REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 63, No. 2, 2022, Pages 379–395
Published online: October 12, 2022
https://doi.org/10.33044/revuma.2149

ON THE DIFFERENTIAL AND VOLTERRA-TYPE INTEGRAL OPERATORS ON FOCK-TYPE SPACES

TESFA MENGESTIE

Abstract. The differential operator fails to admit some basic structures including continuity when it acts on the classical Fock spaces or weighted Fock spaces, where the weight functions grow faster than the classical Gaussian weight function. The same conclusion also holds in some weighted Fock spaces including the Fock–Sobolev spaces, where the weight functions grow more slowly than the Gaussian function. We consider modulating the classical weight function and identify Fock-type spaces where the operator admits the basic structures. We also describe some properties of Volterra-type integral operators on these spaces using the notions of order and type of entire functions. The modulation operation supplies richer structures for both the differential and integral operators in contrast to the classical setting.

1. Introduction

The boundedness and compactness of the differential operator $Df = f'$ have been studied on various function spaces defined over the unit disc or finite domains; see for example [3, 4, 5, 10] and the references therein. The operator is known to act in unbounded way in many Banach spaces including the classical Hilbert space $L^2(\mathbb{R})$ defined over an infinite domain. In [21], Ueki showed that the operator is unbounded on the classical growth type Fock space. We continued the study in [18] and verified its unboundedness on all classical Fock spaces and weighed Fock spaces where the weight functions grow faster than the Gaussian weight function. Later in [16], we drew the same conclusion on the Fock–Sobolev spaces, which are typical examples of weighted Fock spaces with weight function growing more slowly than the Gaussian function. Following these, the author in [13] posed the question of how fast the weight function should grow in order for the corresponding weighted Fock spaces to support a bounded differentiation operator. To shed light on the question, we considered the spaces $\mathcal{F}^p_{\psi_m}$, which consist of entire functions f for

2020 *Mathematics Subject Classification.* Primary 47Exx, 30H20, 47G10; Secondary 46E22, 47B32.

Key words and phrases. Fock-type spaces; differential operator; Volterra-type integral operator; bounded; compact; order; type.

which

$$\|f\|_{\psi_m} = \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\psi_m(z)}\, dA(z) \right)^{\frac{1}{p}} < \infty,$$

where $\psi_m(z) = -|z|^m$, $m > 0$, and dA denotes the Lebesgue area measure on the complex plane $\mathbb{C}$. We proved among others the following main result.

Theorem 1.1 ([13, Theorem 1.1])**.** *Let $0 < p, q < \infty$.*

(i) *If $p \le q$, then $D : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded if and only if*

$$m \le 2 - \frac{pq}{pq + q - p} \tag{1.1}$$

and compact if and only if the inequality in (1.1) *is strict.*

(ii) *If $0 < q < p$, then the following statements are equivalent:*
- (a) *$D : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded.*
- (b) *$D : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is compact.*
- (c) *$m < 1 - 2\left(\frac{1}{q} - \frac{1}{p}\right)$.*

The result showed that the spaces $\mathcal{F}^p_{\psi_m}$ support a bounded differential operator when the weight function ψ_m actually grows much more slowly than the classical Gaussian function. In particular, when $p = q$, we note that D is bounded only when m is at most one. A natural question is what happens to the structures when we modulate the classical weight function by some positive parameters α instead of changing the rate of its growth. The first main theme of this note is to investigate this question and study the interplay between the exponents of the Fock spaces and the modulating parameters. Interestingly, it turns out that such an approach offers a rich supply of Fock-type spaces which support the operator D enriched with basic structures.

1.1. **The differential operator on the Fock-type spaces $\mathcal{F}^p_\alpha$.** We may first specify our working spaces. For a positive parameter α and $0 < p < \infty$, the Fock-type space $\mathcal{F}^p_\alpha$ consists of all entire functions f for which

$$\|f\|_{(p,\alpha)} = \left(\frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) \right)^{\frac{1}{p}} < \infty.$$

Then, our first main result asserts that D acts continuously between these spaces if and only if it is compact as precisely stated below.

Theorem 1.2. *Let $0 < p, q, \alpha, \beta < \infty$. Then the following statements are equivalent:*

(i) *$D : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is bounded.*

(ii) *$\alpha < \beta$ and $p \le q$.*

(iii) $D:\mathcal{F}_\alpha^p\to\mathcal{F}_\beta^q$ *is compact. In this case,*

$$\sqrt{\frac{\beta\alpha}{(\beta-\alpha)e}}\le\|D\|\le\sqrt{(2\pi q)^{\frac{1}{q}}e^{\frac{\alpha\beta}{\beta-\alpha}}}\sqrt{\frac{(\beta\alpha)^{\frac{2}{q}}}{(\beta-\alpha)^{\frac{3}{q}}}}. \tag{1.2}$$

Note that condition (ii) is a strict inclusion property $\mathcal{F}_\alpha^p\subset\mathcal{F}_\beta^q$. Consequently, further spectral and dynamical structures of the operator make no sense on the spaces $\mathcal{F}_\alpha^p$ as the operator is not bounded when $\alpha=\beta$. Thus, the novelty to enrich D with some basic structures is the fact that it acts between two different underlying spaces, that is, $\beta\neq\alpha$. In contrast, following the description in Theorem 1.1, several dynamical properties of D have been recently studied on the spaces $\mathcal{F}_{\psi_m}^p$ [6].

When $\alpha=\beta=1$, the spaces $\mathcal{F}_\alpha^p$ correspond to the classical Fock spaces, and Theorem 1.2 reiterates that the operator D has no bounded structure in its action on them.

1.2. **The Volterra-type integral operator.** In this section we study the effect of modulating the weight function on the structures of the Volterra-type integral operator as compared to the existing results on the classical setting. We recall that for holomorphic functions f and g, the Volterra-type integral operator V_g is given by

$$V_gf(z)=\int_0^z f(w)g'(w)\,dw.$$

The theory of the Volterra-type integral operator has attracted much research interest, especially in the last two decades. Several properties of the operator acting on various spaces of analytic functions have been extensively studied. See for example [1, 2, 7, 8, 9, 18, 19, 20]. In [7, 17], it was shown that a symbol g induces a bounded operator V_g on the classical Fock spaces if and only if it is a polynomial of degree at most two. The same conclusion was drawn on the Fock–Sobolev spaces [16, 15] which are typical examples of weighted Fock spaces with weight function growing more slowly than the Gaussian function. The study was continued in [8, 18] on weighted Fock spaces with weight function growing faster than the classical weight, where it was shown that the operator admits a richer operator-theoretic structure than on the classical setting.

Although many studies have already been done on this class of operators, there are still some interesting settings where the basic structures of the operators are unknown. The second theme of this note is to investigate such structures on Fock-type spaces generated by modulating the Gaussian function by positive parameters again. It turns out that such spaces provide a richer structure of the operator in contrast to the classical spaces.

