.7	56	1.90	10.66	0.95	5.21
$\geqslant$ 1 items	55%	X_S	22.9	870	0.35	1.40	0.18	0.58
1 item	40%	X_{S1}	17.8	493	0.50	2.10	0.25	0.90
2 items	11%	X_{S2}	31.0	1222	0.33	1.24	0.16	0.50
$\geqslant$ 3 items	4%	X_{S3}	52.8	3861	0.17	0.60	0.08	0.23

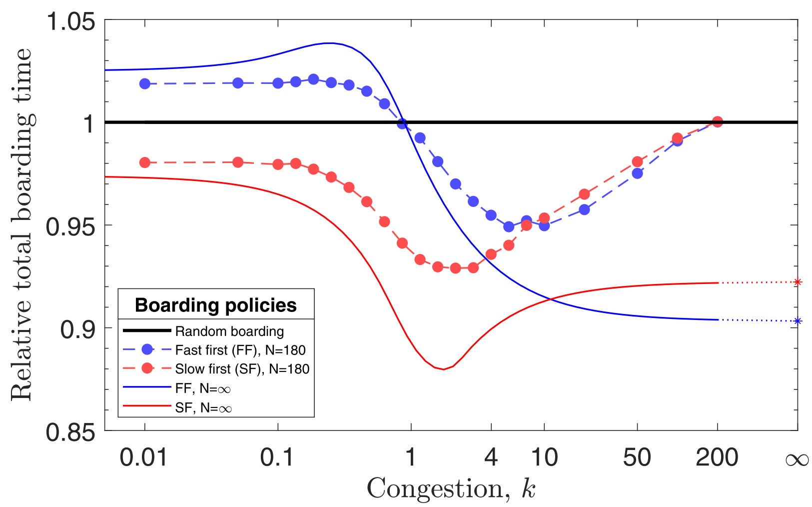


Fig. 4. Discrete event simulations with realistic number of passengers N confirm the asymptotic results that slow first (SF) is superior to fast first (FF) for moderate values of congestion k. The reverse result that fast-first can outperform slow-first for large values of k occurs for $k > 8.9$. We have used empirical distributions for both the luggage-handling times X_F, X_S and the waiting times W_1, W_2. The proportion of slow passengers is $p = 0.55$, and there are $h = 3$ seats per row segment. Each point in the graph is an average of 100 000 discrete-event runs. Details of the simulation model are given in Appendix E. In the asymptotic estimates ($N = \infty$, solid lines) the effective aisle-clearing time is $\tau^2(q) = 1 + B_q q + C_q q^2$, where both B_q, C_q have constant values within the slow and the fast region.

the ranking of fast first and random boarding depends on the parameter settings (Erland et al., 2019; 2021). In this section we quantify how seat interference affects the total boarding time and the corresponding ranking of the three policies. Only high-level results are presented in this section — detailed calculations of heaviest curves and corresponding heaviest curve weights are found in Appendices B and C.

Analytical comparisons can be made more explicit when the heaviest curves are not piecewise as described in Section 3.2. Let k_c^* be the minimum value of k_c in Eq. (11) taken over all of the three policies, and assume $k \leqslant k_c^*$. We first use the formulation of the weight of the geodesic in Eq. (10) to show that the superiority of slow first is maintained when seat interference is taken into account. Using realistic parameters in Fig. 4, we demonstrate analytically for $N = \infty$, and by simulations with finite $N = 180$, that this is also the case for $k > k_c^*$, except when the congestion is very large.

For the analytical comparisons we maintain the assumption that the effective aisle-clearing time τ_Y^2 can be substituted by $\langle Y^2 \rangle$. We implicitly make the natural assumption that if the passengers are separated into two groups based on perceived luggage-handling time X, then the bin luggage-handling time of the fast passengers is shorter than that of the slow passengers, both in terms of $\langle X_F \rangle < \langle X_S \rangle$ and $\langle X_F^2 \rangle < \langle X_S^2 \rangle$. Likewise, we assume that the two first moments of W_2 are larger than those of W_1, i.e., it takes a longer time to wait for two already seated passengers to give way than for only one. This is also the case for the empirical data in Table 1.

5.1. Slow first is superior for low congestion $k \leqslant k_c^*$

The formulation of maximal curve weight $\mathcal{W}(r^*)$ in Eq. (10) indicates that slow first is superior to fast first and random boarding — also when seat interference is taken into account: With slow first, the average bin luggage-handling time $\langle X_q \rangle$ is decreasing, and this counterbalances the effect of increasing waiting probability $p_W(q)$. Hence, the variance of the aisle-clearing time Y_q on the q domain is reduced. Formally, comparisons of policies can be done in the following way.

