Monatshefte für Mathematik
https://doi.org/10.1007/s00605-022-01768-0

Closed range Volterra-type integral operators and dynamical sampling

Tesfa Mengestie[1]

Received: 20 July 2022 / Accepted: 3 September 2022

Abstract
We solve the closed range problem for Volterra-type integral operator on Fock spaces. Several applications of the result related to the operators invertibility, Fredholm, and dynamical sampling structures from frame perspectives are provided. We further prove a bounded Volterra-type integral operator preserves no frame property. On the contrary, the adjoint operator preserves frame if and only if it is noncompact but fails to preserve both tight frames and Riesz basis.

Keywords Fock space · Closed range · Volterra-type integral · Fredholm · Frame · Dynamical sampling

Mathematics Subject Classification 46BXX · 47BXX · 47GXX · 30DXX

1 Introduction

For holomorphic functions f and g in a given domain, we define the Volterra-type integral operator V_g by

$$V_g f(z) = \int_0^z f(w)g'(w)dw.$$

Various aspects of V_g have been widely studied since 1997 mainly on spaces of holomorphic functions: see for example on Hardy spaces [4, 5, 15], Bergman spaces [6, 8, 14], and Fock spaces [7, 11, 12] and the respective reference therein. The purpose of this note is to take the study further and solve the closed range problem for V_g on the

Communicated by Adrian Constantin.

✉ Tesfa Mengestie
Tesfa.Mengestie@hvl.no

1 Mathematics Section, Western Norway University of Applied Sciences, Klingenbergvegen 8, 5414 Stord, Norway

Fock spaces. We further provide several application of the result concerning invertibility and dynamical sampling structures of the operator from frame perspectives. Note that the closed range problem is one of the basic problems in operator theory which finds lots of connections in various parts of mathematics.

For $1 \leq p < \infty$, the Fock spaces $\mathcal{F}_p$ consist of all entire functions f on the complex plane $\mathbb{C}$ for which

$$\|f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} |f(z)| e^{-\frac{p}{2}|z|^2} dA(z) < \infty,$$

where A denotes the Lebesgue area measure on $\mathbb{C}$. The bounded and compact V_g on Fock spaces had been identified in [7, 11]. In deed, for $p \leq q$, the operator $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if $g(z) = az^2 + bz + c$ for some $a, b, c \in \mathbb{C}$. In this case, compactness is described by the condition $a = 0$. On the other hand, for $p > q$, $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if it is compact, and this holds if and only if $g(z) = az + b$ and $q > 2p/(p + 2)$.

The rest of this note is organized into two parts. In the first part we study the closed range problem for V_g on the Fock spaces. Theorem 1.1 provides a complete answer to this problem. We apply this result and identify conditions under which the operator becomes Fredholm and draw the conclusion that it fails to be surjective and hence not invertible. In the second part, we further apply the result to study more applications related to the dynamical sampling behaviours of the operator from frame perspectives. It is proved that there exists no function f in $\mathcal{F}_2$ for which its orbits under V_g or its adjoint represents a frame family for the space. In addition, we show that the operator fails to preserves frames structures. On the contrary, the adjoint operator preserves frame if and only if it is noncompact but fails to preserve both tight frames and Riesz basis.

Note that if g is a constant, then the operator V_g reduces to the zero operator, and we excluded this case in the rest of the manuscript. We may now state the first main result.

Theorem 1.1 *Let $1 \leq p, q < \infty$ and $V_g : \mathcal{F}_p \to \mathcal{F}_q$ be bounded and hence $g(z) = az^2 + bz + c$ for some $a, b, c \in \mathbb{C}$. Then V_g has a closed range if and only if $a \neq 0$ and $p = q$. The closed range is given by*

$$\mathcal{R}(V_g) = \big\{ f \in \mathcal{F}_p : f(0) = 0 \big\}. \tag{1.1}$$

As will be explained latter, the result equivalently characterizes when V_g is bounded from below. That is, there exists a constant $\epsilon > 0$ such that $\|V_g f\|_q \geq \epsilon \|f\|_p$ for all $f \in \mathcal{F}_p$. In contrast to conditions often given in terms of sampling sets or reverse Carelson measures, our condition here is quite simple to apply.