We may first characterize the bounded and compact $V_g:\mathcal{F}_\alpha^p\to\mathcal{F}_\beta^q$ in terms of the function

$$M_{(\alpha\beta,g)}(z)=\frac{|g'(z)|e^{\frac{\alpha-\beta}{2}|z|^2}}{1+|z|}, \tag{1.3}$$

where α and β are henceforth positive numbers. A more simplified version of the characterization will be given in the subsequent corollaries and Theorem 1.6.

Theorem 1.3.

(i) *Let $0 < p \leq q < \infty$, and let g be an entire function on $\mathbb{C}$. Then $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded if and only if $M_{(\alpha\beta,g)} \in L^\infty(\mathbb{C}, dA)$, and the operator is compact if and only if $M_{(\alpha\beta,g)}(w) \to 0$ as $|w| \to \infty$. In this case,*

$$\|V_g\| \simeq \|M_{(\alpha\beta,g)}\|_{L^\infty}.$$

(ii) *Let $0 < q < p < \infty$, and let g be an entire function on $\mathbb{C}$. Then the following statements are equivalent:*
  (a) *$V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded.*
  (b) *$M_{(\alpha\beta,g)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.*
  (c) *$V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is compact. In this case,*

$$\|V_g\| \simeq \|M_{(\alpha\beta,g)}\|_{L^{\frac{pq}{p-q}}}. \tag{1.4}$$

By the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) we mean that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

Now we consider some of the implications of Theorem 1.3. In contrast to the classical setting, the theorem shows that modulating the classical weight function provides richer operator-theoretic properties to V_g only under the condition that it acts between two different Fock-type spaces again. If it acts on a space, the result remains as in the classical case. We record this as follows.

Corollary 1.4.

(i) *Let $0 < p, q < \infty$, $\alpha > \beta$ and let g be an entire function on $\mathbb{C}$. Then $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded (compact) if and only if g is identically the zero function.*

(ii) *Let $0 < p \leq q < \infty$, $\alpha = \beta$ and let g be an entire function on $\mathbb{C}$. Then $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is*
  (a) *bounded if and only if $g(z) = az^2 + bz + c$ for some $a, b, c \in \mathbb{C}$;*
  (b) *compact if and only if $g(z) = az + b$ for some $a, b \in \mathbb{C}$.*

(iii) *Let $0 < q < p < \infty$, $\alpha = \beta$ and let g be an entire function on $\mathbb{C}$. Then $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded (compact) if and only if $q > \frac{2p}{p+q}$ and $g(z) = az + b$ for some $a, b \in \mathbb{C}$.*

The corollary refines the conditions in Theorem 1.3 when $\alpha \geq \beta$. In this case the symbol g can be a complex polynomial of degree at most two. It remains to find an analogous and simpler description for the case when $\alpha < \beta$, which is the main purpose of the next section.

1.3. **The order and type of the symbol g.** In this section we study the growth of the inducing maps g using the notions of order and type of analytic functions and simplify further the conditions in Theorem 1.3 whenever $\beta > \alpha$. Interestingly, we find that the order and type properties of g completely determine the bounded and compact Volterra-type integral operators acting between two different Fock-type spaces.

For an entire function f and a positive r, we set $M_f(r) = \max\{|f(z)| : |z| = r\}$. Then the order $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\big(\log M_f(r)\big)}{\log r}.$$

If $0 < \rho(f) < \infty$, then we also define the type $\sigma(f)$ of f by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho(f)}}.$$

By definition, all polynomials have order 0, while $f(z) = e^{az}$ has order 1 and type $|a|$, and $h(z) = e^{az^2}$ has order 2 and type $|a|$.

The next result shows that if $V_g : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is bounded, then g can be at most of order 2 and type $\frac{\beta-\alpha}{2}$.

Proposition 1.5. *Let $0 < p, q < \infty, \alpha < \beta$ and let g be an entire function on $\mathbb{C}$.*

(i) *If $p \le q$, then $V_g : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is*
   (a) *bounded if and only if $\rho(g) < 2$ or $\rho(g) = 2$ and $\sigma(g) \le \frac{\beta-\alpha}{2}$;*
   (b) *compact if and only if $\rho(g) < 2$ or $\rho(g) = 2$ and $\sigma(g) < \frac{\beta-\alpha}{2}$.*

(ii) *If $p > q$, then $V_g : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is bounded (compact) if and only if $\rho(g) < 2$ or $\rho(g) = 2$ and $\sigma(g) < \frac{\beta-\alpha}{2}$.*

The result provides an interesting interplay between the order and type of the symbol g to generate a bounded and compact Volterra-type integral operator V_g. We can now give a refined form of Theorem 1.3 for the case when the inducing symbol g is zero free on the complex plane.

Theorem 1.6. *Let $0 < p, q < \infty$, $\alpha < \beta$ and let g be a non-vanishing entire function on $\mathbb{C}$.*

(i) *If $p \le q$, then $V_g : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is*
   (a) *bounded if and only if*

$$g(z) = e^{c+bz+az^2} \tag{1.5}$$

   for some $a, b, c \in \mathbb{C}$ such that $|a| < \frac{\beta-\alpha}{2}$, or $|a| = \frac{\beta-\alpha}{2}$ and either $b = 0$ or $a = -\frac{(\beta-\alpha)b^2}{2|b|^2}$;
   (b) *compact if and only if g has the form in (1.5) and $|a| < \frac{\beta-\alpha}{2}$.*

(ii) *If $p > q$, then $V_g : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is bounded (compact) if and only if g has the form in (1.5) and $|a| < \frac{\beta-\alpha}{2}$.*

We close this section with the following particular case of V_g. When $g(z) = z$, the operator V_g reduces to the classical Volterra operator $Jf(z) = \int_0^z f(w)\,dw$. We also note that $DJf = f$ and $JDf(z) = f(z) - f(0)$ for all $z \in \mathbb{C}$. Thus, it is natural to ask how the conditions for the operator J corresponding to Theorem 1.2 turn out to be. Setting $g(z) = z$ in Theorem 1.3, we deduce the following.

Corollary 1.7.

(i) *Let $0 < p \leq q < \infty$. Then $J : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded (compact) if and only if $\alpha \leq \beta$.*

(ii) *Let $0 < q < p < \infty$. Then $J : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded (compact) if and only if either $\alpha < \beta$ or $\alpha = \beta$ and $q > \frac{2p}{p+q}$.*

It follows that only compact Volterra operators are bounded, and this happens because of the linear factor in the expression in (1.3), which originates from the Littlewood–Paley type description of the spaces as stated in (2.2) below.