As in Eq. (19), assume that the aisle-clearing time is a sum of the time needed to organize overhead bin luggage $X = X_q$ and the waiting time W for already seated passengers to let the passenger pass. Also assume that the squared effective aisle-clearing time $\tau_Y^2(q)$ can be substituted by $\langle Y_q^2 \rangle$.

Theorem 1. *Let the number of passengers N be large, congestion $k \leqslant k_c^*$, and assume $\tau^2(q) = \langle Y_q^2 \rangle$. Then the asymptotic total boarding time of the slow-first policy is smaller than for the random boarding and fast-first policies, also when seat interference is taken into account.*

A simple proof based on the compact expression for maximal curve weight $\mathcal{W}(r^*)$ in Eq. (10) is found in Appendix D.

Fig. 4 shows that the asymptotic boarding time (solid lines) for slow first, random boarding and fast first are ranked in that order for small k. The simulation results (bullets connected by dashed lines) confirm that the ranking is valid for realistic number of passengers $N = 180$ as well, even though the relative difference between policies is less compared to $N = \infty$.

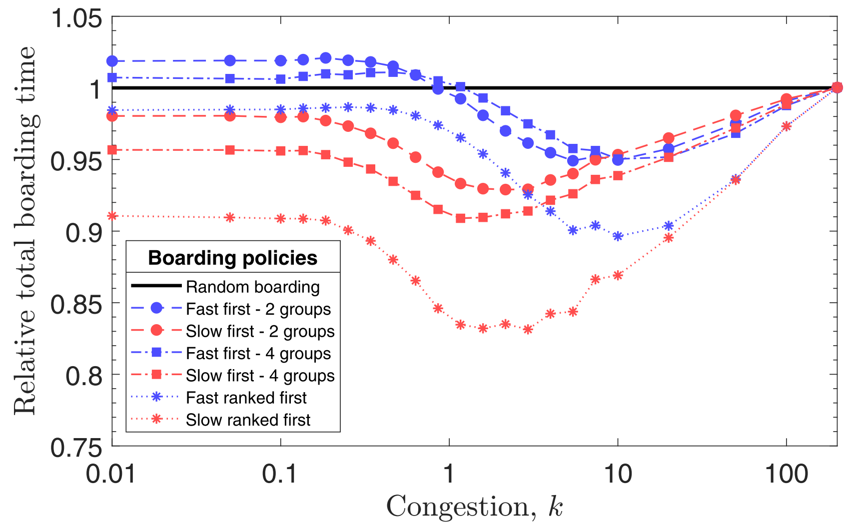


Fig. 5. Discrete event simulations with realistic number of passengers $N = 180$ confirm the asymptotic results for low congestion that the total boarding time is reduced when the passengers are divided into more groups with the slowest groups entering first. The groups are ranked according to their effective luggage-handling time. We have used empirical luggage-handling and waiting time distributions and $h = 3$ seats per row segment. The passengers have been divided into groups based on the number of hand luggage each passenger is carrying. The lower bound "slow ranked first" where the queue is organized based on descending luggage-handling time, can only be attained by clairvoyantly knowing the luggage-handling time of each passenger before boarding. Each point in the graph is an average of 100 000 discrete-event runs. Details of the simulation model are given in Appendix E.

5.2. Slow first is superior for realistic congestion k

In Theorem 1 we utilize the simple formulation of the asymptotic boarding time when $k \leqslant k_c$. With the empirical parameters used in Fig. 4, k_c takes the values 0.076, 0.551, and 0.947 for fast first, random boarding and slow first, respectively. This means that the result in Theorem 1 is strictly speaking only valid for $k \leqslant k_c^* = 0.076$.

For $k > k_c^*$, the calculations of the asymptotic boarding time are based on less explicit calculations, which makes analytical comparisons of policies much harder. Still, the boarding times can be calculated, and in Fig. 4 the superiority of slow first is valid for values of k that cover a range that is much larger than realistic values of $k \approx 4$. For $k = 4$ and $N = 180$, the slow-first policy yields 6.4% less boarding time than random boarding while the improvement with fast first is 4.5%. With less than full occupancy, the congestion k is typically smaller, and the relative improvement with slow first compared to random boarding would be even higher.

Also note that the relative ranking between policies in Fig. 4 with respect to asymptotic boarding time ($N = \infty$) is very similar to the simulation-based results with a realistic number of passengers ($N = 180$) for all values of k. Note in particular that fast first overtakes random boarding when $k \approx 0.9$ and subsequently slow first when $k \approx 10$ for both choices of N.

