Lisa Steffensen

Critical mathematics education and climate change

A teaching and research partnership in lower-secondary school

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Scientific environment

This study was conducted at Western Norway University of Applied Sciences, Faculty of Education, Arts and Sports, and Department of Language, Literature, Mathematics and Interpreting.

Research programme:

The study takes place within the Bildung and Pedagogical Practices PhD programme at Western Norway University of Applied Sciences. I have also participated in the Western Norway Graduate School of Educational Research research programme II WNGER II.

Research group:

The study is associated with the research group Lived Democracy at HVL, where Kjellrun Hiis Hauge was the previous research group leader and Suela Kacerja the current one. My research is directly affiliated with the project: "Lived democracy – Classroom engagement on environmental issues and development of critical mathematical competence."

Other scientific environments:

During my PhD research, educational conferences have provided meeting places and communities, as well as courses and lectures. These include, for example, MES, PME, AERA, BERA, ICME, ICTMA, JURE/EARLI, NOMUS, and Erasmus.

In addition, the scientific environment introduced by supervisors, colleagues (e.g. seminar series from the research group LATACME at HVL) and co-authors, even in research articles not included in the thesis (Suela Kacerja, Ragnhild Hansen, Peter Gøtze, Yasmin Abtahi, Richard Barwell), has contributed to expanding my perspectives.

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Being born in Norway has provided me with privileges such as comprehensive welfare, free education, gender equality, little poverty, peace, a working democracy, and lastly, the opportunity to conduct this PhD research. These privileges I value immensely.

Preface

My journey towards this PhD study started way before my date of employment, and my interpretations can be seen in the light of this journey. My educational background constitutes different strings, and looking back, they partly come together in this study. One degree in environmental technology and another in mathematics education come together in the topic of this study: critical mathematics education in the context of climate change.

The culture at my educational institution, partly inspired by the ideas of Stieg Melin-Olsen, has influenced my perspectives: First as a teacher-student, then as a teacher-educator, and now as a participant in the PhD programme. Another significant contribution in my journey towards this PhD-study came during a research project I attended as a student, led by Marit Johnsen-Høines. Here the focus was on investigative dialogical approaches to mathematics education, real-life education, critical mathematical competencies, empowerment and critique, all of which are relevant and could be used as keywords for this study.

As a former mathematics (and science) teacher in lower secondary school, and as a mathematics teacher educator, it is a concern of mine that mathematics education should matter to the students, engage them, and potentially lead to engagement in society. I have always been interested in the socio-political aspects of mathematics, and the choice of climate change as a context is deliberate. The interest concerns the mathematical and scientific-related nature of this topic, but indeed also the social and political aspects. I regard it as vital to prepare students for dealing with complex real-life problems and enable them to be critical citizens in society.

I want to end this preface by quoting one of the students from my empirical study. The first time I spoke to her, she said to me, in a straightforward and somewhat condemning tone: “It’s about time you are here!” I was caught off guard at first and did not know what she meant. When talking to her, it became clear that she considered climate change as an unprioritised topic in school. Although this student did not express anything about the role of mathematics education, this comment has stuck with me throughout this PhD-study. I agree. I do think it is time. Time for mathematics educators to engage in socio-political issues such as climate change and enable students to be critical citizens in a lived democracy.

Abstract

Keywords: Critical mathematics education, critical mathematical competencies, teachers, students, the formatting power of mathematics, climate change, real-world problems, controversies, values, dialogues, critical citizens and lived democracy.

This study took place at Western Norway University of Applied Sciences. The research question is: *How can teachers facilitate students' critical mathematical competencies in a climate change context*? To enable students to act as critical citizens and empower them for a lived democracy is a crucial task for education. From a critical mathematics perspective, students can become aware of mathematics' role in shaping society. They can be capable of critiquing the use of mathematics and applying mathematical competencies to empower themselves both personally and for the greater good of society.

The question contains four research focuses that are addressed in four individual papers. Research focus no. 1 identifies and critically reflects on concepts and perspectives emphasised as important in the literature of two fields – critical mathematics education and post-normal science – and is addressed in a literature overview (paper I). Research focuses 2, 3, and 4 involve a research partnership (papers II, III, and IV) with three teachers and their four classes in lower-secondary school. For about a year, 42 classroom lessons were designed by the teachers to develop students' critical mathematical competencies in a climate change context. Research focus no. 2 involves how teachers' values can influence their teaching by investigating their facilitation and reflections of value-aspect with respect to climate change and school mathematics. Research focus no. 3 identifies the potential for facilitating students' awareness and understanding of the formatting power of mathematics. Lastly, research focus no. 4 identifies how students' critical mathematical competencies can appear in their argumentation. This study has, therefore, a perspective on students' critical mathematical competencies and how teachers facilitate them.

The findings from the four papers are structured and discussed in six themes. In the first theme, *lived democracy and critical citizens*, I discuss how the teachers connect climate change, students' critical mathematical competencies, and democracy. They emphasise critical competencies as a crucial skill for students and treat the students

as critical citizens by engaging them in discussion and debates. In the second theme, *the mathematical formatting of climate change*, I identified, amongst others, how the teachers express it as vital that students identify, understand and reflect on how mathematics can influence how we perceive climate change issues. I also discussed how teachers deliberate or un-deliberate that choices of graphs, numbers or topics can influence students or others.

In the third theme, *critique and critical reflections,* I identify how the teachers facilitate students' critical reflections regarding mathematics-based argumentation in complex scientific issues. In addition, I explore how they prepare them to deal with uncertainties, consider implications of graphs, and include their critical agency in taking justified standpoints. In the fourth theme, *mathematical literacy and kinds of knowing*, I discuss students' intertwined mathematical, technological, and reflective knowing. Examples of how students sometimes struggle to move beyond the mathematical borders of a task are contrasted with how they use their everyday knowledge and relate the task to the real-world. The students' mathematical literacy is discussed in relation to local and global climate change concerns and 21st-century skills.

In the fifth theme, *controversies and values*, I identify how the teachers emphasise the controversies in climate change issues to deliberately create debate and reflections, instead of avoiding the controversies. In the sixth theme, *student-centred and dialogic learning*, I discuss six aspects characterising the learning environment in the research partnership, for instance, student-centred approaches, types of understanding, and the content and qualities of dialogues.

These six themes are relevant when teachers facilitate students' critical mathematics competencies in a climate change context. They are neither exhaustive nor exclusive but can provide a foundation for teachers and researchers who consider including complex real-life problems in the mathematics classroom and aim at developing students as critical citizens in a lived democracy.

List of Tables and Figures

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List of publications included in the PhD thesis

Paper I:

Steffensen, L. (2017). Critical Mathematics Education and Post-Normal Science: A literature overview. *Philosophy of mathematics education journal*, *32*.

Paper II:

Steffensen, L., Herheim, R., & Rangnes, T. E. (in press). Wicked problems and critical judgment in school mathematics. In N. S. Kennedy & E. Marsal (Eds.), *Dialogical Inquiry in Mathematics Teaching and Learning: A Philosophical Approach*. Berlin: LIT Verlag.

Paper III:

Steffensen, L., Herheim, R., & Rangnes, T. E. (accepted). The mathematical formatting of how climate change is perceived: Teachers' reflection and practice. In A. Andersson & R. Barwell (Eds.), *Applying critical perspectives in mathematics education*. Leiden & Rotterdam: Brill | Sense.

Paper IV:

Steffensen, L. (2020). Climate change and Students' critical competencies: A Norwegian study. In J. Anderson & Y. Li (Eds.), *Integrated Approaches to STEM Education: An International Perspective*. Switzerland: Springer publishing.

Contents

1 Introduction

Mathematics influences our lives in numerous ways, from small everyday tasks to urgent global challenges, and it is vital to enable students with the necessary mathematical competencies to deal with these challenges. That being said, what kind of competencies are required, and how can these be facilitated? Before starting this PhD-study, I was interested in how mathematics education could be meaningful in a student's life and become something more than school knowledge. I pondered how mathematical competencies could empower students to make a difference for themselves and society. To be an engaged and active citizen can involve taking a standpoint, being reflective and critical, or potentially taking action, for instance, by adjusting a way of life. I was thus interested in how mathematics education could encourage students to be critical citizens that engage in meaningful contexts for themselves and a lived democracy.

Recently, students world-wide have engaged in school strikes demanding action on climate change by governments, decision-makers, private companies, and others (Kohli, 2019; Marris, 2019; Wearden & Carrington, 2019). Climate change is an urgent socio-political issue (United Nations, 2020) and can be regarded as one such meaningful context. Climate change involves complexity, uncertainty, knowledge from several disciplines, and ethical questions. Barwell (2013) emphasises that the issues of climate change requires other forms of citizen participation, partly inspired by the ideas from post-normal science. Post-normal science was developed by Funtowicz and Ravetz (1993, 2003) and provides insights into how urgent and complex societal problems can be addressed when knowledge is uncertain and values are conflicting. An essential aspect of post-normal science is that scientific knowledge is not sufficient to provide answers to complex problems and that citizens should participate more in the decision-making process. Although there is a scientific consensus on anthropogenic climate change (Cook et al., 2013; Hulme, 2009; Oreskes, 2018), politicians and policymakers struggle to interpret and make decisions regarding how society should deal with it. Such indecisiveness, combined with the complexity, can lead to public confusion, uncertainty and disinterest among citizens. Mathematics education has a role to play in preventing such a lack of interest and enabling students to deal with complex problems.

From a Critical Mathematics Education (CME) perspective, students can become aware of the nature of mathematics; identify how mathematics can be used; critique its use in society; and, apply mathematical competencies to empower themselves both personally and as critical citizens (Ernest, 2002; Skovsmose, 1994). If students are to be empowered as critical citizens, how can mathematics education facilitate this? Some aspects of such facilitation are described by Ernest (2002); mathematics education should include socially relevant topics, the use of factual and real data, classroom discussions, and allowing conflicts of opinions. A Norwegian mathematics classroom typically involves teachers instructing students; students working individually with textbook-task training for competencies to solve routine tasks; and less student interaction and exchange of ways of thinking and argumentation (Bergem, Kaarstein, & Nilsen, 2016; Vavik et al., 2010). Internationally, Echazarra, Salinas, Méndez, Denis, and Rech (2016) report a similar trend that those teaching strategies referred to as traditional still dominate in most countries. This classroom reality differs from the one Ernest (2002) emphasises. To address this, I decided to design an empirical study, situated within CME, to enhance the understanding of how to facilitate students' critical mathematical competencies when dealing with complex problems.

The amount and role of mathematics in climate change call for mathematics education to be involved. As emphasised by Barwell (2013), mathematics contributes to *describing*, *predicting* and *communicating* climate change. Describing climate change can include measures of global temperatures, sea-levels, or degrees of glacial melting. Measurements are relevant for understanding climate change, but also often disputed. Lloyd (2018) describes one such disagreement where climate researchers came to different conclusion using the same raw data. The processes of predicting climate change makes use of advanced mathematical modelling. Oreskes (2018) underlines that prediction is by definition, uncertain, thus while most climate scientists agree on anthropogenic climate change, debates exist regarding its predicted tempo and how severely the impact will be. Communicating climate change involves text, pictures, charts, graphs, and can be used by, for example, scientists, politicians, journalists, and students for various reasons. Interest organisations can sometimes spread organised denial of anthropogenic climate change (Krange, Kaltenborn, & Hultman, 2019). If teachers or students use graphs or other data influenced by these interest organisations, they could be misled and potentially make judgments based on misinformation. Citizens need to understand and reflect on these descriptions, predictions and communications to develop informed argumentation and standpoints on climate change issues. Informed, critical, and engaged citizens are vital to a democratic way of living.

Facilitating students' critical mathematics competencies in the context of climate change corresponds with the Norwegian Education Act (1998), where the preamble states that students should "learn to think critically and act ethically and with environmental awareness" (§ 1-1). Critical thinking is a core activity in several subjects. The preamble is elaborated in the Core Curriculum (Utdanningsdirektoratet, 2017), where critical thinking is described as enabling students to take a standpoint on important questions in their own life. Utdanningsdirektoratet underlines three interdisciplinary topics: democracy and citizenship, sustainable development, and public health and wellbeing. The first two are particularly relevant for this study. Climate change is an essential aspect of sustainable development, as the United Nations (2020) emphasises by including climate action as one of the 17 goals for sustainable development.

In the Norwegian mathematics curricula, Utdanningsdirektoratet (2019) highlights that students should be able to do the following: explore and analyse findings from real data from nature and society; evaluate these; become aware of the assumptions and premises of mathematical models; formulate their own arguments; argue for how mathematical representation can be used to promote different standpoints; and, participate in the public debate. Public debates on climate change often involve mathematical information combined with a variety of standpoints. If students, as a part of their mathematics education, critically reflect on links between mathematical information and how claims are presented and promoted, they can develop their critical awareness of how mathematics can be used and abused in society and thereby become critical citizens in a lived democracy.

Internationally, the Organisation for Economic Co-operation and Development (OECD) relates mathematical literacy to climate change. For instance, in the PISA 2003 framework, they point to mathematical literacy as an essential competency when dealing with "global warming and the greenhouse effect" (2003, p. 24). In the

PISA 2018 framework, OECD (2018a) accentuates students' global competence as "students' ability to interact with the wider world" (p. 165) to examine climate change risk critically. They further describe climate issues as a relevant scientific context for the development of mathematical literacy. Lastly, in the PISA 2021 framework (draft), mathematical competencies are described as evolving from performing basic arithmetic operations to addressing complex areas such as climate change (OECD, 2018b).

Therefore, if the goal is that students shall develop critical mathematical competencies vital to their role as critical citizens in the context of climate change, the question remains: how can this be facilitated through mathematics education? In this PhD-study, I elaborate on this question in an empirical study, where teachers facilitate students' critical mathematical competencies in the context of climate change. The research partnership took place in lower secondary school and consisted of three mathematics teachers and their four classes, where the teachers planned and conducted the teaching.

1.1 Research question and focus

The overarching research question for this PhD-study concerns how mathematics education can contribute to students becoming critical citizens who engage in societal and complex problems like climate change. The research question is formulated as follows:

How can focusing on climate change facilitate students' critical mathematical competencies?

The concept of critical mathematical competencies is not clearly defined in the research literature. Alrø and Johnsen-Høines (2016), Kennedy (2018), and Mamolo (2018), and Sikunder (2015) refer to aspects such as understanding the world, empowering, critical citizens, society, mathematical literacy, and CME. A related concept is critical democratic competence. Based on the work of Blomhøj (1992, 2003), Hansen and Hana (2012) describe this as "the ability to evaluate, analyse, and criticise the use of mathematics in society". The ability to *apply mathematics* can be added to this latter description. To clarify how I understand the concept of critical mathematical competencies[1] in this study, and based on literature from CME, I highlight four aspects: (1) The formatting power of mathematics, (2) Mathemacy and mathematical literacy, (3) Critique and critical reflections, and (4) Values.