2. Proofs of the results

Before we begin the proofs, we record some known facts that are required in the sequel. For each entire function f and $0 < p < \infty$, the subharmonicity of $|f|^p$ implies the point estimate

$$|f(z)| \leq e^{\frac{\alpha|z|^2}{2}} \left(\int_{D(z,1)} |f(w)|^p e^{-\frac{p\alpha|w|^2}{2}}\, dA(w) \right)^{\frac{1}{p}} \leq e^{\frac{\alpha|z|^2}{2}} \|f\|_{(p,\alpha)}, \tag{2.1}$$

where $D(z,1)$ is a disc of radius 1 and center z. This shows that point evaluations are bounded linear functionals. Thus, the Fock space $\mathcal{F}_\alpha^2$ is a reproducing kernel Hilbert space with kernel function at each point $w \in \mathbb{C}$ given by

$$K_{(w,\alpha)}(z) = e^{\alpha\overline{w}z} \qquad \text{and} \qquad k_{(w,\alpha)}(z) = \frac{K_{(w,\alpha)}(z)}{\|K_{(w,\alpha)}\|_{(2,\alpha)}} = e^{\alpha\overline{w}z - \frac{\alpha}{2}|w|^2}.$$

Moreover, from a straightforward computation we get that, for all $0 < p < \infty$,

$$\|K_{(w,\alpha)}\|_{(p,\alpha)} = e^{\frac{\alpha}{2}|w|^2}.$$

Another useful result is due to Constantin [7], who describes the Fock-type spaces in terms of derivatives; namely that

$$\|f\|_{(p,\alpha)}^p \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z). \tag{2.2}$$

2.1. **Proof of Theorem 1.2.** Since (iii) obviously implies (i), we plan to show that (i) implies (ii) and that (ii) implies (iii). Thus, suppose that $D : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded. Then, applying the operator to the reproducing kernel $K_{(w,\alpha)}$, we have

$$\begin{aligned} \|K_{(w,\alpha)}\|_{(p,\alpha)}^q \|D\|^q &\geq \frac{q\beta}{2\pi} |w\alpha|^q \int_{\mathbb{C}} \left| K_{(w,\alpha)}(z) \right|^q e^{-\frac{q\beta}{2}|z|^2}\, dA(z) \\ &= |w\alpha|^q \left(\frac{q\beta}{2\pi} \int_{\mathbb{C}} \left| e^{\beta\left(\frac{\alpha\overline{w}}{\beta}\right)z} \right|^q e^{-\frac{q\beta}{2}|z|^2}\, dA(z) \right) \\ &= |w\alpha|^q \left\| K_{\left(\frac{\alpha w}{\beta},\beta\right)} \right\|_{(q,\beta)}^q \end{aligned}$$

for all $w \in \mathbb{C}$. It follows that

$$\|D\| \geq \sup_{w\in\mathbb{C}} \frac{|w\alpha| \left\| K_{(\frac{\alpha w}{\beta},\beta)} \right\|_{(q,\beta)}}{\left\| K_{(w,\alpha)} \right\|_{(p,\alpha)}} = \sup_{w\in\mathbb{C}} |w\alpha| e^{\frac{\alpha}{2}|w|^2\left(\frac{\alpha}{\beta}-1\right)},$$

from which the necessity of $\alpha < \beta$ holds when $|w| \to \infty$. Moreover,

$$\|D\| \geq \sup_{w\in\mathbb{C}} |w\alpha| e^{\frac{\alpha}{2}|w|^2\left(\frac{\alpha}{\beta}-1\right)} = \sqrt{\frac{\alpha\beta}{(\beta-\alpha)e}},$$

which is the lower estimate in (1.2).

Next, we proceed to show the other requirement, $p \leq q$. Assume on the contrary that $p > q$. We plan to follow a classical technique whose original idea goes back to Luecking [12]. Consider a sequence (λ_j) in $\mathbb{C}$ such that $\inf_{j\neq i} |\lambda_j - \lambda_i| > 0$. Then, for any given positive number r, there exists a positive integer N_r such that any disc of radius r contains at most N_r points of the sequence (λ_j). Furthermore, we assume that the sequence (λ_j) is dense in $\mathbb{C}$ in the sense that there exists a small positive number ϵ such that every point of $\mathbb{C}$ is contained in some disc $D(\lambda_j, \epsilon)$. It follows that the function

$$F = \sum_{j\geq 1} a_j k_{(\lambda_j,\alpha)}$$

belongs to $\mathcal{F}_\alpha^p$ for every ℓ^p sequence (a_j) with norm estimate $\|F\|_{(p,\alpha)} \lesssim \|(a_j)\|_{\ell^p}$. See [23, Theorem 2.34] or [11, 22] for details.

If $(r_j(t))_j$ is the Rademacher sequence of functions on $[0,1]$ chosen as in [12], then the sequence $(a_j r_j(t))$ also belongs to ℓ^p with $\|(a_j r_j(t))\|_{\ell^p} = \|(a_j)\|_{\ell^p}$ for all t. Consequently, the parameterized functions

$$F_t = \sum_{j\geq 1} a_j r_j(t) k_{(\lambda_j,\alpha)}$$

belong to $\mathcal{F}_\alpha^p$ with norm estimates $\|F_t\|_{(p,\alpha)} \lesssim \|(a_j)\|_{\ell^p}$. Furthermore, an application of Khinchine's inequality [12] yields

$$\left(\sum_{j\geq 1} |a_j|^2 |k'_{(\lambda_j,\alpha)}(z)|^2\right)^{\frac{q}{2}} \lesssim \int_0^1 \left|\sum_{j\geq 1} a_j r_j(t) k'_{(\lambda_j,\alpha)}(z)\right|^q dt. \tag{2.3}$$

Making use of (2.3), and subsequently of Fubini's theorem, we further have

$$\begin{aligned}
&\int_{\mathbb{C}} \left(\sum_{j\geq 1} |a_j|^2 |k'_{(\lambda_j,\alpha)}(z)|^2\right)^{\frac{q}{2}} e^{-\frac{q\beta}{2}|z|^2}\, dA(z)\\
&\quad\lesssim \int_{\mathbb{C}}\int_0^1 \left|\sum_{j\geq 1} a_j r_j(t) k'_{(\lambda_j,\alpha)}(z)\right|^q dt\, e^{-\frac{q\beta}{2}|z|^2}\, dA(z)\\
&\quad= \int_0^1 \int_{\mathbb{C}} \left|\sum_{j\geq 1} a_j r_j(t) k'_{(\lambda_j,\alpha)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}\, dA(z)\, dt\\
&\quad\simeq \int_0^1 \|DF_t\|_{(q,\beta)}^q\, dt \lesssim \|(a_j)\|_{\ell^p}^q.
\end{aligned}$$