5.3. Ambiguous ranking of slow first and fast first for high congestion

Fig. 4 shows that for very large values of k fast first is superior to both slow first and random boarding. Such high congestion values are unrealistic under normal circumstances. However, if social distancing rules during the COVID-19 pandemic were enforced combined with full occupancy, the congestion could be as high as $k = 8$ (Bachmat et al., 2021). Still, with a realistic number of passengers $N = 180$, the gain is marginal, and there is no relative difference for very large values of k since then each passenger would fill up the whole aisle. However, when $N = \infty$, the relative difference between fast first and slow first approaches a non-negligible limiting value with fast first taking 2.1% less time than slow first. From a theoretical perspective this is remarkable since the difference vanished when seat interference was not taken into account (Erland et al., 2021). A tangible physical explanation for why seat interference has this effect is yet to be concluded upon.

6. Towards optimal ordering of the passenger queue

In this section we use the formulation of the maximal curve weight in Eq. (10) to show that for policies that do not take the designated row position of passengers into account, the total boarding time is minimized if the queue positions of passengers are according to descending luggage-handling time. Corresponding policies that require detailed ordering of the queue based on both row numbering and individual aisle-clearing times, are identified and shown to be asymptotically optimal when the number of passengers tends to infinity (Willamowski & Tillmann, 2022). However, the application of policies like this would require the clairvoyant knowledge of the luggage-handling time of every passenger, in addition to individual compliance with the designated ordering. The former is impossible, the latter is impractical. However, we use the optimal solution as a guiding principle to devise a new boarding policy that contains neither of the deficiencies of the optimal policy.

6.1. Slow ranked first is superior with low congestion k

As noted in Theorem 1, slow first is superior to both fast first and random boarding for small k. As shown in Fig. 4, the superiority of slow first increases rapidly with k, but only up to a certain threshold. Above that threshold, the heaviest curves could be piecewise and the comparisons between policies are more challenging to analyze. Nevertheless, we show that the slow-first policy can be improved for small k (a proof for Corollary 1 is found in Appendix D).

Corollary 1. *Assume* $\tau_Y^2(q) = \langle Y_q^2 \rangle$ *and* $k < k_c$ *for all possible orderings of the queue. Then the asymptotic total boarding time is minimized if the queue is ordered according to decreasing luggage-handling time, i.e., "slow ranked first".*

The simulation results in Fig. 5 with empirical luggage-handling and waiting-time distributions, confirm that slow ranked first is superior to the other policies when the number of passengers is $N = 180$. When the queue is ordered according to decreasing luggage-handling time, the ranking also imposes a variability reduction in aisle-clearing time Y_q for fixed q, and this seems to amplify the superiority of such policies. Substituting $\tau^2(q)$ by $\langle Y_q^2 \rangle$ is strictly speaking only valid when Y_q is constant, while $\tau^2(q)$ is somewhat larger than $\langle Y_q^2 \rangle$ for all known cases with stochastic Y_q (Erland et al., 2021). Hence, policies with higher variability in aisle-clearing time within groups — like random boarding and fast first — tend to have higher total boarding time.

Conversely, this variability reduction could explain why the fast-ranked-first policy is superior to both random boarding and fast first in Fig. 5. However, the variability reduction in fast first compared to random boarding do not counterbalance that the aisle-clearing time increase faster in q for fast first, and hence fast first is inferior to random boarding for small values of k.

6.2. A practical improvement — Slow groups first with more groups

The optimal slow-ranked-first policy in Corollary 1 is not practically implementable as it requires clairvoyance to know the luggage-handling time of each passenger before boarding. Still, we can devise a policy that is better than the slow-first policy.

The empirical data in Table 1 suggest that more luggage items tend to increase the luggage-handling time. Hence, a practical implementation of slow ranked first is to divide the passengers into four groups based on the number of luggage items they carry. The group with the most luggage items enter the queue first, and then the rest enters the queue according to decreasing number of luggage items.

The simulation results in Fig. 5 seem to confirm that the best boarding strategy is to let the passengers enter the queue based on descending number of luggage items. The improvement with four groups compared to with two groups is most pronounced when the congestion is $k \approx 1$. However, there is also significant improvement with more realistic congestion-values, and the boarding time compared to random boarding for slow first with 2 groups and 4 groups is 93.6% and 92.2%, respectively (when $k = 4$). Still, there is a sizable gap down to the non-implementable slow-ranked-first policy, which yields 84.2%.