As first immediate consequence of the result, we observe that the classical integral operator $If(z) = \int_0^z f(w)dw$ and Hardy operator $Hf(z) = \frac{1}{z}\int_0^z f(w)dw$ have no closed ranges on Fock spaces. As another consequence of Theorem 1.1, we record the next corollary about Fredholm Volterra-type integral operators. Recall that a bounded operator T in a Banach space is said to be Fredholm if its range $\mathcal{R}(T)$ is closed and both Ker T and Ker T^* are finite dimensional. If T is Fredholm, its index is the number given by $dim(KerT) - dim(KerT^*)$. It is known that every bounded operator with closed range has an inverse called the pseudo-inverse, or the Moore-Penrose inverse. Since Ker V_g^* is the orthogonal complement of the range of V_g, (1.2) implies that Ker $V_g^* = \mathbb{C}$.

Corollary 1.2 *Let $1 \leq p < \infty$ and V_g is bounded on $\mathcal{F}_p$ and hence $g(z) = az^2+bz+c$ for some $a, b, c \in \mathbb{C}$. Then the following statements are equivalent.*

(i) $a \neq 0$;

(ii) *V_g is Fredholm of index one and its Fredholm inverse is given by*

$$V_g^{-1} f(z) = \begin{cases} \lim\limits_{w \to \frac{-b}{2a}} \frac{f'(w)}{2aw+b} & z = -b/2a \\ \frac{f'(z)}{2az+b}, & z \neq -b/2a. \end{cases} \tag{1.2}$$

Note that the differential operator $Df = f'$ is not bounded on Fock spaces [10]. Thus, the well definedness and the boundedness of V_g^{-1} on $\mathcal{F}_p$ comes from the requirement $a \neq 0$ and the estimate in (1.3) below.

We remark that applying integration by part in the definition of the operator V_g above gives the relation

$$M_g(f) = f(0)g(0) + V_g(f) + J_f(g),$$

where $Mg(f) = gf$ is the multiplication operator and $J_g f = V_f(g)$ is the Volterra companion integral operator. It is known that $J_g : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if M_g is bounded, and this holds only when g is a constant where the constant being zero for $p > q$. Thus, these operators have obviously closed ranges on $\mathcal{F}_p$ and will not be a point of further discussion in the rest of our consideration.

We give a word on notation. The notion $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

1.1 Proof of Theorem 1.1

Assuming that g is not a constant, we first show V_g is an injective map. Let f_1 and f_2 in $\mathcal{F}_p$ such that $V_g f_1 = V_g f_2$. Taking derivative on both sides we notice $f_1(z) = f_2(z)$ for all $z \in \mathbb{C}$ except possibly at points where g' vanishes. But since f_1 and f_2 are entire, it follows that $f_1 = f_2$. Consequently, as known from an application of Open Mapping Theorem, an injective bounded operator has closed range if and only if it is bounded from below (see for example [1, Theorem 2.5]). Thus, we proceed to use this equivalent reformulation as a tool to prove the claim. Another important tool in our work is the estimate

$$\|f\|_p^p \simeq |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p (1+|z|)^{-p} e^{-\frac{p\alpha}{2}|z|^2} dA(z) \tag{1.3}$$

which holds for all entire functions f [7]. Suppose now that $a \neq 0$ and $p = q$. Then for every $f \in \mathcal{F}_p$,

$$\|V_g f\|_p^p \simeq \int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} |f(z))|^p e^{-\frac{p}{2}|z|^2} dA(z) \simeq \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \simeq \|f\|_p^p,$$

which readily shows V_g is bounded from below. Hence, the conditions in the theorem are sufficient.

Conversely, suppose for the sake of contradiction V_g is bounded from below and $a = 0$. Applying the operator to the normalized kernel function $k_n = K_n/\|K_n\|_2$ and estimate (1.3)

$$\begin{aligned}\|V_g k_n\|_q^q &\simeq \frac{1}{\|K_n\|_2^q} \int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+|z|)^q} |K_n(z))|^q e^{-\frac{q}{2}|z|^2} dA(z)\\ &= \frac{n^{-q}|b|^q}{\|K_n\|_2^q} \int_{\mathbb{C}} \frac{|nK_n(z))|^q}{(1+|z|)^q} e^{-\frac{q}{2}|z|^2} dA(z) \simeq \frac{|b|^q \|K_n\|_q^q}{n^q \|K_n\|_2^q} = \frac{|b|^q}{n^q} \to 0\end{aligned}$$

as $n \to \infty$. This contradicts boundedness from below. Therefore, $a \neq 0$.