Now since the discs $D(\lambda_j, 2\epsilon)$ cover $\mathbb{C}$, we estimate

$$\sum_{j\geq 1} |a_j|^q \int_{D(\lambda_j,2\epsilon)} (1+|z|)^q \, dA(z)$$
$$\simeq \sum_{j\geq 1} |a_j|^q \int_{D(\lambda_j,2\epsilon)} (1+|z|)^q \frac{|k'_{\lambda_j}(z)|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|\lambda_j|)^q} \, dA(z)$$
$$\lesssim \max\{1, N_{\max}^{1-q/2}\} \int_{\mathbb{C}} \Big(\sum_{j\geq 1} |a_j|^2 |k'_{\lambda_j}(z)|^2 \Big)^{\frac{q}{2}} e^{-\frac{q\beta}{2}|z|^2} \, dA(z) \lesssim \|(a_j)\|_{\ell^p}^q,$$

where we also used that $1+|z| \simeq 1+|\lambda_j|$ whenever $z \in D(\lambda_j, 2\epsilon)$. Applying duality between the spaces $\ell^{p/q}$ and $\ell^{p/(p-q)}$, we get

$$\sum_{j\geq 1} \Big(\int_{D(\lambda_j,2\epsilon)} (1+|z|)^q \, dA(z) \Big)^{\frac{p}{p-q}} < \infty.$$

On the other hand, for each $z \in D(\lambda_j, 3\epsilon/2)$,

$$(1+|z|)^q \lesssim \int_{D(\lambda_j,2\epsilon)} (1+|w|)^q \, dA(w),$$

and hence

$$\int_{\mathbb{C}} (1+|z|)^{\frac{qp}{p-q}} \, dA(z) \leq \sum_{j\geq 1} \int_{D(\lambda_j,3\epsilon/2)} (1+|z|)^{\frac{qp}{p-q}} \, dA(z)$$
$$\lesssim \sum_{j\geq 1} \Big(\int_{D(\lambda_j,2\epsilon)} (1+|w|)^q \, dA(w) \Big)^{\frac{p}{p-q}} < \infty,$$

which is a contradiction as $p > q$ and $1+|z|$ cannot be $pq/(p-q)$ integrable over the whole complex plane $\mathbb{C}$.

Assume now that (ii) holds and let f_n be a sequence of functions in $\mathcal{F}_\alpha^p$ such that $\sup_n \|f_n\|_{(p,\alpha)} < \infty$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Then, for $R > 0$,

$$\|Df_n\|_{(q,\beta)}^q = \frac{q\beta}{2\pi} \int_{|z|\leq R} |f'_n(z)|^q e^{-\frac{q\beta}{2}|z|^2} \, dA(z) + \frac{q\beta}{2\pi} \int_{|z|>R} |f'_n(z)|^q e^{-\frac{q\beta}{2}|z|^2} \, dA(z)$$
$$\lesssim \sup_{|z|\leq R} (|f'_n(z)|^q) \int_{|z|\leq R} e^{-\frac{q\beta}{2}|z|^2} \, dA(z)$$
$$+ \sup_{|z|>R} \Big((1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2} \Big) \int_{|z|>R} \frac{|f'_n(z)|^q e^{-\frac{q\alpha}{2}|z|^2}}{(1+|z|)^q} \, dA(z).$$

The sequence $f'_n \to 0$ uniformly on compact subsets of $\mathbb{C}$. Thus

$$\sup_{|z|\leq R} (|f'_n(z)|^q) \int_{|z|\leq R} e^{-\frac{q\beta}{2}|z|^2} \, dA(z) \to 0$$

as $n \to \infty$. On the other hand,

$$\begin{aligned}
&\sup_{|z|>R} \Big((1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2}\Big) \int_{|z|>R} \frac{|f_n'(z)|^q e^{-\frac{q\alpha}{2}|z|^2}}{(1+|z|)^q}\, dA(z) \\
&\quad \lesssim \sup_{|z|>R} \Big((1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2}\Big) \bigg(|f_n(0)|^q + \int_{|z|>R} \frac{|f_n'(z)|^q e^{-\frac{q\alpha}{2}|z|^2}}{(1+|z|)^q}\, dA(z)\bigg) \\
&\quad \lesssim \|f_n\|_{(q,\alpha)}^q \sup_{|z|>R} \Big((1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2}\Big),
\end{aligned}$$

where, for the last inequality, we used (2.2). Since $p \leq q$, applying the inclusion property on Fock spaces and the assumption on the sequence f_n, we have

$$\begin{aligned}
&\|f_n\|_{(q,\alpha)}^q \sup_{|z|>R} \Big((1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2}\Big) \\
&\quad \leq \|f_n\|_{(p,\alpha)}^q \sup_{|z|>R} \Big((1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2}\Big) \lesssim \sup_{|z|>R} (1+|z|)^q e^{-\frac{q}{2}(\beta-\alpha)|z|^2}.
\end{aligned}$$

Then, for $\beta > \alpha$, letting $R \to \infty$ in the last inequality, we observe that Df_n converges to zero in the space $\mathcal{F}_\beta^q$.

It remains to verify the upper estimate in (1.2). Thus, let $\alpha < \beta$ and $f \in \mathcal{F}_\alpha^p$. Applying Cauchy's derivative formula

$$f'(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(z+w)}{w^2}\, dA(w),$$

it follows that

$$\begin{aligned}
\|Df\|_{(q,\beta)}^q &= \frac{q\beta}{2\pi} \int_{\mathbb{C}} |f'(z)|^q e^{-\frac{q\beta}{2}|z|^2}\, dA(z) \\
&\leq \frac{q\beta}{(2\pi)^{q+1}} \int_{\mathbb{C}} \bigg(\int_{|w|=1} |f(z+w)||dA(w)|\bigg)^q e^{-\frac{q\beta}{2}|z|^2}\, dA(z).
\end{aligned}$$

The inner integral is estimated as

$$\begin{aligned}
&\int_{|w|=1} |f(z+w)||dA(w)| \\
&\qquad \leq \int_{|w|=1} e^{\frac{\alpha}{2}|z+w|^2} \Big(\sup_{z\in\mathbb{C}} |f(z+w)| e^{-\frac{\alpha}{2}|z+w|^2}\Big) |dA(w)| \\
&\qquad \leq \|f\|_{(p,\alpha)} \int_{|w|=1} e^{\frac{\alpha}{2}|z+w|^2} |dA(w)| \leq 2\pi e^{\frac{\alpha}{2}} \|f\|_{(p,\alpha)} e^{\frac{\alpha}{2}|z|^2+\alpha|z|},
\end{aligned}$$

where we used the inequality

$$\sup_{z\in\mathbb{C}} |f(z+w)| e^{-\frac{\alpha}{2}|z+w|^2} \leq \|f\|_{(p,\alpha)}.$$