7. Discussion and outlook

In this paper we have generalized a method that applies space-time geometry as a tool for estimating the boarding time of a large family of group-based airplane boarding policies. First, we represent the solution of the connected optimization problem in a more compact way than in previous literature. Then, the representation enables us to analyze and compare several policies when seat interference is taken into account. Finally, we can quite easily deduce that when the congestion is small, the total boarding time is minimized if the passengers enter the airplane according to descending luggage-handling time. The latter has previously only been shown when the passengers have been divided into two groups without seat interference, but the current result enables us to find better solutions for the more complex situation with more than two groups.

The analytical results are based on asymptotic leading order estimates with large number of passengers ($N \to \infty$). Even though such estimates tend to overestimate the boarding time, the ranking of different policies with a realistic number of passengers is strikingly well predicted by the method. Simulations confirm that this is the case when seat interferences are included as well (see, e.g., Fig. 4).

Other policies that have not been treated in this paper can also be compared by the same framework. This includes the outside-in policy (Marelli et al., 1998), where the window seat passengers board first, followed by the middle seat passengers, and finally aisle seat passengers. This means that no-one will have to wait for fellow passengers to rise up to let them pass to a seat closer to the window, and the probability of waiting $p_W(q) \equiv 0$. Hence, the outside-in policy is asymptotically equal to random boarding without seat interference. Under random boarding with seat interference and 6 seats per row, 39% of the passengers will have to wait for already seated aisle or middle passengers to rise up to let them pass. With $N = 180$ passengers, random boarding yields 17% longer boarding time than the outside-in policy when empirical distributions are applied. I.e., the seat interference gives a non-negligible contribution to the boarding time. This is comparable to the 12% observed increase in a single small-scale experiment with $N = 72$ passengers by Steffen & Hotchkiss (2012).

Customer satisfaction and the fact that passengers tend to dislike too detailed instructions, are probably the reason why group-based policies are so widely applied by airlines, and such policies are also the target of this article. Nevertheless, there have been trials with policies that instruct passengers on a more detailed level, and the Steffen method (Steffen, 2008) is probably the most well-known of such policies. It should be noted that further improvements have been developed by mixed-integer programming techniques that among other parameters take individual aisle-clearing time into account in order to minimize the total boarding time (Willamowski & Tillmann, 2022). It is, e.g., proved that all individual aisle-clearing times must be known in order to devise an optimal by-seat policy.

Airplane boarding policies tend to be compared in terms of the total boarding time, from the first person entering the airplane till the last person sits down. However, from a managerial perspective, other objectives must also be kept in mind in order to ensure customer satisfaction. One such goal is the minimization of the time each person has to wait in the queue to get seated. Seen from this perspective, Bachmat et al. (2021) have shown that fast first is superior to both random boarding and slow first when seat interference is not taken into account. One could hypothesize that fast first would be even more superior in this respect when seat interference is taken into account, but this remains to be verified by further analysis.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Transformations that prove a flat metric

The infinitesimal curve weight in Eq. (1) when $\tau(q, r(q)) = \tau(q)$, is given by the metric

$$\begin{aligned} d\mathcal{W}^2 &= \tau^2(q)\left[dqdr - k(r-1)dq^2\right] \\ &= \tau^2\left(\frac{\ln(x)}{k}\right)dxdy \\ &= d\tilde{x}dy. \end{aligned} \tag{23}$$

The second line in Eq. (23) appears when the transformations in Eq. (7) are applied. When $\tau(q)$ is a constant, the metric is flat and the geodesics are straight in xy-space (Bachmat, 2014).

When $\tau(q)$ depends on q, the third line in Eq. (23) is obtained by the following transformation (from xy-space to $\tilde{x}y$-space),

$$\tilde{x} = \int \tau^2\left(\frac{\ln(x)}{k}\right)dx.$$

So, we still have a flat metric that gives straight geodesics in $\tilde{x}y$-space for all continuous $\tau(q)$.

If we, e.g., assume the special case that $\tau^2(q) = 1 + Bq$ [i.e., $h = 2$ in Eq. (21)], the explicit transformation reads

$$\tilde{x} = \int 1 + B\frac{\ln(x)}{k}dx = x + \frac{Bx}{k}[\ln(x) - 1].$$

Appendix B. Random boarding with seat interference — explicit results

In this section we derive the asymptotic total boarding times for a specific version of the random boarding policy by explicit computation of the heaviest curve and corresponding weight. The computations depend on the value of k.