The formatting power of mathematics (1) contributes to shaping our society, in sometimes invisible ways. Barwell (2013, 2018) and Skovsmose (1994) draw attention to how mathematics could be an essential construct in social issues such as climate change and format how people perceive these. A vital part of students' critical mathematical competencies is to develop an awareness and critical reflections regarding how the mathematical formatting of climate change can take place.

The research question is based on the understanding that students' mathematical literacy and mathemacy (2) should include competencies above and beyond mastering basic skills. Students should be enabled to reflect, make argumentation, make well-founded judgements, and have the competencies to apply learning outcomes to meet complex challenges. Skovsmose (1994) described mathemacy as taking justified stands as a means for social and political reforms as well as for self-empowerment. Mathematical literacy can be described as "the capacity of individuals to reason mathematically and solve problems in a variety of 21st century contexts" (OECD, 2018b, p. 6). 21st-century skills constitute a variety of competencies[2]. For instance, highlighted by Care, Kim, Vista, and Anderson (2018), Partnership for 21st Century Learning (2019), and OECD (2018b) are concepts such as critical thinking, reflection, communication and collaboration, research and inquiry, problem-solving, ethical obligations, global awareness, and environmental literacy.

The research question builds on the understanding that students should have opportunities to critique and critically reflect (3) *on* and *with* mathematics. They can be invited to ask questions; explore; look for multiple explanations; and, learn to question what is included/not included. Furthermore, they should be encouraged to question what graphs and numbers are based on, the underlying assumptions, the purpose, how data is collected, how data is simplified, and how the use of mathematics can impact our opinions and judgments. Reflective knowing, that is, the competence to take a justified stand (Skovsmose, 1994), is related to these

[1] In paper IV, the concept of critical mathematical competencies is also discussed.
[2] It does not exist a global, fixed or unified set of competencies which defines the 21st-century skills.

exemplifications of critique and critical reflections. Critique, critical reflections and reflective knowing are essential parts of students' critical mathematical competencies. Hauge and Barwell (2017) accentuate that a critical approach in mathematics education concerns students' democratic participation through an emphasis on critiquing by e.g. using a mathematical method to examine environmental problems. They suggested facilitating students' "discussion of the meaning and consequences of calculations of global temperature changes, discussion of possible actions, and discussion of the role of mathematics, such as the limitations of the statistical methods" (p. 28). Critiquing real-world issues can broaden students' critical reflections to involve more than pure mathematics by including real-world aspects.

Lastly, the research question is based on the understanding that values (4) are vital for students' critical democratic competencies. They can become aware of own and others' ethical considerations and reflect on how values and ethical considerations can be involved in seemingly neutral mathematics. Moreover, they can identify how values are embedded in societal issues such as climate change. Ernest (2019) underlined that mathematics education could empower learners as critical and mathematically literate citizens, and use mathematics to raise ethical concerns, e.g. on "care for the earth and the environment" (p. 86). He points to that if this is done well, it is both an asset to the student and society.

Facilitating students' critical mathematical competencies can either have a focus on teacher's facilitation or students' critical mathematical competencies. I focus on both these aspects and consider them interrelated. Teachers' facilitation can be described by students' statements, or students' competencies can be aided and described by teachers' facilitation. To answer the research question and give consideration to the perspectives of both teachers and students, four research focuses were designed and addressed in four individual papers (listed chronologically):

(1) Identify and critically reflect on concepts and perspectives emphasised as important in the literature from two fields, CME and post-normal science (paper I).

(2) Explore how teachers´ values can influence their teaching by investigating their facilitation and reflections of value-aspect in regard to climate change in school mathematics (paper II).

(3) Identify the potential for facilitating students' awareness and understanding of the formatting power of mathematics (paper III).

(4) Identify how students' critical mathematical competencies can appear in their argumentation (paper IV).

Focus (1) involves systematic approaches to acquiring an overview of the existing research literature concerning CME and post-normal science. During the process of constructing the literature overview, I developed insights into areas such as theoretical perspectives, methodology and methods, terminology, and key journals. I also designed and refined the research question and the four research focuses. This literature overview also provided insight into potential research gaps. A limited number of empirical studies from the mathematics classroom were found, and these supported the choice of conducting a qualitative empirical study. This empirical study is based on the understanding that mathematics teachers and researchers need to be engaged in enabling students as critical citizens who can participate in the extended peer communities described in post-normal science. The findings of the literature overview partly serve as a basis for the three following papers. For instance, one of the findings was the identification of 19 key concepts or phrases within the two fields. Two of these, *controversies* and *values*, were used as a conceptual framework in paper II, "Wicked problems and critical judgment in school mathematics". Three others, *the formatting power of mathematics*, *uncertainty*, and *critical citizens*, were used as theoretical perspectives in paper III, "The mathematical formatting of how climate change is perceived: Teachers' reflection and practice". Another three, *three kinds of knowings*, *critical reflections*, and *(inquiry-based) dialogues*, were used in the conceptual framework of paper IV, "Climate Change and Students' Critical Competencies: A Norwegian Study".

Focuses (2), (3) and (4) involve a research partnership with three teachers and their 15-16 year old students. The methodological framework for these three papers was inspired by action research. Focuses (2) and (3) concern teachers' facilitation, how teachers' values can impact their facilitation, and how teachers could facilitate students' awareness of the formatting power of mathematics, while focus (4) addresses students' critical mathematical competencies.

Inspired by Herheim (2012), an overview of the research question, research focus and title of paper are shown in Table 1.

Table 1. The title of the papers, the four research focuses, and the research question

Title of paper	**Focus of paper**
I. Critical Mathematics Education and Post-Normal Science: A literature overview	To identify and critically reflect on concepts and perspectives emphasised as important in the literature from two fields: critical mathematics education and post-normal science.
II. Wicked problems and critical judgment in school mathematics	To explore how teachers´ values can influence their teaching by investigating their facilitation and reflections of value-aspect in regard to climate change in school mathematics.
III. The mathematical formatting of how climate change is perceived: Teachers' reflection and practice	To identify the potential for facilitating students' awareness and understanding of the formatting power of mathematics
IV. Climate Change and Students' Critical Competencies: A Norwegian Study	To identify how students' critical mathematical competencies can appear in their argumentation
RESEARCH QUESTION: HOW CAN FOCUSING ON CLIMATE CHANGE FACILITATE STUDENTS' CRITICAL MATHEMATICAL COMPETENCIES?	

Table 1 recurs throughout the thesis, gradually adding columns that provide overviews of methodology, methods, analysis, and findings.

One purpose of this study is to explore and describe if and how tackling a complex and urgent socio-political issue such as climate change in the mathematics classroom can enable students to act as critical citizens in a lived democracy. Another purpose is to provide insights into teachers practice and how CME can be applied in the classroom. According to Skovsmose (1994), the integration of CME in the classroom should not be reduced to certain activities, topics, or a set of teaching principles. The activities, topics, or teaching principles might constitute certain qualities, and it can be relevant to gain insights into these qualities. Both a literature overview and an empirical study can provide valuable insights into this context. The insights can be relevant and valuable for mathematics researchers and educators, pre-and in-service teachers, and teachers.

CME consists of a variety of concerns and topics, such as social justice, power, and language. Researchers like Coles, Barwell, Cotton, Winter, and Brown (2013), Wolfmeyer and Lupinacci (2017), and Radakovic, Weiland, and Bazzul (2018), Yasukawa (2007) indicate links between mathematics education and sustainability. Other researchers connect climate change and mathematics education, for instance, Abtahi, Gøtze, Steffensen, Hauge, and Barwell (2017), Barwell (2013, 2018), Hansen (2010), Hauge and Barwell (2015, 2017), and Hauge et al. (2015). My review of the research literature in this particular field shows limited empirical studies in this area (see Appendix A for an overview of previous literature).

1.2 Philosophical assumptions

It is important to be transparent and bring forth the underlying assumptions one holds as a researcher. Creswell (2013) highlights four philosophical assumptions important for transparency: ontology, epistemology, axiology, and methodology. Ontology is described as “the nature of reality and its characteristics”, epistemology as “what counts as knowledge and how knowledge claims are justified”, and axiology as “the role of values in research” (p. 20). In the following, aspects of the underlying philosophical[3] assumptions relevant for this study are described.

When studying the teaching and learning of climate change, ontological, epistemological, and axiological questions can arise. Whether climate change is real or not is an ontological question, and something people disagree about (Esbjörn-Hargens, 2010). However, as Hulme (2009) points out, disputes about climate change may often be presented as scientific disputes, but they in fact concern epistemological disagreements such as how one establishes knowledge or the values in knowledge-seeking. This study is based on the understanding that anthropogenic climate change is real and independent of its human conception and the observed reality of climate change as socially constructed through human observations and

[3] Aspects of underlying assumptions regarding methodological are reflected upon in the chapter 3.

interpretations. These assumptions comply with the ideas from critical realism. Cornell and Parker (2010) underline that critical realism involves an epistemology recognising social dimensions of knowledge and an ontology asserting an objective reality of earth systems while recognising the complexity and difficulty of the prediction of phenomena. This study is founded on a subjectivist ontology: describing reality as something being constructed by individuals and groups (Wood & Smith, 2016). The reality of climate change is shaped by political and social power, e.g. through the formatting power of mathematics. Based on this foundation, it is essential to empower people and their ability to critique aspects of society.

I regard knowledge as value-loaded and as socially constructed. This view is in contrast to classical epistemology, where the view on knowledge is associated with "justified, true belief" (Hollis, 1994, p. 9). Emphasising teachers' facilitation of students' critical mathematical competencies is based on the view that participants construct meanings within a situation. Participants' views are considered relevant and formed through social interactions such as discussions. This view is consistent with ideas from social constructivism, where Creswell (2013) describes that individuals develop subjective meanings of the experiences they have of the world they live in. This view of knowledge as socially constructed influenced this PhD-study's design, for instance, when choosing a research partnership with three teachers and their students, where participants' social interactions can form views on how to facilitate CME in the classroom or standpoints on climate change issues.

Related to my view of knowledge, mathematics and mathematics education is neither neutral nor value-free as socially constructed. Mathematics can be considered a language that in different ways interprets or represents reality. These interpretations and representations depend on humans and are thus socially constructed. Skovsmose (2012a) argues that many people in society celebrate and trust the rationality of mathematics. Such confidence in rationality was challenged by critical theories rooted in the Frankfurt School. In the Frankfurt School, traditional views on science were rejected, and they were critical to the ideal of "scientific and technological knowledge to control nature and the development of a calculative, impersonal kind of reasoning" (Alvesson & Sköldberg, 2009, p. 147). Mathematical rationality should not automatically triumph ethical considerations. Furthermore, mathematics can have anti-democratic consequences if citizens consider mathematics as value-free (Ernest, 2009). For instance, if politicians present selected statistics according to their political agenda, and citizens have a non-critical filter toward mathematics, citizens could be influenced accordingly.

Values can influence and act as a filter of how we interpret information and read research articles (Corner, Markowitz, & Pidgeon, 2014). For instance, values could influence the choice of particular graphs (as discussed in paper III), or influence how we solve mathematical tasks. Nurse and Grant (2019) found that political interest influenced how people used their numeracy abilities and solved mathematical problems. An absolutist approach to the philosophy of mathematics, by which perceptions of mathematics are objective and neutral, do not include the human dimension. Stemhagen (2009) underlines that if students are encouraged to acknowledge their ability to construct new knowledge and evaluate the value of their constructions, they could develop a mathematical agency and epistemological empowerment beyond the mere mastering of calculation. Thus awareness of the normative aspects of the school is, for me, essential. Schools and curricula are normative, e.g. as seen in the Education Act (1998) stating that students should "act ethically and with environmental awareness" (§ 1-1). Whether these norms concern environmental awareness, enabling the workforce to satisfy the free-market capitalism, or promoting neoliberalist ideas on economic growth, this is relevant for all teachers to reflect on.

1.3 The context of real-world problems

Real-world problems range from tasks with weak links to the problem, accompanied by artificial questions (e.g. How many legs have 20 polar bears?), to tasks with strong connections to the actual real-world problems (e.g. What actions should we take on climate change?). Teachers can sometimes equate real-world tasks with contextualised problems such as pizza-cutting, as described by Simic-Muller, Fernandes, and Felton-Koestler (2015). The relatively large span between the objects of such tasks is underlined by Jurdak (2016a, 2016b) who argues that school mathematics and real-world problem solving should be more interconnected. Justification for such interconnectedness is rooted in the discrepancy between solving problems in the mathematics classroom and the real world. Jurdak (2016) highlights that "school problem solving practices to be more embedded in real world problem solving" (p. 68). Researcher such as Busse (2005), Maaß (2006) and

Christiansen (2001) emphasise that such interconnectedness is not without challenges. When students do modelling tasks based in real-world situations, they either neglect the mathematical aspects, neglect the real-world aspects, try to balance these two aspects, or disassociate them.

I consider approaches that do not neglect or disassociate aspects of the problem as relevant for students' critical mathematical competencies. Such holistic approaches can be in agreement with a transdisciplinary approach, described by English (2016b) as knowledge and skills from several disciplines applied to real-world problems to improve learning. Teachers who implement real-world problem-solving in the mathematics classroom could, therefore, consider how developing students' mathematical competencies includes competencies such as calculating as well as learning about the real-life problem with all its implications.

Climate change is sometimes referred to as a wicked or a super wicked problem (Levin, Cashore, Bernstein, & Auld, 2012). Rittel and Webber (1973) characterise a wicked problem as one that is unique with no definite formulation and solutions that are good-or-bad (rather than true-or-false); it is not known when solutions are found, there is no ultimate test of solutions and no set of well-described potential solutions, as these can be considered a symptom of another problem and explained in numerous ways. Levin et al. (2012) expands this characterisation by referring to climate change as a super wicked problem, since time is running out to deal with the problems, there is no central authority who can make decisions, and decisions made at present can have an irreversible impact on future generations. Climate change exhibits many characteristics that make it especially wicked. For instance, there are the cross-sectional aspects, it is a global problem, and those who cause the problem might not be those who are most affected by it. In addition, it involves a tension between rich and poor states, there are no quick technological fixes, and it challenges existing practices (e.g. a very carbon-depended way of living). In wicked problems, mathematical representation can quickly become complicated, unclear, or misleading. When students face wicked problems, they need a different set of competencies than those used when solving more straightforward mathematical tasks. Engaging students in wicked problems in the mathematics classroom could be one approach to bridging the gap between the school environment and real-world problems as described by Jurdak (2016a, 2016b). Learning with real-world problems could contribute to developing students' critical mathematical competencies, thereby helping them become critical citizens in a lived democracy.