Next, we assume $a \neq 0$ and show that $p = q$ whenever range of V_g is closed. If $p > q$, then as already indicated above, boundedness of the operator is achieved only when $a = 0$. Thus, it remains to check for $p < q$. Since $a \neq 0$, note that for every $f \in \mathcal{F}_q$ boundedness from below implies

$$\|V_g f\|_q^q \simeq \int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+|z|)^q} f(z))|^q e^{-\frac{q}{2}|z|^2} dA(z) \simeq \|f\|_q^q \geq \epsilon \|f\|_p^q \tag{1.4}$$

for some $\epsilon > 0$. The last inequality in (1.4) clearly indicates we need to compare the norm of functions in $\mathcal{F}_p$ and $\mathcal{F}_q$. We may consider the sequence $f_n(z) = z^n, n = 1, 2, ...$ in $\mathcal{F}_p$. Using polar integration and Stirling's approximation formula

$$\|f_n\|_p^p = p \int_0^\infty r^{np+1} e^{-pr^2/2} dr = \Big(\frac{1}{p}\Big)^{np/2} \Gamma\Big(\frac{np+2}{2}\Big) \simeq \Big(\frac{n}{e}\Big)^{\frac{np}{2}} \sqrt{n}. \tag{1.5}$$

See also [16, p.40]. It follows from this and (1.4), the estimate

$$\|V_g f_n\|_q \simeq \|f_n\|_q \geq \epsilon \|f_n\|_p$$

holds only

$$\|f_n\|_q / \|f_n\|_p \simeq n^{\frac{1}{2q} - \frac{1}{2p}} \geq \epsilon$$

for all $n \in \mathbb{N}$. This gives a contradiction when $n \to \infty$.

It remains to verify (1.1). From the proof made above, we already have $p = q$ and hence $\mathcal{R}(V_g) \subseteq \mathcal{F}_p$. On the other hand, for each $h \in \mathcal{F}_p$, we consider the function f_h

defined by

$$f_h(z) = \begin{cases} \lim\limits_{w \to \frac{-b}{2a}} \frac{h'(w)}{2aw+b} & z = -b/2a \\ \frac{h'(z)}{2az+b}, & z \neq -b/2a. \end{cases}$$

Observe that f_h is entire and by (1.3), it belongs to $\mathcal{F}_p$ and $V_g f_h = h$. Therefore, the other inclusion $\mathcal{F}_p \subseteq \mathcal{R}(V_g)$ holds and completes the proof.

2 Dynamical sampling with V_g and V_g^*

We now turn our attentions to some applications of Theorem 1.1 on dynamical sampling from frame perspectives. Dynamical sampling deals with representations of frames $\{f_n\}_{n=0}^{\infty}$ in the form $\{T^n f\}_{n=0}^{\infty}$ for some linear operator T defined on a given Hilbert space $\mathcal{H}$ where

$$\{T^n f\}_{n=0}^{\infty} = \left\{f,\ Tf,\ T^2 f,\ T^3 f,\ ...\right\}$$

is the orbit of $f \in \mathcal{H}$ under the operator T. Recall that a family (f_j), $j \in I$ of vectors in a Hilbert space $\mathcal{H}$ is a frame if there exist positive constants A and B such that for any $g \in \mathcal{H}$

$$A\|g\|_{\mathcal{H}}^2 \leq \sum_{j \in I} |\langle g, f_j \rangle_{\mathcal{H}}|^2 \leq B\|g\|_{\mathcal{H}}^2. \tag{2.1}$$

The constants A and B are called the lower and upper bounds of the frame respectively. It is called a tight frame when $A = B$. Frames are generalizations of bases and their main applications comes from the fact that a frame can be designed to be redundant while still providing a reconstruction formula for each vector in the space. Thus, identifying methods that generate new frames has been an interesting problem in frame theory. A special type of frame is Riesz basis. A family (f_j), $j \in I$ of vectors in $\mathcal{H}$ is a Riesz basis if it is complete and there exist constants $0 < A \leq B < \infty$ such that for any $c_j \in \ell^2(I)$

$$A \sum_{j \in I} |c_j|^2 \leq \left\| \sum_{j \in I} c_j f_j \right\|_{\mathcal{H}}^2 \leq B \sum_{j \in I} |c_j|^2. \tag{2.2}$$

We may start with the following important lemma which connects the closed range problem with dynamical sampling in frame theory.