Integrating with polar coordinates we get

$$\begin{aligned}\|Df\|_{(q,\beta)}^q &\le \frac{q\beta}{2\pi} e^{\frac{q\alpha}{2}} \|f\|_{(p,\alpha)}^q \int_{\mathbb{C}} e^{-\frac{q(\beta-\alpha)}{2}|z|^2+q\alpha|z|}\, dA(z)\\ &= q\beta e^{\frac{q\alpha}{2}} \|f\|_{(p,\alpha)}^q \int_0^\infty r e^{-\frac{q(\beta-\alpha)}{2}r^2+q\alpha r}\, dr\\ &\le q\beta e^{\frac{q\alpha}{2}} \|f\|_{(p,\alpha)}^q \frac{\sqrt{2\pi}\alpha e^{\frac{q\alpha^2}{2(\beta-\alpha)}}}{\sqrt{q(\beta-\alpha)^3}}\\ &= \sqrt{\frac{2\pi q}{(\beta-\alpha)^3}}\beta\alpha e^{\frac{q\alpha\beta}{2(\beta-\alpha)}} \|f\|_{(p,\alpha)}^q,\end{aligned}$$

from which the right-hand estimate in (1.2) follows, and this completes the proof. □

2.2. **Proof of Theorem 1.3.** (i) Let $0 < p \le q < \infty$. We first prove the necessity of the condition for boundedness. Considering the action of the operator on the normalized reproducing kernels, and applying (2.2) and (2.1), we obtain

$$\begin{aligned}\|V_g\|^q \ge \|V_g k_{(w,\alpha)}\|_{(q,\beta)}^q &\gtrsim \int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+|z|)^q} |k_{(w,\alpha)}(z)|^q e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &\ge \int_{D(w,1)} \frac{|g'(z)|^q}{(1+|z|)^q} |e^{\alpha\overline{w}z}|^q e^{-\frac{q}{2}(\beta+\alpha)|z|^2}\, dA(z) \gtrsim \frac{|g'(w)|^q}{(1+|w|)^q} e^{\frac{q}{2}(\alpha-\beta)|w|^2}\end{aligned}$$

for all $w \in \mathbb{C}$. This gives

$$\|V_g\|^q \gtrsim \sup_{w\in\mathbb{C}} \frac{|g'(w)|^q}{(1+|w|)^q} |e^{\frac{q}{2}(\alpha-\beta)|w|^2} = \sup_{w\in\mathbb{C}} M_{(\alpha\beta,g)}^q(w). \tag{2.4}$$

Conversely, the relations (2.4) and (2.2) imply

$$\begin{aligned}\|V_g f\|_{(q,\beta)}^q &\simeq \int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+|z|)^q} |f(z)|^q e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &\le \sup_{z\in\mathbb{C}} \left(\frac{|g'(z)|^q}{(1+|z|)^q} e^{\frac{q(\alpha-\beta)}{2}|z|^2} \right) \int_{\mathbb{C}} |f(z)|^q e^{-\frac{\alpha q}{2}|z|^2}\, dA(z)\\ &= \frac{2\pi}{q\beta} \|f\|_{(q,\alpha)}^q \sup_{z\in\mathbb{C}} \frac{|g'(z)|^q}{(1+|z|)^q} e^{\frac{q(\alpha-\beta)}{2}|z|^2} \lesssim \|f\|_{(p,\alpha)}^q,\end{aligned}$$

where the last inequality follows by the inclusion property. From this and (2.4), we get that

$$\|V_g\| \simeq \sup_{w\in\mathbb{C}} M_{(\alpha\beta,g)}(w).$$

Next, suppose that $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is compact. Since $k_{(w,\alpha)}$ is uniformly bounded on $\mathcal{F}_\alpha^p$ and converges to zero as $|w| \to \infty$, we have

$$M_{(\alpha\beta,g)}(w) \lesssim \|V_g K_{(w,\alpha)}\|_{(q,\beta)} \to 0$$

as $|w| \to \infty$. Therefore, $M_{(\alpha\beta,g)}(w) \to 0$ as $|w| \to \infty$.

To prove the sufficiency of the condition, let (f_n) be a sequence of functions in $\mathcal{F}_\alpha^p$ such that $\sup_n \|f_n\|_{(p,\alpha)} < \infty$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Then, for $R > 0$, applying (2.2) again we have

$$\begin{aligned}\|V_g f_n\|_{(q,\beta)}^q &\simeq \int_{\mathbb{C}} |f_n(z)|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &= \int_{|z|\le R} |f_n(z)|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &\quad + \int_{|z|>R} |f_n(z)|^q M_{(\alpha\beta,g)}^q(z) e^{-\frac{\alpha q}{2}|z|^2}\, dA(z)\\ &\lesssim \|V_g 1\|_{(q,\beta)}^q \max_{|z|\le R} |f_n(z)|^q + \|f_n\|_{(q,\alpha)}^q \sup_{|z|>R} M_{(\alpha\beta,g)}^q(z).\end{aligned}$$

Note that, by (i), the condition obviously implies boundedness. Consequently, $\|V_g 1\|_{(q,\beta)} \le \|V_g\| \|1\|_{(p,\alpha)}$ and hence

$$\|V_g f_n\|_{(q,\beta)}^q \le \|1\|_{(p,\alpha)}^q \max_{|z|\le R} |f_n(z)|^q + \|f_n\|_{(p,\alpha)}^q \sup_{|z|>R} M_{(\alpha\beta,g)}^q(z),$$

where in the last inequality we use the inclusion property on Fock spaces. Now let $n \to \infty$ and then $R \to \infty$ in the above relation to deduce that $\|V_g f_n\|_{(q,\beta)} \to 0$ as $n \to \infty$. Thus, V_g is compact as asserted.

(ii) If $M_{(\alpha\beta,g)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$, applying (2.2) and Hölder's inequality we have

$$\begin{aligned}\|V_g f\|_{(q,\beta)}^q &\simeq \int_{\mathbb{C}} \frac{|g'(z)|^q |f(z)|^q}{(1+|z|)^q} e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &= \int_{\mathbb{C}} |f(z)|^q M_{(\alpha\beta,g)}^q(z) e^{-\frac{\alpha q}{2}|z|^2}\, dA(z)\\ &\le \left(\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\right)^{\frac{q}{p}} \left(\int_{\mathbb{C}} M_{(\alpha\beta,g)}^{\frac{pq}{p-q}}(z)\, dA(z)\right)^{\frac{p-q}{p}} \lesssim \|f\|_{(p,\alpha)}^q.\end{aligned}$$

This verifies that (b) implies the statement in (a) and the one-side estimate

$$\|V_g\| \lesssim \|M_{(\alpha\beta,g)}\|_{L^{\frac{pq}{p-q}}}. \tag{2.5}$$

Next, we show that (b) implies (c). Let (f_n) be a sequence of functions in $\mathcal{F}_\alpha^p$ such that $\sup_n \|f_n\|_{(p,\alpha)} < \infty$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Then, for $R > 0$,

$$\begin{aligned}\|V_g f_n\|_{(q,\beta)}^q &\simeq \int_{\mathbb{C}} |f_n(z)|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &= \int_{|z|\le R} |f_n(z)|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\ &\quad + \int_{|z|>R} |f_n(z)|^q M_{(\alpha\beta,g)}^q(z) e^{-\frac{\alpha q}{2}|z|^2}\, dA(z)\\ &\lesssim \|V_g 1\|_{(q,\beta)}^q \max_{|z|\le R} |f_n(z)|^q + \int_{|z|>R} |f_n(z)|^q M_{(\alpha\beta,g)}^q(z) e^{-\frac{\alpha q}{2}|z|^2}\, dA(z).\end{aligned}$$