The context of climate change involves competencies and knowledge from multiple subjects, in particular those related to *realfag*. *Realfag* is a term used in the Norwegian curriculum and includes the subject mathematics, natural science, physics, chemistry, biology, geology, and astronomy[4]. For practical reasons, the related term STEM (Science, Technology, Engineering, and Mathematics) is used in this thesis. The interdisciplinarity of climate change involves different aspects of how competencies can be perceived. I particularly point out competencies relating to critical perspectives that are relevant to critical citizens in a climate change context. Maaß, Geiger, Ariza, and Goos (2019) state that complex problems like climate change have caused different stakeholders to "emphasising the urgent need to identify and improve the competencies young people will require to meet the demands of their futures—personal, civic and workplace" (p. 2). To identify and improve such competencies is not a straightforward task, and I refer to 21st-century skills[5] and key competencies, partly inspired by Maaß et al. (2019). Key competencies were described by the European Commission (2019) as a combination of knowledge, skills, and attitudes to gain work, social and personal fulfilment, and participate in active citizenship. Knowledge concerns established concepts, facts and theories; skills involve the ability to use existing knowledge; while attitudes concern the disposition to (re)act to ideas and situations. These key competencies, together with the 21st-century skills, are essential for students' critical mathematical competencies.

Competencies related to the STEM-field are described as important by the Norwegian Ministry of Education and Research (2015) in their strategy for *realfag,* and underlined as crucial for providing a qualified workforce for society (Bybee, 2013; English, 2016a, 2016b, 2017; Maaß et al., 2019). Although enabling a qualified workforce is vital for society, this study gives attention to personal and civic concerns as critical citizens critiquing society. The ability to foster critical reasoning and enquiry supported by the use of knowledge from various disciplines was underlined by Geiger (2019) as essential.

[4] In lower-secondary school only mathematics and natural science are *realfag*; however, technology is integrated into the curriculum of mathematics and natural science.
[5] 21st century skills are described in sub-chapter 1.1

Another aspect concerns the degree of integration. Integrated approaches to teaching and learning complex problems use competencies from different subjects (Sanders, 2008). As highlighted by English (2016a), integrated STEM education should not be four disciplines taught in isolation but should deal with problems equivalent to how real-world problems are handled: in integrated ways. Students should be able to pose, solve and interpret questions and solutions mathematically, as well as analyse, reason and communicate. Problems like climate change calls for an increased emphasis on integrated STEM approaches “reflecting the interdisciplinary solutions required in tackling today’s complex economic, social and environmental problems” (English, 2017, p. s7). Relatedly, Tan, Teo, Choy, and Ong (2019) emphasise that an integrated STEM approach should build on “solving complex, persistent and extended real-world problems using practices unique to the four disciplines while drawing on the connections within and between disciplines” (p. 7).

Linked to the previous aspect is a third aspect: how mathematics is utilised in STEM education or interdisciplinary approaches. English (2016a), Geiger (2019), and Maaß et al. (2019) argue that the role of mathematics is underutilised and overlooked within STEM education, especially regarding critical aspects. When including climate change, an awareness of the role of mathematics is relevant, and although I relate to integrated approaches, this study focuses its attention on mathematics.

In this introduction, the background for the research question and focus was presented. The rest of this thesis is structured into three chapters. Chapter 2 presents the theoretical perspectives and conceptual framework. Here perspectives from CME are particularly relevant. Chapter 3 provides an overview of the research methodology and research methods, where the research partnership with the three teachers and their students is described and reflected upon. In chapter 4, the findings are presented and discussed. They are first briefly presented chronologically by papers before being discussed according to six identified themes: lived democracy and critical citizens, the mathematical formatting of climate change, critique and critical reflections, mathematical literacy and three kinds of knowing, controversies and values, and student-centred and dialogic learning. In this last chapter, I point out the limitations of the study and implications for future research, before ending with some concluding comments.

2 Theoretical perspectives

This chapter presents the theoretical foundation for this study. Theoretical perspectives from CME, the broader fields of mathematics education and education, the STEM-field, and post-normal science were essential when designing the research and formulating the research question. These perspectives contribute to the conceptual framework and provide a framework for analysing and reflecting on results and findings. The different role of theories in this study relates to what Hiebert and Grouws (2007) highlight: “Theories are useful because they direct researchers’ attention to particular relationships, provide meaning for the phenomena being studied, rate the relative importance of the research questions being asked, and place findings from individual studies within a larger context” (p. 373). The theoretical perspectives vary in the four papers; for instance, post-normal science is prominent in paper I and less emphasised in papers II, III, and IV, and perspectives from STEM are used more extensively in paper IV. This combined theoretical foundation provides insights that would perhaps be missed if only one field were used. The inclusion of several theoretical perspectives is underlined by Lester (2010) as relevant when using realistic, complex situations because the problem itself typically involves a variety of theories.

The role of practice and collaboration with teachers and students is a key part of this PhD-study. This view is consistent with Skovsmose’s (1994) emphasis on both educational theory *and* practice: “Philosophy may provide clarification and supply new interpretations, but basic guidelines for educational reforms must be produced by educational practice” (p. 8). Educational practices are both teachers’ and students’ reflective practices in the classroom, and three of the papers address educational practices. Papers II and III concern teachers’ educational practices and their reflections, while paper IV involves students’ practices.

Table 2 presents an overview of the titles, research focuses, and theoretical and conceptual frameworks for each of the papers. This table is an extension of Table 1. The rest of this chapter is structured into three parts: the first presents the theoretical foundation and previous research; the second the conceptual framework; and the third includes some concluding reflections on the theoretical perspectives.

Table 2. An overview of each paper's title, research focus, and theoretical and conceptual framework

Title of paper	Research focus	Theoretical perspectives and conceptual framework
Critical Mathematics Education and Post-Normal Science: A literature overview	*1. To identify and critically reflect on concepts and perspectives emphasised as important in the literature from two fields, CME and PNS.*	**Theoretical perspectives** Two theoretical fields, Critical Mathematics Education (CME) and Post-normal Science (PNS) were used to identify concepts emphasised in the literature as important.
Wicked problems and critical judgment in school mathematics	*2. To explore how mathematics teachers´ values can influence their teaching by investigating their facilitation and reflections on value-aspects regarding climate change.*	**Conceptual framework** Two main concepts were used to analyse and discuss the data: controversies and values.
The mathematical formatting of how climate change is perceived: Teachers' reflection and practice	*3. To identify the potential for facilitating students' awareness and understanding of the formatting power of mathematics.*	**Theoretical perspectives** Three main concepts were used to analyse and discuss the data: the formatting power of mathematics, uncertainty, critical citizens.
Climate Change and Students' Critical Competencies: A Norwegian Study	*4. To identify how students' critical mathematical competencies can appear in their argumentation.*	**Conceptual framework** Theoretical perspectives from CME and STEM were used. The conceptual framework constitutes of three main concepts: critical reflections, mathematical, tecnological and reflective knowing, and inquiry-based dialogues.

2.1 Critical Mathematics Education

In this PhD-study, I take a critical mathematics perspective. Inspired by researchers like Skovsmose (1994), Yasukawa, Skovsmose, and Ravn (2012), Ernest (2015), Sriraman, Roscoe, and English (2010), Gutstein (2006), and I challenge the view that mathematics education, mathematics, and its applications, are universal, objective, neutral, culture-free, and value-free. One reason to challenge this view is how the perception of mathematical representation as neutral can potentially hide (political) choices made through the process of collecting and presenting the data. Another reason is how mathematics education can itself hide values, for instance, through the curriculum. Mellin-Olsen's (1987) argues that openness and reflectiveness about how politics is embedded in mathematics education, the curriculum, or the examination system was necessary. To critically reflect on mathematics education, mathematics, and its applications, can offer awareness of the political nature of these areas to both teachers and students.

Climate change is used as a context for facilitating students' critical mathematical competencies. This choice builds on the socio-political turn in mathematics education, a term coined by Gutiérrez (2013) to describe an increased focus on CME in recent mathematics education research. This turn connects mathematics education with complex socio-political issues such as conflicts, controversies, social justice, inequality, and global crisis. Barwell (2013, 2018) proposes CME as a theoretical perspective that can conceptualise how teaching and learning mathematics can engage in climate change. He considered the role that mathematics education can play in both understanding, formatting, and responding to climate change, and suggests one "get behind the data to consider political issues" (2013, p. 13). To get behind the data can involve investigating the role mathematics can play in creating climate change (for instance by facilitation of the technology) and in constructing our understanding of climate change. Barwell suggests that students could engage in climate change, for example by working with quantities of climate and emissions data or weather station data, and by communicating with climate scientists, politicians, or representatives of the community.

Mathematics education can enable students to interpret and challenge socio-political issues. For instance, Frankenstein (1998, 2010a) mentions several goals regarding curriculum, e.g. to understand: (1) The mathematics, (2) The mathematics of political knowledge, (3) The politics of mathematical knowledge, and (4) The politics of knowledge. All these goals are relevant for this PhD-study: Students' understanding of the mathematics used in climate change (e.g. statistics and graphs) is relevant for understanding climate change[6] and reflecting on how mathematics deepens our understanding of climate change. Reflecting on how these data came about is also relevant, as is considering how knowledge (mathematics) is learned in school.

Climate change can be a matter of social justice; for instance, the consequences of climate change are not distributed equally. One example that illustrates this is that the predicted physical effects of climate change in Norway, such as flooding, drought, and storms, will not be as harsh as in other regions. As a well-developed country, Norway can adapt to changes in a more sufficient way than the least-developed countries (CICERO, 2018). At the same time, Norway profits significantly by the extracting of oil and gas, and by that, is part of a continued greenhouse gas-emission. The social justice perspective was underlined by Skovsmose (2014), who described CME in terms of concerns: "To address social exclusion and suppression, to work for social justice in whatever form possible, to try open new possibilities for students, and to address mathematics critically in all its forms and application" (p. 116). Although the social justice issue is relevant, I emphasise addressing mathematics critically, by critiquing with mathematics, to develop students' critical mathematical competencies and enable critical citizenship.

Skovsmose (2012a) elaborates three views of mathematics that are relevant for this study: (1) "mathematics is essential for understanding nature", (2) "mathematics is a powerful resource for technological invention", and (3) "mathematics is a pure rationality which operates almost as an intellectual game, divorced from other human activities" (p. 49). Mathematics is, along with other disciplines, crucial for our understanding of climate change. In such a view lies a recognition that when mathematics is used to describe, predict, and communicate about climate change, it becomes a potent part of formatting our understanding of climate change (Barwell, 2013, 2018; Skovsmose & Yasukawa, 2009). Technological development has created a positive impact on society, e.g. in medicine, but it has also facilitated technology that

[6] I also regard that students need not always understand the advanced mathematics behind, for instance, climate models in order to reflect on these issues, in line with findings emphasised by Hauge (2016a); Hauge et al. (2015).

causes problems such as climate change (Hauge & Barwell, 2017). Skovsmose (2012a) refers to the wonders and horrors of mathematics in action. A crucial part of CME is to consider how mathematics can both be a powerful resource as well as contribute to the destruction of the environment. Lastly, the view on mathematics as pure rationality is problematic when mathematics is involved in complex issues such as climate change. For instance, choices are made by mathematicians involved in measuring global temperature and CO_2-levels and when projecting climate change models (see discussion in paper III). It is therefore relevant to draw attention to this aspect when facilitating students' critical mathematical competencies. Summarised, I emphasise addressing mathematics critically as a way of achieving insights into climate change, as a powerful resource in technological development, and as pure rationality.

I combined and coordinated theoretical perspectives to develop a conceptual framework for studying the inclusion of climate change in mathematics teaching. The term combining is used when "the theoretical approaches are only juxtaposed" (Prediger, Bikner-Ahsbahs, & Arzarello, 2008, p. 173), while the term coordinating is used when the elements in the conceptual framework are "well fitting" and coordinated in a "harmonious way" (Wedege, 2010, p. 67). Within mathematics education, there is a diversity of theories, and they exist in complicated networks (Bikner-Ahsbahs, 2016; Prediger et al., 2008).

Several perspectives and concepts developed within mathematics education concern specific ways of teaching. As pointed to in the introduction, a Norwegian classroom can be characterised by a teacher giving instructions and students working with tasks, referred to as the exercise discourse by Mellin-Olsen (1991) and the exercise paradigm by Skovsmose (2001, 2011). Skemp (2006) and Mellin-Olsen (1981) argue that such teaching and learning is based on an instrumental approach to mathematical understanding. It also concerns what types of tasks students work with. Tasks can, for instance, be problem-solving tasks related to the real-world, as emphasised by Jurdak (2016a). Another perspective is how one perceives mathematics and the potential students can have for learning. For instance, Boaler (2015) emphasises that mathematics is an area with the potential for a growth mindset. These perspectives from mathematics education are often intertwined with perspectives from CME, and rather than distinguish between these two fields, I acknowledge this interrelationship.

The teachers' role in developing educational practices concerning socio-political issues is crucial. Giroux (2013) argues that educational reforms seem to place little confidence in the ability of teachers to provide "intellectual and moral leadership for our youth" (p. 165). It is essential to bring forth teachers voices from this empirical research, and papers II and III thus have a focus on the teachers.

Climate change issues can differ from other problems included in the mathematics classroom. For instance, no single, correct approach exists for reaching the right answer. Therefore, teachers who facilitate such problems encounter other challenges than those with more traditional tasks. These challenges can also be faced in the real world. Funtowicz and Ravetz (1999, 2003) suggest that issues like climate change, with characteristics such as urgency, complexity, uncertainty, values, and a plurality of perspectives, need other approaches to the science-policy interface than do more «normal» scientific problems, as science alone cannot provide answers to what is the best decision. They refer to complex science-related problems where "typically facts are uncertain, values in dispute, stakes high, and decisions urgent" (2003, p. 1) as post-normal issues. Approaches and solutions designed to solve "normal" problems are insufficient because of the plurality of perspectives, and they argued that a post-normal approach was required. This approach can involve citizens critically assessing the relevance and validity of the research and contributing with insight and perspectives in the public debate. Moreover, policymaking and actions should consider the uncertainties, values and the plurality of legitimate perspectives in complex issues like climate change.

Funtowicz and Ravetz (1993, 2003) argue that post-normal science should involve a wide range of participants, not only experts. They refer to such involvement as an extended peer community. An extended peer community may include local inhabitants with local knowledge, investigative journalists, members of pressure groups, people "with a stake in the dialogue on the issue" (1993, p. 739) or those with a "desire to participate in the resolution of the issue"(2003, p. 7). If citizens should contribute to these extended peer communities, they can benefit from learning to deal with complexity and uncertainty in school.