B1. Random boarding with $k = 0$

The effective aisle-clearing time $\tau_Y(q)$ can be approximated by Eq. (21). We simplify by setting $\langle X_q^2 \rangle = 1$ (such that, e.g., $\tau_Y(q) = \sqrt{1 + Bq}$ for $h = 2$). Then Eq. (5) gives

$$h = 2 : r^*(q) = \frac{q + \frac{B}{2}q^2}{1 + \frac{B}{2}}$$
$$h = 3 : r^*(q) = \frac{q + \frac{B_1}{2}q^2 + \frac{C}{3}q^3}{1 + \frac{B_1}{2} + \frac{C}{3}}.$$

The corresponding maximal curve weights in Eq. (6) are

$$h = 2 : \mathcal{W}(r^*) = \sqrt{1 + \frac{B}{2}}$$
$$h = 3 : \mathcal{W}(r^*) = \sqrt{1 + \frac{B_1}{2} + \frac{C}{3}}. \quad (24)$$

Note that without seat interference, $B = 0 = B_1 = C$, and we obtain the estimates obtained in, e.g., Erland et al. (2019).

B2. Random boarding with $h = 2$ and $0 < k < k_c$

When we for $h = 2$ assume $\tau(q) = \sqrt{1 + Bq}$, the heaviest curve and its corresponding weight in Eqs. (9) and (10), respectively, gives

$$y^*(x) = \frac{1}{k}\frac{(x-1) + \frac{B}{k}[x(\ln(x) - 1) + 1]}{(e^k - 1) + \frac{B}{k}\left[e^k(k-1) + 1\right]} - \frac{1}{k}$$
$$\mathcal{W}(y^*) = \sqrt{\frac{e^k - 1}{k} + \frac{B\left[e^k(k-1) + 1\right]}{k^2}}. \quad (25)$$

The corresponding $r^*(q)$ in the normalized queue-row diagram on the unit square can be obtained by the transformations in Eq. (7). Since r^* must be inside the unit square, we must require that k does not exceed a certain value k_c, defined implicitly by Eq. (11),

$$k_c : \quad e^{k_c} = 2 - B\left(e^{k_c} - \frac{e^{k_c} - 1}{k_c}\right). \quad (26)$$

E.g., with random boarding $B = 0.21$ in Table 1, and then $k_c = 0.64$.

B3. Random boarding with $h = 2$ and $k > k_c$

If $k > k_c$ as defined by Eq. (26), the heaviest curve $y^*(x) = y_P^*(x)$ is a piecewise curve as defined by Eqs. (12) to (14). The curve departs from the baseline in $x = x_k$, and when $h = 2$, from Eq. (14), x_k is implicitly given by

$$x_k = \frac{e^k[k + B(k-1)]}{2\left[k + B\left(\ln(x_k) - \frac{1}{2}\right)\right]}. \quad (27)$$

The curve is smoothly continued by the curve defined by Eqs. (12) and (13). Hence, the heaviest curve and its corresponding weight in Eq. (15) are given by

$$y_P^*(x) = \begin{cases} -\frac{1}{kx} & 1 \leqslant x < x_k \\ -\frac{(e^k - x)(1 - \frac{B}{k}) + \frac{B}{k}\left[e^k k - x\ln(x)\right]}{x_k{}^2[k + B\ln(x_k)]} & x_k \leqslant x < e^k. \end{cases}$$
$$\mathcal{W}(y_P^*) = \frac{2\sqrt{k}}{3B}\left[(1 + Bq_k)^{\frac{3}{2}} - 1\right] + \sqrt{\frac{1 + Bq_k}{k}}, \quad (28)$$

where $q_k \equiv \ln(x_k)/k$.

If we assume a realistic congestion $k = 4$ and $h = 2$ seats per row segment, the outside-in policy with $B = 0$ gives $x_k = 27.30$ in Eq. (27) (corresponding to $q_k = 0.827$), while random boarding with $B = 0.21$ (see Table 1) gives $q_k = 0.829$.

Appendix C. Analysis of fast-first and slow-first policies

C1. Heaviest curve weight for fast first

We assume that $\tau^2(q)$ has the second order polynomial form in Eq. (21). For fast-first, there is a discontinuity in $\tau(q)$ when $q = 1 - p$ where the parameters of $\tau^2(q)$ change values as indicated by Table 1. We therefore denote $\tau(q) = \tau_F(q)$ when $q < (1 - p)$ in the first part of the queue where there are only fast passengers, and as the slow passengers enter the queue, we denote $\tau(q) = \tau_S(q)$ when $q > (1 - p)$.