Lemma 2.1 *Let $\mathcal{H}$ be a Hilbert space and T be a bounded linear operator on $\mathcal{H}$. If $\{T^n f\}_{n=0}^{\infty}$ is a frame for some $f \in \mathcal{H}$, then*

(i) *T is surjective.*
(ii) *$\|(T^*)^n g\|_{\mathcal{H}} \to 0$ as $n \to \infty$ for all $g \in \mathcal{H}$.*

While the proof of part (ii) is available in [3], part (i) follows easily since $\{T^n f\}_{n=0}^{\infty}$ is a frame for each $h \in \mathcal{H}$, there exists sequence (c_n) such that

$$h = \sum_{n=1} c_n T^n f = T\Big(\sum_{n=1} c_n T^{n-1} f\Big)$$

from which the claim follows. Note here that the frame property implies $R(T) = \mathcal{H}$ is closed and this in particular connects us with our result in Theorem 1.1.

Theorem 2.2 *Let V_g is bounded on $\mathcal{F}_2$ and hence $g(z) = az^2 + bz + c$ for some $a, b, c \in \mathbb{C}$. Then neither $\{V_g^n f\}_{n=0}^{\infty}$ nor $\{(V_g^*)^n f\}_{n=0}^{\infty}$ is a frame for any choice of f in $\mathcal{F}_2$.*

Proof Suppose, for the sake of contradiction, that there exists an $f \in \mathcal{F}_2$ such that $\{V_g^n f\}_{n=0}^{\infty}$ is a frame. Then by Lemma 2.1, V_g is surjective which contradicts Theorem 1.1.

Next we consider the case with the adjoint operator. Suppose that there exists an $h \in \mathcal{F}_2$ such that $\{(V_g^*)^n h\}_{n=0}^{\infty}$ is a frame again. Then the range of V_g^* is closed. By Theorem 1.1 and Closed Range Theorem, this holds if and only if $a \neq 0$. On the other hand, set $f(z) = z$ and consider the iterations

$$\begin{aligned}
V_g f(z) &= \int_0^z (2aw + b)w\,dw = \frac{2a}{3}z^3 + \frac{b}{2}z^2,\\
V_g^2 f(z) &= \int_0^z (2aw + b)\Big(\frac{2a}{3}w^3 + \frac{b}{2}w^2\Big)dw = \frac{2^2a^2}{3\cdot 5}z^5 + \frac{10ab}{2\cdot 3\cdot 4}z^4 + \frac{b^2}{2\cdot 3}z^3,\\
V_g^3 f(z) &= \int_0^z (2aw + b)\Big(\frac{2^2a^2}{3\cdot 5}w^5 + \frac{10ab}{2\cdot 3\cdot 4}w^4 + \frac{b^2}{2\cdot 3}w^3\Big)dw\\
&= \frac{2^3a^3}{3\cdot 5\cdot 7}z^7 + \frac{132a^2b}{6!}z^6 + \frac{18ab^2}{5!}z^5 + \frac{b^3}{4!}z^4,
\end{aligned}$$

and

$$\begin{aligned}
V_g^4 f(z) &= \int_0^z (2aw + b)\Big(\frac{2^3a^3}{3\cdot 5\cdot 7}w^7 + \frac{132a^2b}{6!}w^6 + \frac{18ab^2}{5!}w^5 + \frac{b^3}{4!}w^4\Big)dw\\
&= \frac{2^4a^4}{3\cdot 5\cdot 7\cdot 9}z^9 + \frac{a^3(384 + 1148b)}{8!}z^8 + \frac{324a^2b^2}{7!}z^7 + \frac{28ab^3}{6!}z^6 + \frac{b^4}{5!}z^5.
\end{aligned}$$

Continuing the iteration,

$$\begin{aligned}
V_g^n f(z) = \int_0^z (2aw + b)V_g^{n-1} f(w)dw &= c_{2n+1}a^n z^{2n+1} + c_{2n}a^{n-1}z^{2n}\\
&+ \cdots + c_{n+2}ab^{n-1}z^{n+2} + c_{n+1}b^n z^{n+1},
\end{aligned}$$

where the sequence c_k is of the form p_k/q_k, $q_k \leq k!$ and p_k is a sequence of numbers some of them involve multiples of b with $p_{2n+1} = 2^n$ and $p_{n+1} = b^n$. Now,