A criticism of post-normal science concerns the view that these extended peer communities can undermine research conducted by scientists or other highly qualified people. However, as Bremer et al. (2018) emphasises, within these extended peer communities, it is important to acknowledge "that each perspective is unique and important, but will be critically scrutinized" (p. 261). Students or other non-researchers have different roles to play than researchers. Young people can sometimes have other perspectives or other concerns than older people, or laypeople can contribute with local knowledge from their community. An essential aspect of such communities is to include value-perspective and the uncertainty connected to complex problems. In terms of education, Hauge and Barwell (2017) argue that the ideas from CME combined with the ideas from post-normal science could prepare students to attend in such extended peer communities. In school, students can interact with different participants such as scientists, politicians, public media, and the general public to understand and act on climate change. The role of CME is to develop students' critical mathematical competencies so students can, amongst others, participate more effectively in extended peer communities.

2.2 Conceptual framework

In this part, the conceptual framework is presented and discussed. The concepts are chosen partly from the literature overview, partly by a careful reading from the fields emphasised in sub-chapter 2.1, and based on the observed qualities in the research partnership and analytic process. Some concepts, such as critique, are explosive: "any attempt to clarify them brings us to use other concepts just as complex as the ones we set out to clarify" (Skovsmose, 2012b, p. 359). A clarification of the concepts is provided in the following text as is how they contribute to addressing the research question and the four research focuses found in Table 1. Some concepts have a more prominent place than others. For instance, lived democracy and critical citizens relate to the research question and the foundation for this PhD-study, while concepts like dialogues were used as analytical tools in paper IV.

2.2.1 Lived democracy and critical citizens

Lived democracy and democratic practices can be understood as more than formal systems of government. It is about ways of associated living, a socio-political engagement, and it is infused in our everyday life. Dewey (2011) emphasises that "democracy is more than a form of government; it is primarily a mode of associated living, of conjoint communal experience" (p. 50). A lived democracy and democratic practices should also involve education and matters of social justice (Chomsky, 2003; Dewey, 2011; Lund & Carr, 2008). Mathematics education researchers and teachers can contribute to ensuring that a lived democracy is included in the mathematics classroom. Breivega, Rangnes, and Werler (2019) describe lived democracy as a way of living, that learning should happen through democratic participation. Such participation can take place both *inside* and *outside* school. Participation *inside* the classroom can include students working with authentic and controversial problems as a part of a lived democracy. For instance, teachers can facilitate students' discussions and debates on real-world problems that are relevant for a student's personal life and emphasise the ability and willingness to listen to peers' argumentation, as well as allow students to investigate mathematical and scientific information. Although these activities take place within a school context, this can be one way of including students' everyday lives outside school and a way of exercising democratic values inside the classroom.

There are multiple ways of participating as a form of lived democracy. The recent school strike for climate actions initiated by Greta Thunberg (Kohli, 2019; Marris, 2019; Wearden & Carrington, 2019) can be considered as one example of participation. Young people can also engage in blogging and environmentally-friendly consumer behaviour (Wagner, Johann, & Kritzinger, 2012). Wagner et al. (2012) underlines that a low voter turnout could be an indicator of the health of a democracy and an alleged crisis in democracy. Although voter participation is important for a lived democracy, in this PhD-study, the attention is directed towards other forms of associated living, for instance, understanding, reflection, argumentation, and discussions on complex socio-political issues in the classroom.

Teachers are essential for students' lived democracy. They can facilitate students' opportunities to be engaged as critical citizens and be a critical voice in education. Teachers can engage and critique education itself, and the role that education (and teachers) can play for students' opportunities to participate in a just, lived democracy (Giroux, 2002, 2011) is significant. Giroux (2011) argues that educators should "reclaim schooling as an emancipatory project" to enable "critical thought, agency, and democracy itself" (p. 43). In this study, I focus on teachers' facilitation for students' critical thoughts and agencies relevant to a lived democracy.

To develop students as critical citizens in their lived democracy is a key aspect of the this study's rationale. A critical citizen is someone who can provide a constructive critique that is vital for democracy and their own life, and education has a role to play in enabling students as critical citizens. Giroux (1989) asserts that school should educate critical citizens "who can think, challenge, take risks, and believe that their actions will make a difference in the larger society" (p. 214). He suggests that schools should provide opportunities for shared experiences, to fight for equal quality of life, and to work in social settings emphasising care and concerns. Facilitating such opportunities in the mathematics classroom can involve different contexts. Climate change can be one context that can provide opportunities for students to think, challenge, and act in society, out of concerns for others and quality of life. Climate change can facilitate reflections on real-world issues in the mathematics classroom, which is relevant when developing students as critical citizens. Critical thinking, independent critical judgments, critical awareness of aspects of society, individual and social empowerment, and a long-term goal towards a more just and democratic society, are essential for developing critical citizens (Ernest, 2002).

Developing students' critical competencies as critical citizens is not unique to CME. It has its roots in thoughts from the Frankfurt School and critical theory, as seen in the work of, e.g. Freire and Macedo (2005). These ideas involve political and societal condition and changes for citizens in society. Critical thinking can involve enabling citizens' understanding of mathematics-based information, e.g. of numbers, tables, or graphs. Critical judgments were discussed in paper II, and a relevant part of such judgments is to understand positions in debates. In particular, when controversies are apparent, as the case can be in climate change issues, it becomes relevant to comprehend argumentation for different views, as a first step to take a justified stand. Understanding the consequences of choices or actions (or no-action) is crucial in order to make critical judgments. OECD (2018a) underlines in the PISA 2018 framework that: "only those students who have some degree of knowledge of the consequences of climate change can fully understand conflicting positions in a debate on the reduction of carbon emission in cities" (p. 191). Critical awareness can involve the ability to identify how mathematics is used in a debate on climate change to substantiate different views and is thus vital for being critical citizens.

Another aspect of critical citizens is the capability to make your voice heard in society. Mathematics education, along with other fields, has a responsibility to enable students to participate in, for instance, to raise your opinions in the classroom or public debates, to make well-founded argumentation, and to critique aspects of society. Skovsmose (2008) claims that citizens should not only be enabled to understand and "receive from authorities as a functional receptive consumer but also to 'talk back' to authorities" (p. 14). To talk back to authorities can involve engaging students to critique and making well-founded argumentation, based on mathematical, scientific, and ethical considerations.

Teachers should consider students as citizens. Although I might use terms such as develop and enable when writing about critical citizens in this PhD-study, I understand students as already (critical) citizens in an ongoing and continuous process and not only as critical citizens-in-the-making. The term citizens-in-the-making, introduced by Marshall (1950), could imply considering students as not yet citizens, and the role of education could become a form of transformation. Biesta, Lawy, and Kelly (2009) addressed concerns for such a role of education as a developmental trajectory, with assumptions of critical citizens as the outcome (see Figure 1). A critical citizen is then only achieved when completed the trajectory outlined.

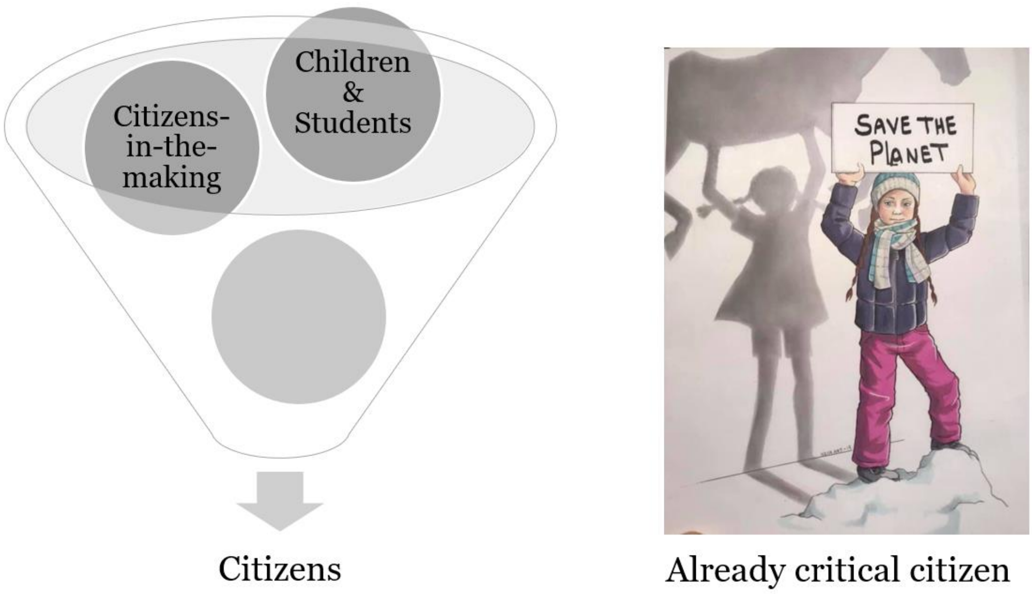

Figure 1. To the left are citizens-in-the-making portrayed as a developmental trajectory, and to the right is recognising that young people are already critical citizens (here by Greta Thunberg & Pippi. The picture is by Andersson (2019), and adapted with permission).

Currently, young people all over the world contribute towards shedding light on the challenges of climate change and demanding action. They act as critical citizens by making their voices heard. They challenge and talk back to authorities, and they believe that their actions can improve their own and other's quality of life. In Figure 1, a picture of Greta Thunberg[7] (Andersson, 2020) with a link to Pippi, the strongest girl in the world, represents these young people as critical citizens. Regarding climate change issues, long-term consequences affect young people to a larger extent than older people, and it is even more important to acknowledge the fact that they are already (critical) citizens when facilitating students' critical mathematical competencies.

2.2.2 The formatting power of mathematics

Students who understand how mathematics format their lives could potentially be empowered on issues relevant to a lived democracy and become critical citizens. The concept of *the formatting power of mathematics*, coined by Skovsmose (1994), refers to the role of mathematics in not only describing the world but in creating our world. Skovsmose (1994) accentuates that "Mathematics has an important social influence; it follows that to understand this formatting power becomes an essential aspect of critical mathematics education" (p. 207). Yasukawa et al. (2012) expresses similar ideas: "mathematical thinking affects how we view, interpret and negotiate our surroundings, and in many ways (sometimes invisible) shapes our society" (p. 268). Also, Greer (2009) underlines that it is essential to make students aware of the way mathematics are intertwined in practically all human activities, and potentially be used as an agenda. By the formatting powers of mathematics, I understand how mathematics can shape our society, and how people's intentional and unintentional use of mathematics can format our understanding and behaviour.

It can be challenging to identify how and where this formatting takes place. The mathematics is often hidden in mathematical models where calculations are invisible. Therefore, it follows that to identify and become aware of the mathematical formatting, which is relevant when facilitating students' critical mathematical competencies. In terms of climate change, it is crucial to become aware of and

[7] By referring to Thunberg, I do not intent to diminish young environmental activist located geographically or historically elsewhere. I do acknowledge that all play a crucial role. However, by using Thunberg as an example, my intention is to refer to the movement receiving an increased attention world-wide, by being "louder and more coordinated" than previously (Marris, 2019, p. 471).

understand the mathematical formatting of climate change (Barwell, 2013, 2018). Mathematics based statistics and models can influence how we perceive climate change and related challenges. In public debates on climate change, one can observe an extensive use of graphs and numbers (as discussed in paper III). The use of mathematics can deliberately (or un-deliberately) influence people to downplay the magnitude of the problem. This type of influence is a challenge for critical citizens in a lived democracy.

2.2.3 Mathematical literacy, mathemacy, and three kinds of knowing

In the research literature, there is a variety of concepts associated with mathematical literacy and mathemacy[8]. In this study, these two concepts are used interchangeably. They were chosen because both are established in the mathematics education research, curriculum, and white papers, partly overlapping; however, they may involve slightly different meanings. In PISA 2021, mathematical literacy is defined as follows:

> *Mathematical literacy is an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st-century citizens. (OECD, 2018b, p. 7)*

Three aspects are of particular interest for this study: the emphasis of a real-world context, the role that mathematics plays in the world, and the reflective 21st-century citizen. For instance, what characterises a real-world context, or what does it involve when students should know the role that mathematics can play? Is it to praise this role, or does it involve critiquing the formatting powers of mathematics? Moreover, when being a constructive, engaged, and reflective 21st-century citizens, what does this mean? Barwell (2013) problematises how OECD framed mathematical literacy[9],

[8] Related conceptions are e.g. numeracy, criticalmathematical numeracy (Frankenstein, 2010b), critical literacy in mathematics (Gutstein, 2006), and reformist critical mathematics (Brantlinger, 2014), matheracy (D'Ambrosio, 2003, 2008, 2010).
[9] Barwell used the PISA 2003 Assesment Framework (OECD, 2003).

as an ability that "would help clarify, formulate or solve a problem" (p. 8). He argues that due to the complex nature of problems like climate change, the rather narrow perspective on mathematical literacy from OECD is no longer tenable. In the definition from PISA 2021, it can still be argued that the focus is on "to solve problems" rather than on critique. Traditionally, the role of mathematics education has been connected to solving problems. However, mathematics should not only be a tool to interpret and solve problems but an approach to critically reflect *on* and *with* mathematics concerning issues such as climate change.

The concept of mathematical literacy can be interpreted narrowly concerning arithmetic-related competencies and basic calculation skills, or with a more broad definition. A broad interpretation differs slightly from more pure mathematical competencies, moving away from school mathematics as a product where algorithm and techniques are in focus and towards more process-oriented mathematics where argumentation, reasoning and communicating is essential. When facilitating and enabling students' mathematical literacy in this broader meaning, Jablonka (2003, 2015) underlines the social aspects of mathematical literacy. Social aspects could be competencies to interpret and read socio-political setting as open to change, support environmental awareness, and evaluate mathematics itself. These broad conceptions of mathematical literacy are associated with the political, moral, and ethical dimensions of mathematics and *mathemacy*.

Mathemacy, described by Skovsmose (1994), is linked to Freire's *literacy*. Literacy involves more than just reading and writing, and it entails a critical engagement in society. Freire and Macedo (2005) express literacy as "a set of practices that functions to either empower or disempower people" (p. ix). Freire (1998) argues for a problem-posing approach to facilitate literacy. He urges teachers to consider their students as conscious, thinking beings, rather than consider students as empty containers in need of deposit (a banking-education). Mathemacy is the capability of moving beyond calculation and formal techniques and "a capacity of making responses and as reading the world as being open to change" (Skovsmose, 1994, p. 94). Mathemacy should "be rooted in the spirit of critique" and "a preconception for social and cultural emancipation" (Skovsmose, 1994, p. 27). In terms of climate change, mathemacy can be more than the ability to interpret graphs correctly; it can involve critically investigating graphs to consider potential consequences for people and who can be empowered to act toward a more environmental-just society. In this study, mathematical literacy is used in the broadest meaning of the word, equal to Freire's literacy and Skovsmose's mathemacy, where the ideas of enabling and empowering people to critically evaluate and cope in a society highly influenced by mathematics.