Typical shapes of the heaviest curves for the fast-first policy are shown in Fig. 6(a–d). The corresponding heaviest weights $\mathcal{W}^* \equiv \mathcal{W}(r^*)$ can be summarized as follows:

$$k < k_c : \mathcal{W}^* = \sqrt{\int_0^{1-p} \tau_F^2(q)e^{kq}dq \quad + \quad \int_{1-p}^1 \tau_S^2(q)e^{kq}dq}$$
$$k_c < k < k_{F1} : \mathcal{W}^* = \sqrt{k}\int_0^{q_k} \tau_F(q)dq \quad + \quad \frac{\tau_F(q_k)}{\sqrt{k}}$$
$$k_{F1} < k < k_{F2} : \mathcal{W}^* = \sqrt{k}\int_0^{1-p} \tau_F(q)dq$$
$$+\sqrt{\int_{1-p}^1 \tau_S^2(q)e^{k[q-(1-p)]}dq}$$
$$k_{F2} < k : \mathcal{W}^* = \sqrt{k}\int_0^{1-p} \tau_F(q)dq \quad + \quad \sqrt{k}\int_{1-p}^{q_k} \tau_S(q)dq$$
$$+\frac{\tau_S(q_k)}{\sqrt{k}} \quad (29)$$

Here q_k is given by Eq. (14), and k_c by Eq. (11). k_{F1} and k_{F2} are the upper and lower limiting values for k when the piecewise curve departures smoothly from the baseline in the fast and the slow region, respectively. These are found by setting $q_k = 1 - p$ and solving Eq. (14), i.e.,

$$k_{F1} : \lim_{q \to (1-p)^-} \tau_F^2(q) \quad = \quad k_{F1}\int_{1-p}^1 \tau_S^2(q)e^{k_{F1}[q-(1-p)]}dq$$
$$k_{F2} : \lim_{q \to (1-p)^+} \tau_S^2(q) \quad = \quad k_{F2}\int_{1-p}^1 \tau_S^2(q)e^{k_{F2}[q-(1-p)]}dq.$$

C2. Heaviest curve weight for slow first

The heaviest curve and the corresponding weight can also be computed when $\tau(q)$ is non-decreasing. However, these are usually more specialized cases, and here we only show the solutions of specific cases of the slow-first policies. For slow first, there is a discontinuity in $\tau(q)$ when $q = p$, and the notation of τ_F, τ_S corresponds to the one used in Appendix C.1 above.