On the other hand, applying Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{|z|>R} |f_n(z)|^q M^q_{(\alpha\beta,g)}(z) e^{-\frac{\alpha q}{2}|z|^2}\, dA(z)\\
&\quad \le \bigg(\int_{|z|>R} |f_n(z)|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\bigg)^{\frac{q}{p}} \bigg(\int_{|z|>R} M^{\frac{pq}{p-q}}_{(\alpha\beta,g)}(z)\, dA(z)\bigg)^{\frac{p-q}{p}}\\
&\quad \lesssim \|f_n\|^q_{(p,\alpha)} \bigg(\int_{|z|>R} M^{\frac{pq}{p-q}}_{(\alpha\beta,g)}(z)\, dA(z)\bigg)^{\frac{p-q}{p}} \lesssim \bigg(\int_{|z|>R} M^{\frac{pq}{p-q}}_{(\alpha\beta,g)}(z)\, dA(z)\bigg)^{\frac{p-q}{p}}.
\end{aligned}$$

Collecting all the above estimates, we get

$$\|V_g f_n\|^q_{(q,\beta)} \lesssim \|V_g 1\|^q_{(q,\beta)} \max_{|z|\le R} |f_n(z)|^q + \bigg(\int_{|z|>R} M^{\frac{pq}{p-q}}_{(\alpha\beta,g)}(z)\, dA(z)\bigg)^{\frac{p-q}{p}}.$$

Letting $n\to\infty$ and then $R\to\infty$ in the above relation, we get that $\|V_g f_n\|_{(q,\beta)} \to 0$ as $n\to\infty$. Therefore, condition (iii) holds.

It remains to show that (a) implies (b). Assume that $V_g : \mathcal{F}^p_\alpha \to \mathcal{F}^q_\beta$ is bounded; then

$$\begin{aligned}
\|V_g\|^q \|f\|^q_{(p,\alpha)} \ge \|V_g f\|^q_{(q,\beta)} &\simeq \int_{\mathbb{C}} \frac{|g'(z)|^q |f(z)|^q}{(1+|z|)^q} e^{-\frac{\beta q}{2}|z|^2}\, dA(z)\\
&= \int_{\mathbb{C}} |f(z)|^q e^{-\frac{\alpha q}{2}|z|^2}\, d\mu_{(\alpha\beta,g,q)}(z),
\end{aligned}$$

where $d\mu_{(\alpha\beta,g,q)}(z) = M^q_{(\alpha\beta,g)}(z)\, dA(z)$. It follows that $\mu_{(\alpha\beta,g,q)}$ is a (p,q) Fock–Carleson measure, and by [14, Theorem 2.3] this holds if and only if

$$\widetilde{\mu_{(\alpha\beta,g,q)}}(z) = \int_{\mathbb{C}} |k_{(z,\alpha)}(w)|^q e^{-\frac{\alpha q}{2}|w|^2}\, d\mu_{(\alpha\beta,g,q)}(w) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$$

and

$$\|\mu_{(\alpha\beta,g,q)}\|^q \simeq \|\widetilde{\mu_{(\alpha\beta,g,q)}}\|_{L^{\frac{p}{p-q}}(\mathbb{C},dA)}, \tag{2.6}$$

where $\|\mu_{(\alpha\beta,g,q)}\|$ denotes the norm of the embedding map from $\mathcal{F}^p_\alpha$ to $L^q\big(\mathbb{C}, e^{-\frac{q\alpha}{2}|w|^2} d\mu_{(\alpha\beta,g,q)}\big)$. Furthermore, by (2.1), we have that

$$\begin{aligned}
&\int_{\mathbb{C}} |k_{(z,\alpha)}(w)|^q e^{-\frac{\alpha q}{2}|w|^2}\, d\mu_{(\alpha\beta,g,q)}(w)\\
&\qquad = \int_{\mathbb{C}} |k_{(z,\alpha)}(w)|^q e^{-\frac{\alpha q}{2}|w|^2} \frac{|g'(w)|^q}{(1+|w|)^q} e^{\frac{q(\alpha-\beta)}{2}|w|^2}\, dA(z)\\
&\qquad \ge \int_{D(z,1)} |k_{(z,\alpha)}(w)|^q e^{-\frac{\alpha q}{2}|w|^2} \frac{|g'(w)|^q}{(1+|w|)^q} e^{\frac{q(\alpha-\beta)}{2}|w|^2}\, dA(z)\\
&\qquad \ge |k_{(z,\alpha)}(z)|^q e^{-\frac{\alpha q}{2}|z|^2} \frac{|g'(z)|^q}{(1+|z|)^q} e^{\frac{q(\alpha-\beta)}{2}|z|^2} = M^q_{(\alpha\beta,g)}(z).
\end{aligned}$$

Thus,

$$\int_{\mathbb{C}} M_{(\alpha\beta,g)}^{\frac{pq}{p-q}}(z)\,dA(z) \le \int_{\mathbb{C}} \widetilde{\mu_{(\alpha\beta,g,q)}}(z)^{\frac{p}{p-q}}\,dA(z) < \infty.$$

This and (2.6) yield the other side estimate,

$$\|M_{(\alpha\beta,g)}\|_{L^{\frac{pq}{p-q}}} \lesssim \|V_g\|,$$

which, together with (2.5), gives the estimate in (1.4) and completes the proof. □

2.3. **Proof of Proposition 1.5.** (i) Consider first that $\rho(g) < 2$. Then, for some positive number ϵ,

$$\rho(g) = \limsup_{r\to\infty} \frac{\log\big(\log M_g(r)\big)}{\log r} = 2 - \epsilon.$$

This means that one can find a positive number R_0 such that, for all $r > R_0$,

$$M_g(r) \le e^{r^{2-\epsilon}}.$$

Thus, for all $|z| = r > R_0$, we have $|g(z)| \le e^{r^{2-\epsilon}}$ and

$$\frac{|g'(z)|e^{\frac{\alpha-\beta}{2}|z|^2}}{1+|z|} \le e^{r^2\left(r^{-\epsilon}+\frac{\alpha-\beta}{2}\right)}. \tag{2.7}$$

Observe that the right-hand side above goes to zero as $r \to \infty$. By Theorem 1.3, it follows that the operator is bounded and compact.

If $\rho(g) = 2$ and $\sigma(g) \le \frac{\beta-\alpha}{2}$, arguing as above we have, for all $r > R_0$,

$$\frac{\log M_g(r)}{r^2} \le \sigma(g)$$

and hence

$$\frac{|g'(z)|e^{\frac{\alpha-\beta}{2}|z|^2}}{1+|z|} \lesssim e^{r^2\left(\sigma(g)+\frac{\alpha-\beta}{2}\right)}. \tag{2.8}$$

Therefore, by Theorem 1.3, V_g is bounded whenever $\sigma(g) \le \frac{\beta-\alpha}{2}$ and compact when the inequality is strict.