Knowledge is often associated with truth, absolutism or authority. However, from the perspective of CME, knowledge can be seen as being subjective and open for modifying and changes (Skovsmose, 1994). The use of *knowing*, therefore, suggests a preference towards a process, rather than the more traditional *knowledge of*. Skovsmose (1994) distinguishes between three types of knowing important for mathemacy and CME: mathematical, technological, and reflective knowing. The *mathematical knowing* "refers to the competencies we normally describe as mathematical skills" (p. 100). On the subject of climate change, these competencies could involve the learning and mastery of algorithms relevant to calculate an increase in global temperature or a decrease in sea-ice. *Technological knowing* "refers to the ability to apply mathematics and formal methods in pursuing technological aims" (pp. 100-101). With respect to climate change, such competencies could involve questioning approaches used in measuring the temperature.

Reflective knowing refers to "the competence needed to be able to take a justified stand in a discussion of technological question" (p. 101). It is not necessary to be an expert to take a justified stand, but some basic competencies are valuable. In terms of climate change, such competencies could involve abilities to participate in real-world discussions. When students are encouraged to discuss controversies on climate change in the classroom, more than mathematical knowing is needed; technological and reflective knowing is necessary as well. Reflective knowing considers the societal and ethical aspects and the norms and values related to the issue. All three types of knowing are vital parts of developing students' critical mathematical competencies. Students can learn more than just "the mathematics"; they could be enabled to critically reflect on and with mathematics, as well as combine mathematical, technological and reflective knowing.

2.2.4 Critique and critical reflections

In this study, I do not draw a line between the concepts of critique and critical reflections and use these interchangeably. Being critical can be understood as engaging in critique, making careful and balanced judgments and reasoning, evaluating claims, questioning authorities, not taking explanation for granted, and independent thinking (Ernest, 2002). All these characteristics are essential qualities for students' critical mathematical competencies. Mezirow (1990) highlights that critical reflection involves a critique of assumptions and beliefs. The concept of critique has a unique role in CME and involves processes that identify, interpret, analyse, evaluate and act. Skovsmose and Greer (2012b) highlight different ways of how critique can be understood, for instance, to critique arguments (a logical endeavour), epistemological scenarios (an epistemic endeavour), or real-world activities (a political endeavour). Students can critically examine the logical aspects of mathematics-based arguments in climate change issues. They can critically reflect on what knowledge is and how it is established, or they can critically investigate the political, economic, or social aspects of climate change issues.

Mathematics and critique are sometimes connected to certainty. However, Skovsmose and Greer (2012b) underline that the different forms of critique should not be associated with obtaining certainty. Dealing with uncertainty in climate change issues is relevant when it concerns facilitation for students' critical mathematical competencies. For instance, as highlighted by Hauge (2016b) and Hauge and Barwell (2015, 2017), students can critique an argument involving mathematical choices. It is vital to acknowledge that absolute certainty may never be achieved, and that such an uncertainty need not necessarily be flawed, or automatically lead to the search for more certainty. To wait for more certainty in climate change can potentially lead to irreversible damage.

Critique can involve both reflections and actions, a critical agency. A critical agency can mean that critique is not only reflective but can include attempts to make changes (Skovsmose & Greer, 2012b). Actions can be concrete activities such as eating less meat but could also include students' argumentations. Herheim and Rangnes (2016) argue that argumentation is an action that can contribute to change. Students' argumentation can also include forms of judgments. Atweh (2012) underlines that every critique involves a judgment about "what is good" (p. 332) and thus enters an ethical discourse. Relatedly, Brookfield (2009) discusses critical reflection as a normative aspect. Regarding climate change, Gardiner (2008, 2011) and Wardekker, Petersen, and van der Sluijs (2009) highlight that critique and critical reflections involve values as well as ethical and moral dimensions. Thus, facilitating students' critical reflections in the context of climate change can involve normative aspects and ethical judgments, and teachers can be aware of this aspect when including this topic in the mathematics classroom.

Skovsmose (1992) provides educational meaning to mathemacy and critical reflections by describing six types of questions teachers and students can work with that generate reflections on different levels. Figure 2 visualises these questions. The first two questions focus on "pure" mathematics, while the following two concern the relationship between mathematics and the problem. The fifth question focuses on the general effect of pursuing the problem with the chosen mathematical tool, while the sixth takes a meta-perspective on the process of reflection. Together, these types of questions can guide students before, during, and after lessons and function as a tool for teachers (and researchers) to facilitate and reflect on students' critical mathematical competencies.

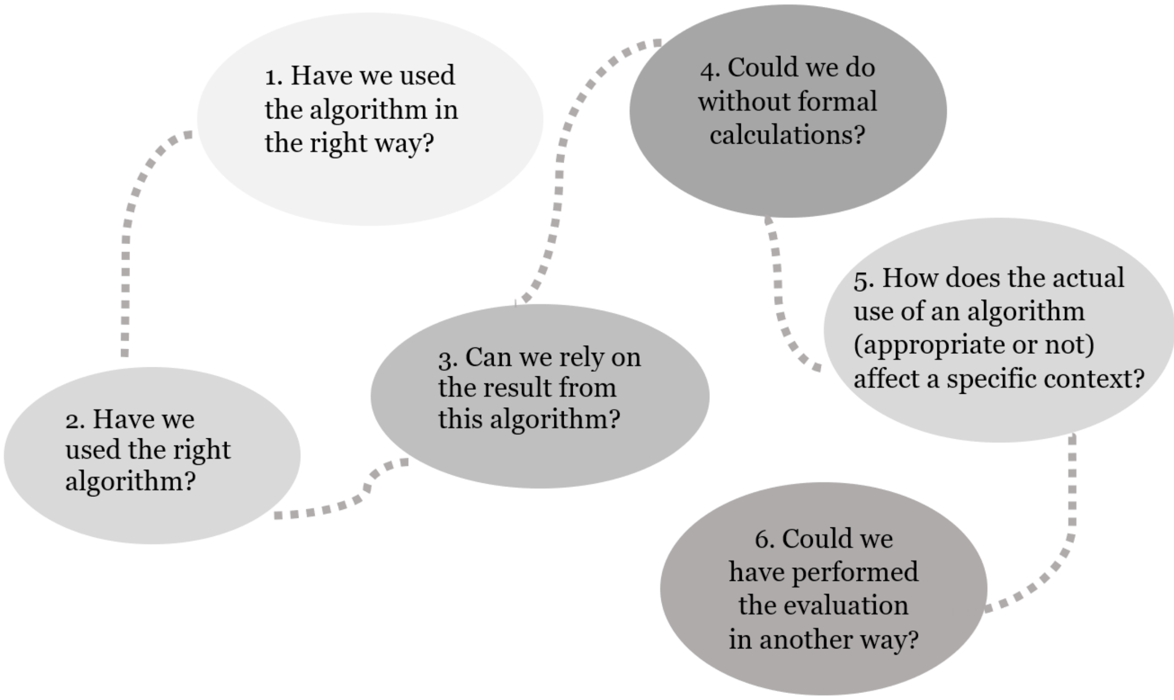

Figure 2. Six types of questions students can ask that generate reflections on different levels (Skovsmose 1992, p. 9).

Figure 2 was used during research partnership meetings to initiate communication about how critical reflections could be included in the mathematics classroom. Hauge, Gøtze, Hansen, and Steffensen (2017, accepted) connect Skovsmose's (1992) six steps of reflections to climate change. Four relevant categories of critical reflections are highlighted concerning; mathematics itself; the relevance and reliability of a mathematics-based argument; the topic of climate change as a result of applying mathematics; and, the role of mathematics in climate change. They discuss whether some types of critical reflections are more present in the classroom than others, for instance, those concerning the algorithm and the mathematics itself rather than the role of mathematics in climate change. Some of the mathematics in climate change are too advanced for students to understand. Teachers can, therefore, find it challenging to use climate change as a context to make critical reflections in the mathematics classroom (Abtahi et al., 2017; Steffensen & Hansen, 2019). However, Hauge et al. (2015) found that "students are capable of reflection on mathematical information, although the underlying mathematics is too advanced to grasp" (p. 1582). Climate change can provide situations where students can use mathematics and mathematics-based argumentation, and in which they can participate in critique and critical reflections.

2.2.5 Values and controversies

From a strictly positivistic view, one can argue that mathematics is neutral, objective, and value-free. As previously described, from a CME-perspective, mathematics and mathematics education is neither value-free nor neutral. For instance, the curriculum can bind teachers "to act ethically and with environmental awareness" (The Education Act, 1998, § 1-1). Values in mathematics education can also be manifest when teachers advocate for specific values and act as "moral agents and value educators" (Seah, 2016, p. 2). Values are an essential part of students' critical competencies and can serve as a foundation as well as provide guidelines.

Freire (2007) describes critical consciousness as representing "the development of critical awareness" (p. 15). He argues that education should include dialogue, investigation and research, and debate real problems. Moreover, the school should have "faith in the student and his power to discuss, to work, to create" (p. 33). Mathematics education should have faith in students' power to be engaged in discussions on socially relevant issues, and these discussions could involve ethical dilemmas involving values and critical consciousness. In real-world public debates about climate change, conflicting perspectives are present, as well as perspectives on righteousness, justice, and fairness (Johansen, Hegdal, & Vetlesen, 2000). Students' discussion in the mathematics classroom should also include such perspectives. Students in Norway feel a moral duty to prevent climate change (Fløttum, Dahl, & Rivenes, 2016). At the same time, they know that their welfare are in part beholden to the petroleum industry. This so-called Norwegian paradox sometimes influences public climate change debates in Norway; while the government argues for actions on climate change, they continue with oil-exploration and exploitation. These conflicting values and interest cause political controversies regarding how to proceed.

Many teachers consider their task limited to convey factual information (Monroe, Plate, Oxarart, Bowers, & Chaves, 2017), and regard teaching about climate change as a matter of giving the students the appropriate scientific knowledge (Hicks, 2014). Facilitating discussions where values, critical consciousness and controversies are apparent may differ from this view. Ernest (2009) and Atweh and Brady (2009) argue that mathematics education should be considered and recognised as a socially responsible discipline because our society is partly enabled by mathematics. Also, Abtahi et al. (2017) suggest that mathematics education could take an ethical and responsible approach to climate change. Including values and controversies in school mathematics through debates and discussions is one way of making mathematics education a more responsible discipline.

Controversies are an essential feature of climate change. A controversial topic is typically disputed, for instance, there are political controversies on how we should deal with climate change. Should nations instantly stop oil exploration? Or corporations pay (higher) carbon tax? Should individuals stop eating meat or only be allowed a certain number of flights? Hess (2009) describes a controversial political issue as "questions of public policy that spark significant disagreement" (p. 37). The controversies in climate change involve scientific and political concerns, such as disputing whether anthropogenic climate change exists. When deciding on what is a controversy, Hess (2009) outlines the "tipping point" for when a topic goes from being a controversy to become a "settled cased" (p. 113). Anthropogenic climate change can be regarded as a settled issue, based on a substantial scientific consensus (Cook et al., 2013; Oreskes, 2018).

Many teachers can avoid controversial topics, reasoned by the fear of indoctrination, discomfort, the need for safety, and a feeling of fairness (Hess, 2009; Hess & McAvoy, 2014). Relatedly, Simic-Muller et al. (2015) found less willingness amongst teachers to teach controversial topics in a real-world-context. Teachers can consider it a challenge to be objective and neutral (Steffensen & Hansen, 2019; Steffensen & Rangnes, 2019). Nevertheless, researchers like Atweh (2012), Gutstein (2006), Hess (2009) emphasise that education should include controversial topics. For instance, Hess (2009) argues that schools should include controversial topics "to help students discuss and envision political possibilities" (p. 6). Facilitating discussions in the classroom on the controversies of climate change could prepare students to participate in public debates in society as informed and critical citizens. An important feature of these discussions is students' acknowledging each other's standpoint and that genuine views and differences should be respected (Gutstein, 2006; Hess, 2009; Hess & McAvoy, 2014). In a lived democracy, different perspectives should be treated with respect. Moreover, citizens should be encouraged to express their views, listen to other standpoints, and potentially adjust their views accordingly; everyone's perspective is important. Disrespecting perspectives that differ from your own, in particular where controversies and values are present, can be deconstructive in debates, unproductive in dealing with challenges, and potentially contribute to a polarised society.

Teachers motivating their students to act is less likely to cause debate in issues with little controversy. Motivating them to action becomes riskier when controversies are involved. However, in climate change, inaction can be a form of action. Teachers who avoid including controversial issues do not choose a neutral path, but actively choose one form of action. Gutstein (2006) suggests that students could investigate and take action on issues in society through the use of mathematics. The recent interest amongst young people seen in the school strikes for climate actions (Kohli, 2019; Marris, 2019; Wearden & Carrington, 2019) could be an entry point for CME to investigate and take forms of action. For instance, as suggested by Steffensen and Kacerja (accepted), this can be done through mathematical modelling when exploring CO_2-footprints and by argumentation and standpoint.

2.2.6 A student-centred and dialogic learning environment

Learning environments can be characterised by teachers show and tell and students' exercise of mathematical tasks from books (Bergem et al., 2016; Echazarra et al., 2016; Vavik et al., 2010). Students are evaluated by how able they are to solve the task. Mellin-Olsen (1991), Skovsmose (2001, 2011), and Maaß (2018) refer to these learning environments as task discourse, exercise paradigm, and teacher-centred transmission-based teaching, respectively.

In contrast to these learning environments stands inquiry-based learning and landscapes of investigation. Inquiry-based learning involves student-centred learning. Students can be involved in observing phenomena, posing questions, carrying out hands-on experiments, seeking explanations, prioritising evidence, interpreting and evaluating solutions, and communicating their solutions (Artigue & Blomhøj, 2013; Maaß, 2018; Maaß & Dorier, 2012; Maaß & Engeln, 2018; Maaß, Swan, & Aldorf, 2017). In landscapes of investigation, there are similar characteristics. The mathematical tasks in a landscape of investigation can refer to pure mathematics, semi-reality, or real-life references (Skovsmose, 2001, 2011). The tasks do not necessarily involve only one correct answer; textbooks are seldom used, the real-life references are meaningful, the teacher becomes a supervisor rather than the one with the correct answer, and students participate in inquiry-oriented discussions. A move towards landscapes of investigation is highlighted by Skovsmose (2001, 2011) as making the students the acting subject in their learning processes, and a move from pure mathematics to real-world problems could facilitate reflections on mathematics and its applications. These kinds of learning environments invite students to formulate questions and explore and look for explanations, qualities that are relevant when facilitating students' critical mathematical competencies. The role of the teacher can shift. Instead of providing explanations and exercises, they can challenge students by probing questions, managing and encouraging discussion where alternative standpoints are present, and helping students connect their ideas (Maaß & Dorier, 2012).