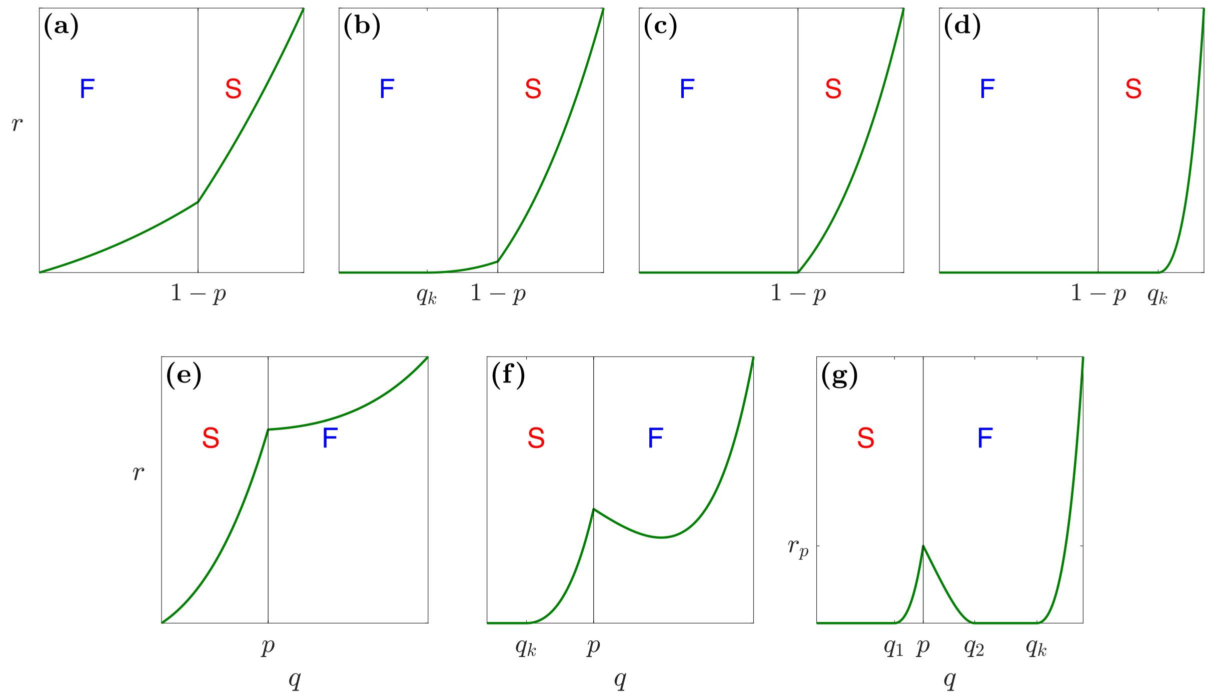


Fig. 6. Illustration of the shape of the heaviest curves for increasing congestion k. (a–d) The fast-first policy. (e–g) The slow-first policy.

Typical shapes of the heaviest curves for the slow-first policy when the proportion p of small passenger is not too small[3] are shown in Fig. 6(e–g). The corresponding heaviest weights can be summarized as follows:

$$\begin{aligned}
k < k_c : \mathcal{W}^* &= \sqrt{\int_0^p \tau_S^2(q)e^{kq}dq \quad + \quad \int_p^1 \tau_F^2(q)e^{kq}dq} \\
k_c < k < k_{S1} : \mathcal{W}^* &= \sqrt{k}\int_0^{q_k} \tau_S(q)dq \quad + \quad \frac{\tau_S(q_k)}{\sqrt{k}} \\
k_{S1} < k : \mathcal{W}^* &= \sqrt{k}\int_0^{q_1} \tau_S(q)dq \; + \; \frac{\tau_S(q_1)[1-(1-r_p)e^{k(q_1-p)}]}{\sqrt{k}} \\
&- \frac{\tau_F(q_2)[1-(1-r_p)e^{k(q_2-p)}]}{\sqrt{k}} \quad + \quad \sqrt{k}\int_{q_2}^{q_k} \tau_F(q)dq \\
&+ \frac{\tau_F(q_k)}{\sqrt{k}}.
\end{aligned} \tag{30}$$

Here q_k is given by Eq. (14), and k_c by Eq. (11).

When $k_{S1} < k$ in Eq. (30), the value $r = r_p$ defines the point where the curve crosses from the slow to the fast region in $q = p$, as indicated in Fig. 6(g). We determine r_p by the value that maximizes $\mathcal{W}^*$. For a given r_p, the curve is smoothly departing the baseline in $q = q_1$. The curve refracts downwards at $q = p$ until touching the baseline smoothly at $q = q_2$. Hence, $q_1 < p < q_2$ are given by the solutions of

$$\begin{aligned}
q_1 : \quad \tau_S^2(q_1)[1-(1-r_p)e^{k(q_1-p)}] \quad &= \quad k\int_{q_1}^{p} \tau_S^2(q)e^{k(q-q_1)}dq \\
q_2 : \quad \tau_F^2(q_2)[1-(1-r_p)e^{k(q_2-p)}] \quad &= \quad -k\int_{p}^{q_2} \tau_F^2(q)e^{k(q-q_2)}dq,
\end{aligned}$$

respectively. This means that r_p, q_1, q_2 are determined by optimization of $\mathcal{W}^*$ in Eq. (30). Finally k_{S1} is defined implicitly as the value of k that gives $q_2 = q_k$.

[3] Small p requires another special case that is not treated here (Erland et al., 2019).

C3. Heaviest curve weight for $k \to \infty$

For $k \to \infty$, we show the heaviest curve weight for non-decreasing $\tau(q)$. However, the result in Eq. (31) can also be extended to cover the slow-first policy. The right-hand-side of Eq. (14) is bounded by

$$k\int_{q_k}^{1} \tau^2(q)e^{kq}dq \quad \leqslant \quad k(1-q_k)\tau^2(q_k)e^{kq_k}.$$

When $k \to \infty$, a solution of Eq. (14) requires that $q_k \to 1$. Then the maximized weight in Eq. (15) is given by

$$\frac{\mathcal{W}(r_p^*)}{\sqrt{k}} \xrightarrow{k\to\infty} \int_0^1 \tau(q)dq, \tag{31}$$

when $\tau(1)$ is finite.