$$\begin{aligned}\|V_g^n f\|_2 &= \|c_{2n+1}a^n z^{2n+1} + c_{2n}a^{n-1}z^{2n} + \cdots + c_{n+2}az^{n+2} + c_{n+1}z^{n+1}\|_2 \\ &\geq \frac{1}{(2n+1)!}\Big| |2^n a^n| \|_2 z^{2n+1}\|_2 - |p_{2n}a^{n-1}| \|z^{2n}\|_2 \\ &\quad - \cdots - |p_{n+2}a| \|z^{n+2}\|_2 - |p_{n+1}| \|z^{n+1}\|_2 \Big| \\ &= \frac{\|z^{2n+1}\|_2}{(2n+1)!}\Big| |2^n a^n| - \frac{|p_{2n}a^{n-1}| \|z^{2n}\|_2}{\|z^{2n+1}\|_2} - \cdots - \frac{|p_{n+2}a| \|z^{n+2}\|_2}{\|z^{2n+1}\|_2} - \frac{|b^n| \|z^{n+1}\|_2}{\|z^{2n+1}\|_2}\Big|.\end{aligned} \tag{2.3}$$

On the other hand, by (1.5)

$$\|z^n\|_2 \simeq \left(\frac{n}{e}\right)^n \sqrt{n}$$

which obviously grows much faster than exponential and factorial sequences. Setting this in (2.3), we observe that

$$\|V_g^n f\|_2 \to 0, \quad n \to \infty$$

as required by part (ii) of Lemma 2.1 only when $a = b = 0$ which is a contradiction, and the claim is proved. □

2.1 Frame preserving V_g and V_g^*

Another interesting operator related question on frame property is as to when V_g preserves frame;in the sense that $V_g f_n$ is a frame whenever f_n is. This is known to be one of the approaches used to construct new frames using tools in operator theory.

A useful result connecting the closed range and the frame preserving problems is the following [2, 9, 13].

Lemma 2.3 *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then T preserves*

(i) *frames on $\mathcal{H}$ if and only if T^* is bounded below on $\mathcal{H}$, and the latter happens if and only if T is surjective on $\mathcal{H}$.*
(ii) *tight frames if and only if there exists a positive constant λ such that $\|T^* f\|_{\mathcal{H}} = \lambda \|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$.*

We may now state our main result for the section.

Theorem 2.4 *Let V_g be bounded on $\mathcal{F}_2$ and hence $g(z) = az^2 + bz + c$ for some $a, b, c \in \mathbb{C}$. Then*

(i) *V_g fails to preserve frame in $\mathcal{F}_2$.*

(ii) V_g^* *preserves frame in* $\mathcal{F}_2$ *if and only if* $a \neq 0$.
(iii) V_g^* *preserves neither tight frame nor Riesz basis in* $\mathcal{F}_2$.

Said differently, the result asserts that the adjoint of a bounded Volterra-type integral operator preserves frame structure on $\mathcal{F}_2$ if and only if it has no compactness property.

Proof Suppose V_g preserves frame on $\mathcal{F}_2$. Then an application of Lemma 2.3 and Theorem 1.1 leads to a contradiction. Statement (ii) follows again from a simple application of Lemma 2.3, Theorem 1.1, and the Closed Range Theorem. Thus we proceed to verify (iii) and suppose V_g^* preserves tight frame. By Theorem 2.4, it follows that $a \neq 0$. On the other hand, by Lemma 2.3, there exists a $\lambda > 0$ such that $\|V_g f\|_2 = \lambda \|f\|_2$ all $f \in \mathcal{F}_2$. Using the function K_0, we obtain

$$\lambda = \frac{\|V_g K_0\|_2}{\|K_0\|_2} = \|az^2 + bz\|_2.$$

Furthermore, considering the sequence of the monomials

$$V_g z^n = \frac{2a}{n+2} z^{n+2} + \frac{b}{n+1} z^{n+1}$$

for all $n \in \mathbb{N}$. Consequently,

$$\|az^2 + bz\|_2 = \lambda = \frac{\|V_g z^n\|_2}{\|z^n\|_2} = \frac{\|\frac{2a}{n+2} z^{n+2} + \frac{b}{n+1} z^{n+1}\|_2}{\|z^n\|_2} \tag{2.4}$$

Using orthogonality of the monomials, we simplify further to deduce that (2.4) holds if and only if

$$\frac{(n^2+4n)|a|^2}{(n+2)^2}\Big(\|z^2\|_2^2\|z^n\|_2^2 - \|z^{n+2}\|_2^2\Big) + \frac{(n^2+2n)|b|^2}{(n+1)^2}\Big(\|z\|_2^2\|z^n\|_2^2 - \|z^{n+1}\|_2^2\Big) = 0.$$

Applying the norm of the monomials in (1.5), the above holds if and only if $a = b = 0$ and hence a contradiction.