Inquiry-based learning and landscapes of investigation are potential ways of enabling students to construct meaning through inquiry-based processes. These inquiries can take the form of "what if"-inquiries and involve global temperature, CO_2-emission, or consumption. In the exercise paradigm and in teacher-centred transmission-based teaching, where much revolves around solving mathematical tasks, there is a possibility that students can develop what Mellin-Olsen (1987) and Skemp (2006) refer to as instrumental understanding. Key phrases that describe this understanding is "rules without reasons" or "the ability to use a rule" (Skemp, 2006, p. 2). Opposed to this is relational understanding, which Skemp (2006) describes as "knowing both what to do and why" (p. 2). A relational understanding can enable students to relate ways of solving one kind of problem and adopting this method to other problems. How students interact with mathematical tasks is relevant for a relational understanding, and it can be relevant to ask questions such as the following:

> *Do they discuss their results when doing investigations? Do they produce their own ideas for further investigations? How much do they relate their mathematical knowledge to situations outside the classroom? Do they have ideas about examining certain situations by means of mathematics? (Mellin-Olsen, 1981, p. 366)*

The questions and concerns raised by Mellin-Olsen (1981) are applicable to reflect on before, during and after lessons. Mathematics teachers can encourage students to discuss their results, and ask what these results tell us, or what this means to us. Mathematics should not be about learning a rule to pass an exam. The connections between students' lives and school mathematics can be relevant to providing a mathematics education that matters to students. The engagement in climate change issues shown by students all over the world (Kohli, 2019; Marris, 2019; Wearden & Carrington, 2019) indicates that climate change matters. The type of learning environment, as well as the context, can have significance for developing students as critical citizens in a lived democracy.

A particular feature of students-centred learning involves dialogues. Dialogues in the mathematics classroom appear in a variety of forms. One form is the Initiative-Response-Feedback (IRF)-dialogue. Sinclair and Coulthard (1975) describe these as teachers asking questions, students replying, and teachers giving feedback. These dialogues relate to teacher-centred learning. In contrast to IRF-dialogues are inquiry-based dialogues. Alrø and Johnsen-Høines (2010, 2012) describe inquiry-based dialogues as constituting qualities such as asking open questions, wondering, making a hypothesis, listening, making inquiries without posing an actual question, and wishing to understand more. The qualities of dialogues impact the learning of mathematics, in particular those involving inquiry, equality, and risk-taking. When I refer to dialogues in this study, I mean inquiry-based dialogues.

Inquiry-based dialogues are highlighted by Herheim and Rangnes (2016) as a way for students to identify and critique the use of mathematics in society and take a stand. To have the ability to participate in dialogues is an essential part of students' critical mathematical competencies. Alrø and Skovsmose (2002) advocate the Inquiry Cooperation Model (IC-model) and present eight aspects of inquiries through dialogues: getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging and evaluating. Dialogues are used as an analytic tool in the papers, particularly in paper IV. The characteristics described by Alrø and Johnsen-Høines (2010, 2012), and the characteristics in the IC-model, contribute to identifying if and how students posed open questions, listening to peers argumentation, or advocate for their standpoint. Dialogic and critical approaches are highlighted by Artigue and Blomhøj (2013) as a theoretical framework that resonates with inquiry-based mathematics education.

When dealing with complex problems, dialogues are particularly relevant. Hauge and Barwell (2017) argued that focusing only on the mathematical procedure is insufficient when dealing with the controversies of climate change. There must be dialogues that engage students in discussions as well as allow them to face competing knowledge-claims, referred to as knowledge conflict by Skovsmose (1994). Hauge and Barwell (2017) exemplify a knowledge conflict when measuring the temperature of the earth; there is no absolute method, and the process involves choices on location and statistical method. These choices can become the object of disputes in public debates as well as the basis of understanding climate change. It is vital for students' critical mathematical competencies that they become aware of knowledge conflicts, and that they learn to negotiate, critique, and reflect through inquiry-based dialogues. Vithal (2003) also emphasises conflict and dialogue as two main components in CME. She describes conflicts that emerge in the classroom as having a different status: "A conflict in mathematical knowing is often given priority over a conflict in reflective knowing by teachers" (p. 344). For instance, conflicts regarding the mathematical aspects of graphs were considered more urgent than the potential implication of the graph. Furthermore, she underlines that different types of conflicts were handled differently. Teachers and students considered conflicts involving debates on reflective knowing as more open than those on mathematical knowing, the latter often expectedly resolved by the teachers.

2.3 Concluding reflection on the theoretical perspectives

In this chapter, I have presented and discussed the theoretical perspectives and the theoretical foundation, with an emphasis on a CME perspective. In the conceptual framework, highlighted concepts were discussed, such as lived democracy, critical citizens, the formatting powers of mathematics, mathematical literacy, critique, values, and dialogic learning environments. Some of these concepts have a more prominent place in this study than others. For instance, critical citizens and lived democracy represent the foundation and background for the study. It is present in all four papers, either explicitly or implicitly. Other concepts have a prominent place in some of the papers and are less emphasised in the others. For instance, controversies and values are highlighted in paper II, the formatting power of mathematics in paper III, and critiquing and dialogical learning in paper IV. Although each of the concepts and theoretical perspectives makes relevant contributions, when put together, they constitute the essence of what this PhD-study is about: how focusing on climate change can facilitate students' critical mathematical competencies. The theoretical perspectives presented here, and in the literature overview (paper I), provide a framework for the research design and an analytical tool to reflect on the results and findings. In the following chapter, the research methodology and research methods are described.

3 Research methodology and research methods

In this chapter, I describe the approaches I have used to understand how teachers can facilitate students' critical mathematical competencies. One approach was to explore relevant literature. The examination of literature was an ongoing process throughout the research: during the literature overview (paper I), the three papers from the research partnership (papers II, III, and IV), and the theoretical chapter in this thesis. The theoretical perspectives were used to generate a conceptual framework for research and analysis. A second approach[10] was an online survey. Two items were published: Abtahi et al. (2017), focusing on teachers' ethical responsibilities, and Steffensen and Hansen (2019), describing challenges and opportunities expressed by teachers. Although findings from this approach are not included in this thesis, this preliminary investigation contributed to gaining an impression of how teachers facilitated climate change in mathematics education and provided valuable insight for the third approach, the research partnership. The research partnership was an empirical study, where I collaborated with three teachers on facilitating students' critical mathematical competencies in the context of climate change.

An overview is presented in Table 3 that builds on Table 1 and Table 2. This overview presents the title of the papers, four research focuses, theoretical perspectives and conceptual framework, methodology, methods, and analysis for each paper. Following Table 3, the text is structured in four parts. The first part describes the research partnership. The second part describes the research methods used, while the third part describes the analysis. The last part reflects on the quality of the research.

[10] I chose not to include the findings from the online survey, mostly for practical reasons. For instance, Steffensen and Hansen (2019) is written in Norwegian and Abtahi et al (2017) had a slightly different focus.

Table 3. Overview of papers, research focuses, theoretical perspectives, and methodology & methods

Title of paper	Research focus	Theoretical perspectives and conceptual framework	Methodology and methods	Analysis
Critical Mathematics Education and Post-Normal Science: A literature overview	*1. To identify and critically reflect on concepts and perspectives emphasised as important in the literature from two fields, CME and PNS.*	**Theoretical perspectives** Two theoretical fields, Critical Mathematics Education and Post-normal science were used to identify concepts emphasised in the literature as important.	**Overview study** Search strategies; inclusion, exclusion, and quality assessment.	Identifying focus areas through key terms and systematisation.
Wicked problems and critical judgment in school mathematics	*2. To explore how mathematics teachers´ values can influence their teaching by investigating their facilitation and reflections on value-aspects regarding climate change.*	**Conceptual framework** Two main concepts were used to analyse and discuss the data; controversies and values.	**Research partnership** Collaborative research in iterative loops, video- and audio recording, interviews, observations, field notes, transcriptions, coding.	Identify patterns of reflections through categorisations, focusing on value-aspects. Operationalisation of theories regarding e.g. values and controversies.
The mathematical formatting of how climate change is perceived: Teachers' reflection and practice	*3. To identify the potential for facilitating students' awareness and understanding of the formatting power of mathematics*	**Theoretical perspectives** Three main concepts were used to analyse and discuss the data; The formatting power of mathematics, uncertainty, critical citizens.	**Research partnership** Collaborative research in iterative loops, video- and audio recording, interviews, observations, field notes, transcriptions, coding.	Identify patterns of reflections through categorisations, focusing on the teachers and mathematical formatting of climate change. Operationalisation of theories regarding, e.g. formatting powers, uncertainty and critical citizens.
Climate Change and Students' Critical Competencies: A Norwegian Study	*4. To identify how students' critical mathematical competencies can appear in their argumentation*	**Conceptual framework** Theoretical perspectives from CME and STEM were used. The conceptual framework constitutes of three main concepts; critical reflections, mathematical, tecnological and reflective knowing, and inquiry-based dialogues.	**Research partnership** Collaborative research in iterative loops, video- and audio recording, interviews, observations, field notes, coding, transcriptions.	Identify patterns of reflections through categorisations, focusing on students' dialogues. Operationalisation of theories regarding e.g., three kinds of knowing, critical reflections, and inquiry-based dialogues.

3.1 A research partnership

The focus of the research partnership was how teachers can facilitate students' critical mathematical competencies. I use the term *research partnership* to describe the collaboration with the teachers. Tiller (2006) describes a research partnership as a partnership between researchers and practitioners where all are equal partners. The participants can have different roles, e.g. regarding practical planning or analyses. The research partnership involves three experienced mathematics and natural science teachers (Max, Kim and Tim) in lower secondary school and their four classes. Kim and Tim had one class each, while Max had two classes, and there were 26–27 students in each class (106 students in total). The collaborative aspect of the research partnership was acknowledged as essential by all participants, compliant with what Atweh (2004) highlights: "Collaborative action research not only contributes to the development of collegiality within the teachers themselves but also between teachers, curriculum developers and academics" (p. 192).

According to Skovsmose and Borba (2004), the theoretical perspectives from CME resonate with the methodologies of "action research, participatory research and participatory action research" (p. 207). In the papers, I used the phrase *inspired by action research* to describe the research partnership. An example of how the research partnership is inspired by action research is given by using a model provided by Skovsmose and Borba (2004). This model (Figure 3) was used in the research partnerships as a developmental and thinking tool. During the meetings, the teachers referred to the c*urrent situation* as something they wanted to change or add to. When they reflected on ways of facilitating students' critical mathematical competencies, this involved an *imagined situation*. Max, one of the teachers, used the wording "in my ideal dream world". Structural and practical constraints, such as how schedules and lessons were organised at the school, how rooms were furnished, and their proximity to outdoor areas impacted these imagined situations into *arranged situations*. The processes in the research partnership involved ethical considerations, theoretical and practical knowledge, pedagogical imagination on how to move between the current situation and the imagined situation, and a willingness to listen, explore, negotiate, and analyse.

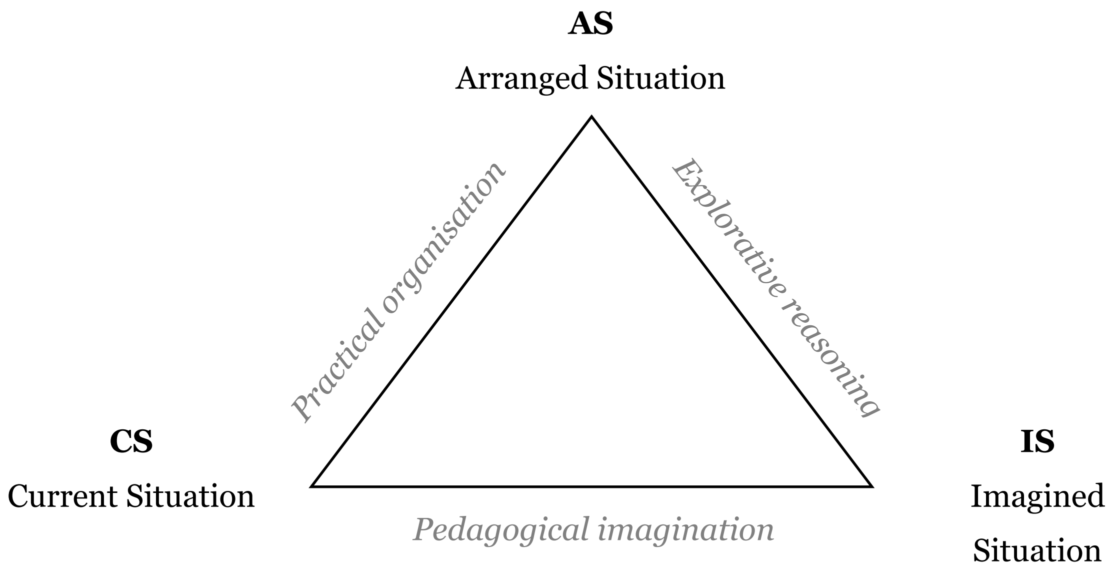

Figure 3. Critical research in a research partnership (Skovsmose and Borba, 2004)

The research partnership focused on educational practices, or what McNiff and Whitehead (2006) refer to as action focuses, where the purpose is to contribute with insights into new practices. Kemmis, Nixon, and McTaggart (2014) underline that changes in practitioners' practices and their understandings of practices is a relevant and interrelated aspect of action research. It is essential to point out that, although I initiated the research and provided the context of facilitating students' critical mathematical competencies, the teachers were the main drivers that initiated the changes in educational practice. During the partnership meetings, we discussed ideas and reflected on our understanding of their practices.

3.1.1 A research partnership through an action-reflection cycle

We had seven Research Partnership Meetings (RPM) (see Appendix C for a more detailed overview). The focus was on planning, evaluating, and reflecting on how students' critical mathematical competencies could be facilitated. The empirical research was done in ongoing cycles of observing, reflecting, acting, evaluating, and modifying, referred to as an action-reflection cycle by McNiff and Whitehead (2006). In Figure 4, the action-reflection cycle for the research partnership is illustrated. Educational designs carried out in the classroom are termed Classroom Activities (CA).

The action-reflection cycle started with an RPM were teachers and I reflected on and explored ideas. Then followed the first CA's, a start-up where students were invited to contribute with ideas on how climate change, mathematics and critical mathematical competencies could be facilitated. Following these first CAs, RPMs and CAs where held interchangeably. The RPMs had a slightly different focus; while some involved detailed planning, others focused on evaluating. Not all three teachers did the same CA, but all of them reflected and planned together before the last CA. Final evaluation with the teachers was the last RPM. Figure 4 visualises how each step of the process builds upon previous planning, actions and reflections. Kemmis et al. (2014) highlights it as a necessity to recognise the capacity of participants in all aspects of the research process, as well as to recognise that all participants aim to improve practices. I acknowledge the capacity of the teachers as vital for the research in aspects such as planning the CAs, reflecting on students' utterances and evaluating the CAs.