Conversely, suppose $\rho(g) > 2$, then there exists a sequence of positive numbers r_n such that

$$\frac{\log\big(\log M_g(r_n)\big)}{\log r_n} \ge \frac{\rho(g)+2}{2}$$

and hence

$$M_g(r_n) \ge e^{r_n^{\frac{\rho(g)+2}{2}}}.$$

It follows that one can find a sequence z_n such that $|z_n| = r_n \to \infty$ and

$$\frac{|g'(z_n)|e^{\frac{\alpha-\beta}{2}|z_n|^2}}{1+|z_n|} \gtrsim e^{r_n^{\frac{\rho(g)+2}{2}}+\frac{\alpha-\beta}{2}r_n^2} \to \infty$$

as $n \to \infty$. Then by Theorem 1.3 again, the operator is not bounded in this case.

Next, if $\rho(g) = 2$ and $\sigma(g) > \frac{\beta-\alpha}{2}$, we show that the operator is not bounded. Indeed, arguing as above, there exists a positive sequence r_n such that

$$\frac{\log M_g(r_n)}{r_n^2} \geq \frac{\sigma(g)}{2} + \frac{\beta-\alpha}{4}$$

and hence $M_g(r_n) \geq e^{\frac{\sigma(g)}{2}+\frac{\beta-\alpha}{4}}$. Thus, we can find a sequence z_n such that $|z_n| = r_n \to \infty$ and

$$\frac{|g'(z_n)|e^{\frac{\alpha-\beta}{2}|z_n|^2}}{1+|z_n|} \gtrsim e^{\left(\frac{\sigma(g)}{2}+\frac{\beta-\alpha}{4}+\frac{\alpha-\beta}{2}\right)r_n^2} \to \infty$$

as $n \to \infty$, and our assertion follows after an application of Theorem 1.3.

The proof of the compactness part follows from a simple modification of the arguments made above.

(ii) The case for $p > q$ follows again easily from the estimates in (2.7) and (2.8), and the second part of Theorem 1.3. □

2.4. **Proof of Theorem 1.6.** (i) (a) Suppose $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded. Then, by Proposition 1.5, we have that $\rho(g) \leq 2$. In addition, since g is non-vanishing on $\mathbb{C}$, an application of Hadamard's product formula yields

$$g(z) = e^{h(z)}, \tag{2.9}$$

where $h(z) = c + bz + az^2$ is a polynomial of degree equal to the order of g, which is at most 2. If $a = 0$, then the necessity of the condition obviously follows from this and Theorem 1.3. Thus we assume that $a \neq 0$ and hence $\sigma(g) = |a|$. By Proposition 1.5 we also have the relation

$$|a| = \sigma(g) \leq \frac{\beta-\alpha}{2},$$

from which part of the necessity condition follows. It remains to show that when $|a| = \frac{\beta-\alpha}{2}$ we have either $b = 0$ or $a = -\frac{(\beta-\alpha)b^2}{2|b|^2}$.

Using the form in (2.9), we estimate the function $M_{(\alpha\beta,g)}$ in (1.3) as

$$M_{(\alpha\beta,g)}(z) = \frac{|g'(z)|e^{\frac{\alpha-\beta}{2}|z|^2}}{1+|z|} \simeq \frac{|2az+b|}{1+|z|}e^{\Re(bz)+\Re(az^2)-\frac{\beta-\alpha}{2}|z|^2} \tag{2.10}$$

for all $z \neq 0$. Setting $a = |a|e^{-2i\theta} = \frac{\beta-\alpha}{2}e^{-2i\theta}$, $0 \leq \theta < \pi$, and replacing z by $e^{i\theta}w$ in (2.10), we have

$$\begin{aligned} M_{(\alpha\beta,g)}(we^{i\theta}) &\simeq \frac{|2awe^{i\theta}+b|}{1+|w|}e^{\Re(bwe^{i\theta})+|a|\Re(w^2)-\frac{\beta-\alpha}{2}|w|^2} \\ &= \frac{|2awe^{i\theta}+b|}{1+|w|}e^{\Re(bwe^{i\theta})+\frac{\beta-\alpha}{2}\left(\Re(w^2)-|w|^2\right)} \end{aligned} \tag{2.11}$$

for all $w \in \mathbb{C}$. In particular, when w is a real number, (2.11) simplifies to

$$M_{(\alpha\beta,g)}(we^{i\theta}) \simeq \frac{|2awe^{i\theta}+b|}{1+|w|}e^{w\Re(be^{i\theta})}. \tag{2.12}$$

By Theorem 1.3, boudedness of V_g implies that the quantity in (2.12) is uniformly bounded and this happens only when $\Re(be^{i\theta}) = 0$. This holds again only if either $b = 0$ or $e^{-i\theta} = \pm ib/|b|$, from which the assertion follows.

Conversely, assume g has the form in (1.5) and the conditions in (i) of the theorem hold. If $|a| < \frac{\beta-\alpha}{2}$, then

$$M_{(\alpha\beta,g)}(z) \simeq \frac{|2az+b|}{1+|z|} e^{\Re(bz)+\Re(az^2)-\frac{\beta-\alpha}{2}|z|^2} \to 0$$

as $|z| \to \infty$, and by Theorem 1.3, the operator is bounded and compact. Assume $|a| = \frac{\beta-\alpha}{2}$. Since the case for $b = 0$ is immediate, we suppose $a = -\frac{(\beta-\alpha)b^2}{2|b|^2}$ and write $a = \frac{\beta-\alpha}{2}e^{-2i\theta}$ as before. Then an easy simplification using the relation $a = \frac{\beta-\alpha}{2}e^{-2i\theta} = -\frac{(\beta-\alpha)b^2}{2|b|^2}$ yields

$$(e^{i\theta}b)^2 = -|b|^2,$$

which shows that $e^{i\theta}b$ is a purely imaginary number. Therefore, we set $e^{i\theta}b = iy$ for some real number y. Furthermore, setting $z = we^{i\theta}$ as before, we note that by Theorem 1.3 V_g is bounded if and only if

$$\begin{aligned}\sup_{w\in\mathbb{C}} M_{(\alpha\beta,g)}(we^{i\theta}) &\simeq \sup_{w\in\mathbb{C}} \frac{|2awe^{i\theta}+b|}{1+|w|} e^{\Re(bwe^{i\theta})+\frac{\beta-\alpha}{2}\left(\Re(w^2)-|w|^2\right)}\\ &\simeq \sup_{w\in\mathbb{C}} e^{\Re(bwe^{i\theta})+\frac{\beta-\alpha}{2}\left(\Re(w^2)-|w|^2\right)} < \infty.\end{aligned}$$

We show this as

$$\begin{aligned}\sup_{w\in\mathbb{C}} e^{\Re(bwe^{i\theta})+\frac{\beta-\alpha}{2}\left(\Re(w^2)-|w|^2\right)} &= \sup_{w\in\mathbb{C}} e^{y\Re(iw)+\frac{\beta-\alpha}{2}\left(\Re(w^2)-|w|^2\right)}\\ &= \sup_{w\in\mathbb{C}} e^{-\Im w\left(y+(\beta-\alpha)\Im w\right)} \lesssim e^{\frac{y^2}{\beta-\alpha}} < \infty,\end{aligned}$$

as required.