Appendix D. Proofs

D1. Proof of Theorem 1

From Eq. (20),

$$\langle Y_q^2\rangle = \langle X_q^2\rangle + 2\langle X_q\rangle\langle W\rangle p_W(q) + \langle W^2\rangle p_W(q).$$

First assume $h = 2$. When $k < k_c^*$, the total boarding time (Eq. (3)) for all three policies is determined by $\mathcal{W}(r^*)$ in Eq. (10), which with $\tau_Y^2(q) = \langle Y_q^2\rangle$ becomes

$$\begin{aligned}
\mathcal{W}^2(r^*) = \int_0^1 \langle X_q^2\rangle e^{kq}dq + 2\langle W\rangle \int_0^1 \langle X_q\rangle p_W(q)e^{kq}dq \\
+\langle W^2\rangle \int_0^1 p_W(q)e^{kq}dq.
\end{aligned} \tag{32}$$

The three boarding policies only differ through X_q, i.e., how the passengers are arranged in the queue, and $\langle X_q^2\rangle$ and $\langle X_q\rangle$ depend on q in the same way as $\tau(q)$ in Eq. (2). The waiting time W in the last term is equal for all policies.

If $k = 0$ and $p_W(q) \equiv 1$, Eq. (32) reduces to

$$\mathcal{W}^2(r^*) = \int_0^1 \langle X_q^2 \rangle dq + 2\langle W \rangle \int_0^1 \langle X_q \rangle dq + \langle W^2 \rangle, \tag{33}$$

and under these assumptions all three terms in Eq. (33) are equal for all three policies. The two integrals are simply the second and first moment, respectively, of the luggage-handling time X of all passengers seen as one group, irrespective of how they are ordered in the queue. This is the case for any policy where the queue ordering is according to luggage-handling time. E.g., with two groups the first integral is given by $\langle X_S^2 \rangle p + \langle X_F^2 \rangle (1-p)$, with a corresponding expression for the second integral.

In Eq. (32), the two first integrands in Eq. (33) are weighted by increasing functions e^{kq} and $p_W(q)e^{kq}$, respectively. This means that moments of X_q that increase in q are weighted more than those decreasing. Consequently, the slow-first policy is superior to both the fast-first and the random boarding policies. The result easily extends to the case where there are $h = 3$ passengers per row segment and there is a distinction between the waiting times W_1 and W_2 as in Eq. (21).

D2. Proof of Corollary 1

For a given luggage-handling time distribution, $\int \langle X_q^2 \rangle dq$ and $\int \langle X_q \rangle dq$ take the same values no matter how the queue is ordered. In order to have the lowest possible penalty by the factors e^{kq} and $p_W(q)e^{kq}$ in the corresponding integrals in Eq. (32), the queue should be ordered according to decreasing luggage-handling time. This slow-ranked-first policy is clearly the ordering that minimizes the curve weight in Eq. (32), when k is sufficiently small.

Appendix E. Simulation model

The simulation is by discrete event modeling with fixed time steps. For each case certain parameters are fixed: the number of passengers N, the congestion k, the policy, the proportion of passengers in each group, the luggage-handling time distribution for each group and the waiting time distributions. Empirical distributions were applied, and these were inferred from data from 296 individual passengers based on video observations from 6 different flights (Steiner & Philipp, 2009). The distributions are discretized as a multiple of the time-step. We have used congestion $k = 4$ with underlying parameters $\rho = 1$, $d = 1$, $w = 2/3$, $n_a = 1$, and $s = 6$ as defined in Section 2.1.

For each case, several of the passenger characteristics are stochastic and several scenarios are generated to reflect this. For each scenario, each of the N passengers is designated to a group, and depending on the policy they are given a queue, row and seat number. In addition, each passenger is attributed a luggage-handling time from the appropriate empirical aisle-clearing time distribution, depending on their group. Any additional waiting time is drawn from the appropriate empirical waiting time distribution, based on the queue number of the other passengers heading for the same row segment. The individual aisle-clearing time is the sum of the luggage-handling time and the waiting time.

For each time step, the queue propagates along the aisle, taking each passenger in turn, starting with the passenger standing furthest into the aisle. Let passenger A be the one under consideration: If passenger A is not at the designated row, the passenger moves towards that row, but must stop short if blocked by the passengers in front. The distance of the aisle that is occupied depends on the distance w between consecutive passengers in the aisle, and the number of standing passengers in between passenger A and the nearest passenger in front that stands waiting at a designated row. When all passengers have moved, a time-step is deducted from the remaining aisle-clearing time of those passengers already standing by their designated rows. If the aisle-clearing time has passed, the passenger sits down. Two passengers cannot clear the aisle at the same row at the same time.

This cycle is repeated until all passengers are seated. The boarding time is given as the number of time steps used.

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