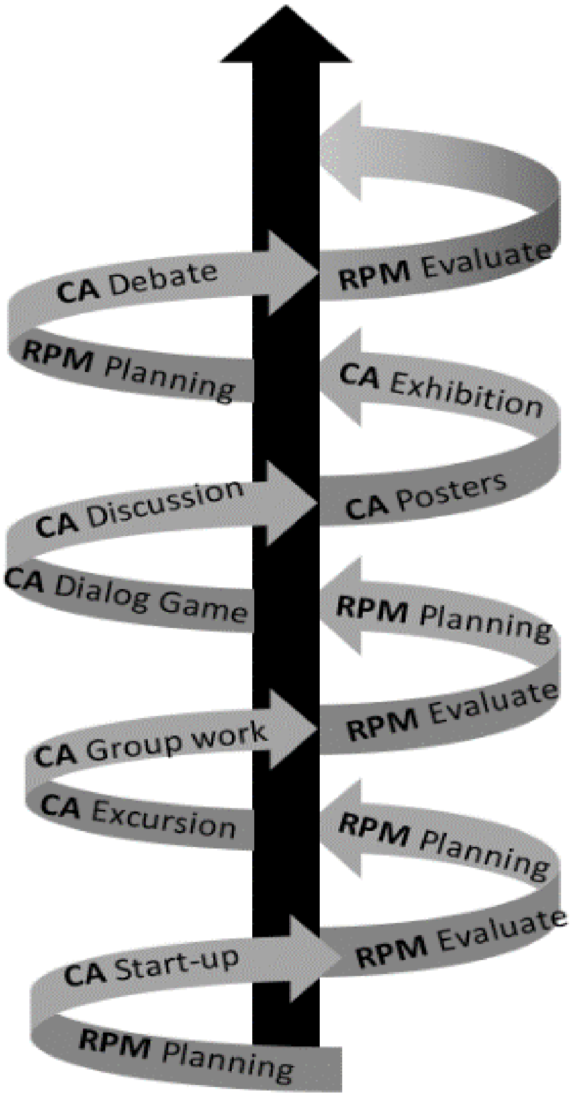

Figure 4. Action-reflection cycles of the research partnership.

The CAs are constituted by seven themes, involving a total of 42 lessons (see Figure 5). All the lessons were organised as group work, varying from 3–6 students in each group. The teachers had three mathematics lessons (3 x 60 min) and two natural science lessons (2 x 60 min) each week in each class. They mainly integrated the teaching in these two subjects. Several of the lessons involved extra teaching recourses, for instance, once a week, the classes were split in half, and sometimes pre-service teachers under training participated. Also, some of the lessons were carried out by teacher temps (due to illness).

The first theme, the *start-up,* was planned and conducted by all three teachers. I was introduced to the students in this start-up. The teachers expressed that the primary purposes were to introduce the topic, invite the students to contribute with ideas on how this topic could be carried out in the classroom, and cover basic concepts regarding climate change (see Appendix D1). By inviting the students to contribute with ideas, the teachers said they wanted to anchor the project within the students' interests so they could be a part of designing the lessons (student involvement). The students' suggestions involved using statistics and graphs, calculating CO_2 for cars, fieldwork, and a panel debate. The teachers' choice of themes resonated with the students' suggestions. By covering the basic concepts, the teachers said they wanted to make sure that the students had some basic knowledge about essential concepts (e.g. to differentiate between climate and weather and between greenhouse-effect and ozone-layer).

Themes (in chronological order)	• **Who and What:**
1. Start-up	• All four classes (4 lessons à 60 min) • Invited students to contribute with ideas
2. Excursion & Report	• All four classes (15 lessons à 60 min) • Measured climate relevant data (CO_2 etc.)
3. Dialogue game	• One class (Kim's. 2 lessons à 60 min) • Discussed and reflected on claims
4. Discussions	• One class (Kim's. 1 lesson à 60 min) • Discussed and reflected on graphs/facts
5. Posters	• Two classes (Max's. 6 lessons à 60 min) • Make posters for an energy-exhibition
6. Energy-exhibition	• All four classes (10 lessons à 60 min) • Presented posters and quiz
7. Dialogue & Debate	• Three classes (Kim's & Max's. 4 lessons à 60 min) • Discussed claims & plenary panel debate

Figure 5. An overview of the themes of the lessons.

The second theme consisted of two parts; *excursion* and a *field report* (see Appendix D2). The theme was planned and conducted by all the teachers and was carried out with reduced class-size. In the excursion, the students collected abiotic and biotic factors, which they later discussed in the classroom, and made a field report. The teachers had several goals, for instance, that the students should experience the "climate researcher role", gain hands-on experience of the different parameters relevant for climate change (e.g. CO_2, O_2, and temperature), acquire practical insight into measuring and measuring uncertainty, collecting and processing data, and presenting it.

The third theme, the *dialogue game,* was planned and conducted by one teacher, Kim (see Appendix D3). It was organised in groups of about five students with a reduced class-size. Kim emphasised that the students should reflect together on questions that did not have straightforward answers. The dialogue game was organised so the students first read a claim and then made an argument on whether they agreed, partly agreed or disagreed with the claim, without the other students commenting. When every group member had made their argument, the students had a second round, now with the possibility to change previous standpoints, possibly as a result of other students' arguments. Kim pointed out that the goal with the dialogue game was not to reach a joint agreement on the claims, but rather to listen and reflect on different perspectives and mathematical argumentation and form personal opinions. Kim had made five claims about global temperature, CO_2-emissions, and Norwegian oil and gas production. The claims could be recognised from public debates. At the end of the lesson, each group made their argumentation in plenary, and there was a teacher-led reflection and summary discussion.

The fourth theme, *discussions*, was planned and conducted by one teacher, Kim, and included one whole class. Here Kim had prepared some fact sheets on climate change in which mathematics was present in the form of graphs and numbers. The students discussed these graphs and numbers, and although not expressed or facilitated, their discussions followed the dialogue-structure from the Dialogue Game.

The fifth theme, *posters*, was planned by all teachers but only conducted by Max, and her two classes (see Appendix $D4_1$). The lessons were carried out with reduced class-size. Max made a task where the students should make a poster. They could choose to focus on essential numbers in climate change, or changes caused by climate change in their local community in the year 2100. The lessons took place in the computer lab, and the posters were made digitally. Two of the posters were to be presented by the students at the energy-exhibition.

The sixth theme, *energy-exhibition*, was planned and conducted by all teachers. The exhibition lasted for two days, and the students and teachers took turns representing the stand. Tim included all his students in the exhibition, while Kim and Max selected a group of students to represent their classes. The teachers made two quizzes with five multiple-choice questions (see Appendix D5). The teachers said that the quiz should actively engage the students in dialogue with the exhibition-participators and that the questions should make the quiz-participants reflect on climate change and mathematics. In addition to the quiz and the two posters made by Max' students, Kim had made a poster himself (Appendix $D4_2$) with the aim to make people critically reflect on climate change issues.

The seventh theme, *dialogue & debate*, was planned by all but conducted by Kim and Max and their three classes. The lesson consisted of two parts: group discussions and a plenary debate (see Appendix D6). The group discussion was organised in the same manner as the previously described dialogue game, while the plenary debate was based on the group discussions. The students discussed issues concerning climate change and mathematics, e.g. CO_2-emission, electric vehicles and road toll.

3.1.2 Strategical sampling and roles in the research partnership

I knew the school and the teachers from previous employment and collaborations. I did a strategical sampling of the school, based on convenience, geographical proximity and willingness to participate. The sampling of teachers was based on their interest to participate and they collaborated regularly. The dialogue between researchers and practitioners is a crucial factor in research (Tiller, 2004). The communication between us was characterised by such as listening, open-mindedness, and friendliness, where all participants felt they gained something from it. Although I regard the teachers as equal researchers, I acknowledge that our different roles give unequal power relations. For instance, while the teachers did this in addition to their regular work, I worked full-time on the research. The fact that we were former colleagues contributed to reducing the traditional gap and unequal power relation;

however, these aspects did not diminish altogether. For instance, at one point, Kim asked me whether students could use computers, which can be interpreted as his considered me as the one in power. In the partnership meetings, we explicitly reflected upon the roles in the research partnership and on the power relations.

The different roles of the participants in the research partnership are worth reflecting on, as researchers, equal teachers/partners, (former) colleagues, observers, or participatory members of the group. Different situations could bring one of the identities more into the foreground. For instance, when interviewing students or teachers, I appeared more as a researcher when I interacted in discussions with students or planning with teachers. These multiple identities can be seen in the context of subjectivity and objectivity. Some situations work better with objectivity, such as when analysing and coding, while other situations work better with subjectivity. Vithal (2004) highlights that in a critical paradigm, the search for an objective "truth" in the study is replaced by findings in this particular context, based on interactions between all participants.

The teachers reflected together. They did not take a spectator's approach, but rather made inquires with each other. The idea of one teacher could turn into another teacher's idea in an alternating exchange of ideas in a transformative social process. Such thoughts on research as a social process are highlighted by McNiff and Whitehead (2006) as vital. All participants can develop inclusional methodologies, and they do not need to share the same set of values or agree on everything. Still, it is critical to recognise the uniqueness of others. The research partnership was a social process that supported participants in further developing ideas, learning and practice. In the initial meeting with the school administration, different concerns were expressed to the teachers (e.g. curriculum goals). The school administration suggested that the research could culminate in students presenting something about mathematics and climate change at an energy exhibition. This annual energy exhibition was a collaboration between the school and the private and public sector in this municipality. Thus, the school administration can be considered as part of an extended research partnership, in the sense that it to a certain degree had influence.

3.2 Research methods

The empirical data consists of audio recordings of seven partnership meetings, video- and audio recording of 42 lessons, written material from teachers and students, and observations and field notes (see Appendix C and Appendix D for overviews and details). This triangulation of methods strengthens the validation of data (Bogdan & Biklen, 2003; Tracy, 2010). Some partnership meetings resembled a focus group interview and provided opportunities for closer observation and further analysis of what the teachers expressed. All meetings were audio-recorded and fully transcribed.

The observations in the classroom were audio and video recorded. I chose three groups from each class to be recorded (two audio-recorded and one video recorded). The choosing was based on whether all students in the group had signed the consent form (see Appendix A2). It was also based on recommendations by the teachers (e.g. willingness to talk). These recordings provided valuable opportunities to observe how students reflected and communicated. The recorded data was uploaded in NVIVO and partly transcribed. A selection of recordings was fully transcribed, based on relevance for the research focus, a strategical sampling of students or the lesson, or for practical reasons (e.g. recording was disrupted by noise). I was present in all the lessons and did all the transcribing.

A semi-structured interview guide was made for interviews with the teachers and students (see Appendix B). In addition to the interviews, I had informal talks individually with teachers and students. These talks provided opportunities to express individual views outside a group setting. Written material from teachers and students included notes (digital or handwritten), lesson and project materials, and presentations. Field notes (e.g. my thoughts and reflections) were written down when observing the lessons. The field notes varied in length, depending on the situations. The research was influenced by practical issues (e.g. sick leave, maternity leave), which resulted in more lessons from one of the teachers compared to the other two.

3.3 Analysis

To answer the overall research question (how can focusing on climate change facilitate students' critical mathematical competencies?) four research focuses (see Table 1) were developed. These were operationalised into concepts that can, to a greater extent, be identified and analysed. For instance, in paper IV, concepts such as critical reflections and dialogues were relevant to identify students' critical mathematical competencies. The qualities of inquiry-based dialogues described by Alrø and Johnsen-Høines (2010, 2012), and the IC-model by Alrø and Skovsmose (2002), were used to identify dialogic qualities in the students' utterances, for instance, how the students were making a hypotheses and inquiries.

The analytic process in the literature overview and the analysis of the empirical data from the research partnership had many similarities. First, it was essential to reduce and make sense of the rather vast amount of available information to get an impression of the research data. Next, it was relevant to check for consistency in the analysis and increase reliability; therefore, multiple methods were used to examine the research data. Examples of these methods were concept mapping, thematising subjects in different ways, statement summaries, data contrasting, and looking for keywords, similarities, differences, and tendencies. Suitable tools like NVivo, Excel worksheets and tables, and concept mapping tools, were used in the analysing process to organise, acquire an overview, transcribe, code and analyse the data.

The unit of analysis was teachers' and students' utterances and actions. For instance, one unit of analysis was the discussions in the partnership meetings. Here we discussed students' and teachers' actions and utterances. Their actions included, for example, teachers' planning and reflections, their choices of learning environments, the design of student-tasks, and students' dialogues. Another unit of analysis was what the students and teachers said and did during the different activities, both at an individual level and group level. When students collaborated, they interacted with one another, and the unit of analysis was the students as a group. However, as highlighted by Frankfort-Nachmias and Nachmias (2008), such an approach can have two fallacies, an ecological and individualistic fallacy. For instance, when my unit of analysis was the group, the findings were not considered applicable to every individual student. Thus, generalisation drawn on behalf of the group was avoided.

The data were organised according to what Creswell (2013) describes as a custom-built and spiralling process involving six procedures: (1) manage and organise data, (2) read, (3) describe, (4) classify, (5) interpret and (6) represent and visualise. The first procedure, (1) to manage and organise the data, was crucial throughout the process of data collecting. All the research data (audio- and video files, word-documents, and pdf-files) were structured in folders in NVivo. The literature was obtained through electronic search engines like Scopus and ERIC. A systematic approach to managing and organising the data was important, both due to a rather substantial amount of data, but also as a way of "getting a sense of the whole database" (Creswell, 2013, p. 183).

The reading of the data (2) involved cycles of in-depth reading and overview readings. When new data emerged, they provided new perspectives and gave existing data new meanings and interpretations. For instance, while transcribing, the reading of the data became more of an in-depth process than when observing or reading written text. Descriptions of data (3) ranged from concise field notes to more detailed reports. For instance, in the literature overview, notes were taken during the inclusion and exclusion process. These notes were later used to condense meanings and contributed to making relevant codes. In the research partnership, a description could be from situations in the classroom, or on immediate thoughts and impressions from the observation, and provided information that might otherwise be forgotten.

Classifying the data (4) and coding was done in NVivo. Condensing meaning, or aggregating text or other data into small codes of information, was done by partly starting with some predefined codes (concepts-driven coding), but mostly by allowing the codes to evolve from the data (data-driven coding). Throughout the coding process, rearranging and refining codes, and identifying themes, were done through cycles. The themes referred to a broader unit of codes, forming a common idea. For instance, in the literature overview, the final coding process resulted in four main themes. Figure 6 shows an excerpt of these themes indicated by numbers, e.g. Risk and Uncertainty was placed in theme one, while Critical citizen and Extended peer community was positioned in theme two. The number of codes and themes differed in the four articles but was kept at a manageable

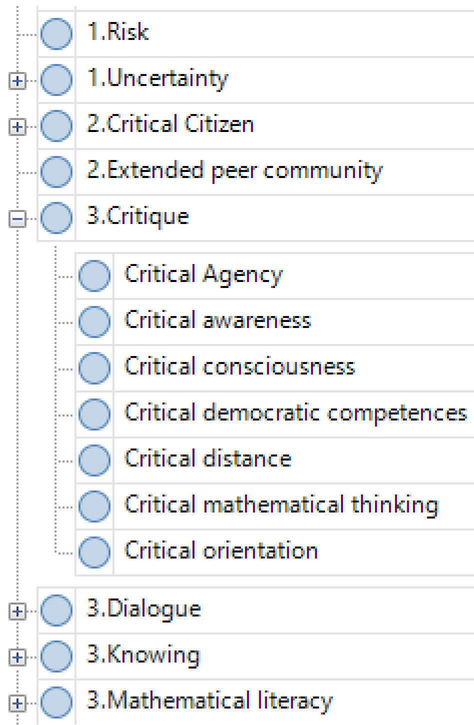

Figure 6. An excerpt of the two-level coding in NVivo.

number with a maximum of three levels of sub-categories. The example in Figure 6 shows a two-level coding with the theme Critique and the seven sub-codes from the literature overview. The labels of the different codes originated both from *in vivo codes*, using specific words to search for, and names composed to best describe the information. An example of in vivo code is *critical consciousness,* used by, e.g. Freire (2007), and *critique* is an example of the best description.