(b) The sufficiency of the condition $|a| < \frac{\beta-\alpha}{2}$ follows from (2.10) and Theorem 1.3. To prove the necessity, note that by part (a) we have $|a| \leq \frac{\beta-\alpha}{2}$. Thus, we need to show that the operator is not compact whenever $|a| = \frac{\beta-\alpha}{2}$. From the arguments made above in (2.12) related to boundedness, the expression

$$M_{(\alpha\beta,g)}(we^{i\theta}) \simeq \frac{|2awe^{i\theta}+b|}{1+|w|} e^{w\Re(be^{i\theta})}$$

is uniformly bounded over the real numbers w only if $\Re(be^{i\theta}) = 0$. It follows that

$$M_{(\alpha\beta,g)}(we^{i\theta}) \simeq 1 \tag{2.13}$$

when $w \to \infty$, and it fails to satisfy the compactness condition in Theorem 1.3. Therefore, $|a| < \frac{\beta-\alpha}{2}$.

(ii) All the estimations made above with $M_{(\alpha\beta,g)}$ are independent of the size of the exponents p and q. If $|a| < \frac{\beta-\alpha}{2}$, then from the estimation in (2.10), we observe that $M_{(\alpha\beta,g)}$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ and hence the sufficiency of the condition follows from Theorem 1.3.

Conversely, it is enough to show that, when $|a| = \frac{\beta-\alpha}{2}$, the operator is not bounded. But this follows by simply arguing as above and noting from (2.12) and (2.13) that the function $M_{(\alpha\beta,g)}$ is not $L^{\frac{pq}{p-q}}$ integrable over $\mathbb{C}$. Then our assertion follows again by Theorem 1.3. □

References

[1] A. Aleman and J. A. Cima, An integral operator on H^p and Hardy's inequality, *J. Anal. Math.* **85** (2001), 157–176. MR 1869606.

[2] A. Aleman and A. G. Siskakis, Integration operators on Bergman spaces, *Indiana Univ. Math. J.* **46** (1997), no. 2, 337–356. MR 1481594.

[3] M. J. Beltrán, Dynamics of differentiation and integration operators on weighted spaces of entire functions, *Studia Math.* **221** (2014), no. 1, 35–60. MR 3194061.

[4] J. Bonet, Dynamics of the differentiation operator on weighted spaces of entire functions, *Math. Z.* **261** (2009), no. 3, 649–657. MR 2471093.

[5] J. Bonet and A. Bonilla, Chaos of the differentiation operator on weighted Banach spaces of entire functions, *Complex Anal. Oper. Theory* **7** (2013), no. 1, 33–42. MR 3010787.

[6] J. Bonet, T. Mengestie and M. Worku, Dynamics of the Volterra-type integral and differentiation operators on generalized Fock spaces, *Results Math.* **74** (2019), no. 4, Paper No. 197, 15 pp. MR 4040629.

[7] O. Constantin, A Volterra-type integration operator on Fock spaces, *Proc. Amer. Math. Soc.* **140** (2012), no. 12, 4247–4257. MR 2957216.

[8] O. Constantin and J. Peláez, Integral operators, embedding theorems and a Littlewood–Paley formula on weighted Fock spaces, *J. Geom. Anal.* **26** (2016), no. 2, 1109–1154. MR 3472830.

[9] D. Girela and J. Peláez, Carleson measures, multipliers and integration operators for spaces of Dirichlet type, *J. Funct. Anal.* **241** (2006), no. 1, 334–358. MR 2264253.

[10] A. Harutyunyan and W. Lusky, On the boundedness of the differentiation operator between weighted spaces of holomorphic functions, *Studia Math.* **184** (2008), no. 3, 233–247. MR 2369141.

[11] S. Janson, J. Peetre and R. Rochberg, Hankel forms and the Fock space, *Rev. Mat. Iberoamericana* **3** (1987), no. 1, 61–138. MR 1008445.

[12] D. H. Luecking, Embedding theorems for spaces of analytic functions via Khinchine's inequality, *Michigan Math. J.* **40** (1993), no. 2, 333–358. MR 1226835.

[13] T. Mengestie, A note on the differential operator on generalized Fock spaces, *J. Math. Anal. Appl.* **458** (2018), no. 2, 937–948. MR 3724708.

[14] T. Mengestie, Carleson type measures for Fock–Sobolev spaces, *Complex Anal. Oper. Theory* **8** (2014), no. 6, 1225–1256. MR 3233976.

[15] T. Mengestie, On the spectrum of Volterra-type integral operators on Fock–Sobolev spaces, *Complex Anal. Oper. Theory* **11** (2017), no. 6, 1451–1461. MR 3674286.

[16] T. Mengestie, Spectral properties of Volterra-type integral operators on Fock–Sobolev spaces, *J. Korean Math. Soc.* **54** (2017), no. 6, 1801–1816. MR 3718425.

[17] T. Mengestie, Volterra type and weighted composition operators on weighted Fock spaces, *Integral Equ. Oper. Theory* **76** (2013), no. 1, 81–94. MR 3041722.

[18] T. Mengestie and S.-I. Ueki, Integral, differential and multiplication operators on generalized Fock spaces, *Complex Anal. Oper. Theory* **13** (2019), no. 3, 935–958. MR 3940399.

[19] T. Mengestie and M. Worku, Isolated and essentially isolated Volterra-type integral operators on generalized Fock spaces, *Integral Transforms Spec. Funct.* **30** (2019), no. 1, 41–54. MR 3871993.

[20] J. Pau and J. Peláez, Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights, *J. Funct. Anal.* **259** (2010), no. 10, 2727–2756. MR 2679024.

[21] S.-I. Ueki, Characterization for Fock-type space via higher order derivatives and its application, *Complex Anal. Oper. Theory* **8** (2014), no. 7, 1475–1486. MR 3261707.
[22] R. Wallstén, The S^p-criterion for Hankel forms on the Fock space, $0 < p < 1$, *Math. Scand.* **64** (1989), no. 1, 123–132. MR 1036432.
[23] K. Zhu, *Analysis on Fock Spaces*, Graduate Texts in Mathematics, 263, Springer, New York, 2012. MR 2934601.

Tesfa Mengestie
Mathematics Section, Western Norway University of Applied Sciences, Klingenbergvegen 8, N-5414 Stord, Norway
Tesfa.Mengestie@hvl.no

Received: September 2, 2020
Accepted: April 1, 2021