Next, we show that V_g^* does not preserves Riesz basis either. Suppose on the contrary it does. Recall that a (f_j), $j \in I$ is a Riesz bases if and only it is a frame and $\omega-$ independent. That is if

$$\sum_{j \in I} c_j f_j = 0$$

for some sequence of scalars (c_j), then $c_j = 0$ for all $j \in I$. In view of this, suppose (f_j), $j \in I$ is a Riesz basis and

$$\sum_{j \in I} c_j V_g^* f_j = 0.$$

for some scalars c_j. Using linearity, for each $h \in \mathcal{F}_2$

$$\left\langle \sum_{j\in I} c_j V_g^* f_j,\ h\right\rangle = \left\langle \sum_{j\in I} V_g^*(c_j f_j),\ h\right\rangle = \left\langle \sum_{j\in I} c_j f_j,\ V_g h\right\rangle = 0.$$

This shows $\sum_{j\in I} c_j f_j$ belongs to the orthogonal complement of the range of V_g. Then by (1.1), $\sum_{j\in I} c_j f_j \in \mathbb{C} = Ker V_g^*$. On the other hand, f_j is a Riesz basis. Hence, $\sum_{j\in I} c_j f_j$ is not necessarily zero.

Funding Open access funding provided by Western Norway University Of Applied Sciences

Data availability The manuscript has no associated data.

References

1. Abramovich, Y.A., Aliprantis, C.D.: An Invitation to Operator Theory. Am. Math. Soc. **2002**, iv+530 (2002)
2. Aldroubi, A.: Portraits of frames. Proc. Am. Math. Soc. **123**, 1661–1668 (1995)
3. Aldroubi, A., Petrosyan, A.: Dynamical sampling and systems from iterative actions of operators. In: Mhaskar, H., Pesenson, I., Zhou, D.X., Le Gia, Q.T., Mayeli, A. (eds.) Frames and Other Bases in Abstract and Function Spaces. Birkhauser, Boston (2017)
4. Aleman, A., Cima, J.A.: An integral operator on Hp and Hardy's inequality. J. Anal. Math. **85**, 157–176 (2001)
5. Aleman, A., Siskakis, A.G.: An integral operator on Hp. Complex Variables Theory Appl. **28**, 149–158 (1995)
6. Aleman, A., Siskakis, A.G.: Integration operators on Bergman spaces. Indiana Univ. Math. J. **46**, 337–356 (1997)
7. Constantin, O.: Volterra type integration operators on Fock spaces. Proc. Am. Math. Soc. **140**(12), 4247–4257 (2012)
8. Constantin, O.: Carleson embeddings and some classes of operators on weighted Bergman spaces. J. Math. Anal. Appl. **365**, 668–682 (2010)
9. Manhas, J.S., Prajitura, G.T., Zhao, R.: Weighted composition operators that preserve frames. Integr. Equ. Oper. Theory, **34**, (2019)
10. Mengestie, T.: A note on the differential operator on generalized Fock spaces. J. Math. Anal. Appl. **458**(2), 937–948 (2018)
11. Mengestie, T.: Product of Volterra type integral and composition operators on weighted Fock spaces. J. Geom. Anal. **24**, 740–755 (2014)
12. Mengestie, T.: Spectral properties of Volterra-type integral operators on Fock–Sobolev spaces. Korean Math. Soc. **54**(6), 1801–1816 (2017)
13. Najati, A., Abdollahpour, M.R., Osgooei, E., Saem, M.M.: More on sums of Hilbert space frames. Bull. Korean Math. Soc. **50**, 1841–1846 (2013)
14. Pau, J., Pelaez, J.A.: Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights. J. Funct. Anal. **259**, 2727–2756 (2010)

15. Pommerenke, Ch.: Schlichte Funktionen und analytische Funktionen von beschrankter mittlerer Oszillation. Comment. Math. Helv. **52**, 591–602 (1977)
16. Zhu, K.: Analysis on Fock Spaces. Springer, New York (2012)