The interpretation of data (5) started during the data collection and continued throughout the process of coding, formatting of themes, analysing, and writing. The interpretation focused on details in the data, as well as on the overall picture. I used both category-based and case-based analysis (Kuckartz, 2019) for further analysis. For instance, in paper III, the category *The formatting powers of mathematics* was used, and a case-summary was made for each of the teachers' utterances. The interpretation was formed as a solitary work, and in collaboration with colleagues, teachers from the research partnership, co-writers, and supervisors.

Representing and visualising the data (6) was relevant throughout the research. Some representations were created to visualise and ease the reading of texts; others were used for clarity and overviews when analysing the data. Some selected figures were refined from thinking-tools during analysis to become tools for communicating with the readers of the papers and this thesis; for instance, Figure 6 was further developed into Table 5 (page 68).

3.4 Concluding reflections on the quality of the study

From the start and throughout this study, choices have been made to address the quality of the research. Here I highlight eight aspects, inspired by Tracy (2010): (1) worthy topic, (2) rich rigour, (3) sincerity, (4) credibility, (5) resonance, (6) significant contribution, (7) ethics, and (8) meaningful coherence. These concepts provide opportunities to discuss choices in regard to the quality of this study.

The first concept, a *worthy topic* (1), concerns the essence of the study. The overall research question concerns students' critical mathematical competencies, which I understand as essential for enabling critical citizens for a lived democracy. Moreover, mathematics education has an essential role to play in challenging undemocratic developments by facilitating students' constructive critique of issues relevant to themselves or society. Mathematics is central in describing, predicting and communicating climate change (Barwell, 2013, 2018), and students need critical mathematical competencies to deal with such wicked problems. The context of climate change brings forth another aspect of this worthy topic: it is a significant challenge for society, and mathematics education and research need to address this.

Rich rigour (2) involves time spent in the field and the theoretical foundation. For instance, the theoretical constructs for the literature overview combine two fields, and the initial literature search resulted in 362 articles, 57 of which were included according to the inclusion and exclusion criteria and quality assessment. Time spent with teachers was approximately 18 months; with students this was about 12 months and included a total of 42 lessons. The relatively long period allowed for connecting and provided multiple settings for facilitating students' critical mathematical competencies. Moreover, it provided the possibility for overview-perspectives as well as more in-depth perspectives.

Sincerity (3) can be achieved by self-reflexivity and transparency. My motivation and background for doing this PhD-study are described in the preface of this thesis. Underlying assumptions are reflected on in subchapter 1.2, and transparency and self-reflexivity are included in the descriptions of the methodology and methods. My background, motivation, and theoretical and methodological choices have influenced the findings and discussions. However, communicating, reflecting and being transparent about different aspects of the research can compensate for any biases I hold as a researcher by giving the reader insights into these.

Credibility (4) refers to thick descriptions, triangulation and crystallisation, multivocality, and member reflections (Tracy, 2010). In the literature overview, a thick description was provided, for instance, by describing the search and exclusion-criteria. However, word-limitations in papers II, III, and IV had some restrictions on the in-depth descriptions. A more thorough and detailed description is provided in this thesis and in the attached teachers material in the appendices. These descriptions can, to a considerable extent, allow readers to explicate culturally situated meanings through additional information. The research partnership provided opportunities for member reflections, questions and critiques of all aspects of the research. For instance, they could read and share input concerning how the findings were presented in the three papers and this thesis.

Resonance (5) involves how research could affect an audience, or how transferable the findings are across different contexts. One example of how I attempted to affect the audience was in paper II. Below is an utterance from a student building up a moral claim on why we should care about our planet:

> *I both agree and disagree with the claim, because I think that we should always save the environment as much as possible. It is our planet and we will be staying here for a very, very long time, so why destroy it already now? Imagine when your grandchildren have grandchildren, do you really want them to wear a breathing mask when they go out because the air is so polluted that it is dangerous to breathe? Or even worse, when your grandson's grandson has grandchildren, do you want that it no longer is possible to live on our dear planet, because we did not bother to drive a little slower, or buy an electric car? I certainly do not want this. I would rather drive a bit slower and let my ancestors live a little longer.*

This utterance was partly chosen because it affected and resonated with me, and potentially the reader, and it was an example of an expression of environmental engagement. This engagement can also show how the research might be transferable across this context with students worldwide showing engagement through climate strikes. Another example of how this research could resonate with other teachers or researchers is that they might identify with how Kim and Max are challenged by climate change controversies.

The *significance of this contribution* (6) concerns providing insight into how wicked problems in the mathematics classroom can occur. It supplements existing research in the field of CME and STEM by offering insight into an empirical practice. For instance, it describes how the teachers facilitated students' critical mathematics competencies and some of their expressed thoughts, ideas, and challenges. It also describes how students dealt with complex challenges, their concerns, and the characteristics of their dialogues. This is insight relevant to and valuable for mathematics researchers and educators, pre-and in-service teachers, and teachers.

Ethical considerations (7) have been carried out throughout this study. When I chose to include climate change in the mathematics classroom, this was an ethical choice, and doing so brings along ethical questions from amongst the students and teachers. This inclusion is in line with researchers such as Abtahi et al. (2017) and Atweh and Brady (2009), who argue that mathematics teachers, educators and researchers have a responsibility to include critical socio-political issues, and as a consequence of this, a social-responsible mathematics education. Additional ethical considerations in the research partnership were anonymity, confidentiality, integrity, and informed consent. Ethical considerations also involve approvals from The Norwegian National Committees for Research Ethics (2014) (see Appendix A1), and informal and formal consents obtained from teachers, teacher temps, and students. It was essential that the teachers' involvement was voluntary and that they had the interest, motivation and willingness to participate. An informal consent and the choice to participate in this PhD-study was underlined as crucial for the agency, freedom and capacity to act in critical educational research by researchers such as Fossheim and Ingierd (2015) and Vithal (2004). Therefore, I emphasised that even though the school administration had agreed to take part on the teachers' behalf, it was possible to decline[11] and withdraw from the research.

A *meaningful coherence* (8) involves using appropriate methods and related research questions, literature and findings to achieve the stated purposes. The study was founded in the field of CME, and as highlighted by Skovsmose and Borba (2004), these theoretical perspectives resonate with the methodologies of action research, which I in this PhD-study relate to the research partnership. The research question on how focusing on climate change can facilitate students' critical mathematical competencies was answered in four individual papers, each holding a specific research focus, as well as in this thesis, which takes a metaperspective and discusses the findings.

[11] Five teachers were invited; two teachers declined for different reasons (e.g. uncertainty regarding content knowledge of either mathematics or natural science and time spent).

4 Findings and discussions

This chapter consists of five parts. In the first part, I present the most important findings from the four papers. In the second part, I take a meta-perspective and discuss the findings across the four papers. This is followed by the third part, where I discuss the limitations of the study. In the fourth part, I reflect on the implications and future research based on this PhD-study. Lastly, in the fifth part, I make some concluding comments.

4.1 Findings from the four papers

Present findings from the four papers briefly and systematically involves limited space for exposing the richness of the empirical findings. However, in this subchapter, together with the next sub-chapter, I outline how the findings from the four papers are interconnected and answer the research question.

An overview, which includes keywords for the findings from the four papers, is presented in Table 4. This table extends Table 1, Table 2, and Table 3, and relates the findings to the research question, theoretical foundation, methodology and analysis. Following Table 4, I present the findings from the papers chronologically: first the literature overview (paper I), then the findings from the research partnership (papers II, III, and IV).

Table 4. Overview of papers, research focus, theoretical perspectives, methodology, methods, and findings

Title of paper	Research focus	Theoretical perspectives	Methodology, methods and analysis	Findings
Critical Mathematics Education and Post-Normal Science: A literature overview	1. To identify and critically reflect on concepts and perspectives emphasised as important in the literature from two fields, CME and PNS.	**Theoretical perspectives** Two theoretical fields, Critical Mathematics Education (CME) and Post-normal Science (PNS), were used to identify concepts emphasised in the literature as important.	**Overview study** Search strategies; inclusion, exclusion, and quality assessment. Identifying focus areas through key terms and systematisation.	**Identified concepts:** wicked problems, uncertainty, complexity, controversy, risks, transdisciplinary, critical citizen, extended peer community, mathematical literacy, reflective knowing, critical agency, critique, dialogue, formatting power, power, responsibility, ethics, value, democratisation, and global society.
Wicked problems and critical judgment in school mathematics	2. To explore how mathematics teachers´ values can influence their teaching by investigating their facilitation and reflections on value-aspects regarding climate change.	**Conceptual framework** Two main concepts were used to analyse and discuss the data: controversies and values.	**Research partnership** Collaborative research in iterative loops, video- and audio recording, interviews, observations, field notes, transcriptions, and code. Identify patterns of reflections through categorisations.	Teachers values can include philosophical dimensions, e.g. values in wicked problems. Wicked problems are perceived as a challenge by teachers. Teachers challenge each other's perspectives, and their values influence the facilitation of students' critical mathematical competencies.
The mathematical formatting of how climate change is perceived: Teachers' reflection and practice	3. To identify the potential for facilitating students' awareness and understanding of the formatting power of mathematics.	**Theoretical perspectives** Three main concepts were used to analyse and discuss the data: the formatting power of mathematics, uncertainty, and critical citizens.	**Research partnership** Collaborative research in iterative loops, video- and audio recording, interviews, observations, field notes, transcriptions, and code. Identify patterns of reflections through categorisations.	Develop an awareness of how mathematics is used in argumentation, allow enough time and opportunities for reflections on complex problems and collaborative research.
Climate Change and Students' Critical Competencies: A Norwegian Study	4. To identify how students' critical mathematical competencies can appear in their argumentation.	**Conceptual framework** Three main concepts were used to analyse and discuss the data: critical reflections, mathematical, tecnological and reflective knowing, and inquiry-based dialogues.	**Research partnership** Collaborative research in iterative loops, video- and audio recording, interviews, observations, field notes, transcriptions, and code. Identify patterns of reflections through categorisations.	Students' critical mathematics competencies appeared when they engaged in inquiry-based dialogues, while they critiqued presuppositions, through (intertwined) mathematical, technological and reflective argumentation.

4.1.1 Findings from paper I: Critical mathematics education and post-normal science: A literature overview

In this literature overview, the focus is to *identify and critically reflect on concepts and perspectives emphasised as important in the literature of two fields, critical mathematics education and post-normal science*. The findings are categorised into four main areas: (1) Important characteristics of tasks; (2) Pupils' role in society; (3) Competences recognised as important; and, (4) Aspects of democracy. Table 5 provides an overview of the four categories, with associated questions, and the identified concepts and perspectives.

Table 5. Overview of four main areas

Main area	Questions	Identified concepts
1. Important characteristics of tasks	What characterises complex real-world problems?	Wicked problems, Uncertainty, Complexity, Controversy, Risks Multi/Inter/Transdisciplinary
2. Pupils' role in society	Why should pupils (be prepared to) contribute to complex problems in society?	Critical Citizen, Extended peer community
3. Competences recognised as important	What kind of competences do pupils need when reflecting on complex real-world problems (in the mathematics classroom)?	Mathematical literacy, Reflective knowing, Critical agency, Critique and Dialogue
4. Aspects of democracy	Why is it important to bring real-world problems into mathematics education from a democracy perspective?	Power, Formatting Power, Responsibility, Ethics, Value Democratisation and Global Society

In the literature from post-normal science, characteristics of real-life problems are emphasised and described. Perspectives such as uncertainty and values are suggested to be included in debates and processes regarding these problems in extended peer communities. In the literature from CME, ways of facilitating competencies recognised as necessary in a democracy are emphasised, and notions such as the formatting power of mathematics are argued to play a key role. In both fields, aspects such as critique and pluralistic dialogue are present and highlighted as important when dealing with complex issues in society. The combined perspective on CME and post-normal science contribute to the understanding of how mathematics education can contribute to students' processes of becoming critical citizens. The identified concepts and perspectives also work as a valuable starting point for exploring how mathematics teachers can facilitate students' critical mathematical competencies.

4.1.2 Findings from paper II: Wicked problems and critical judgments in school mathematics

In this paper, we (Steffensen, Herheim and Rangnes) focus on the value-aspects of climate change and how critical judgments in the mathematics classroom can be facilitated. We explore *how teachers' values can influence their teaching by investigating their facilitation and reflections of value-aspect in regard to climate change in school mathematics*. We reflect on findings from when students and teachers worked with an exhibition (a poster and a quiz) and dialogue game, and include reflections from the partnership meetings.

We found that the teachers' values influenced how they facilitated students' learning. For instance, they did not use mathematics textbooks or teacher-centred methods. Instead, they designed tasks from scratch or tweaked traditional mathematical tasks so that students could discuss the aspects of a graph, as an alternative to finding a quick answer to the task. The teachers facilitated students' collaborative work in groups and communication in dialogues and discussions. They stressed that students should listen to each other's arguments and potentially change their argumentation as a result of other students' reasoning.

The teachers' values were also seen in that they emphasised the controversies in climate change issues, instead of avoiding them. For instance, Kim designed a poster by using visual tools (graphs and a cartoon) to deliberately create debate and reflection. They highlighted the use of numbers in climate change for students to reflect on. In the partnership meetings, the teachers reflected (together) on how mathematics can be used purposely, and they challenged each other standpoints and used numbers and graphs.

In the paper, we discuss whether the teachers' values influenced their use of numbers and graphs to highlight features of climate change when designing a quiz. They chose different topics to point out either the small temperature changes during the last 20 years or the extensive sea ice melting in the Arctic. The students used numbers (e.g. percentage of countries' renewable energy) to make value judgments. Also, they built up moral claims, using mathematical and scientifical facts provided as a starting point.

4.1.3 Findings from paper III: The mathematical formatting of how climate change is perceived: Teachers' reflection and practice

This paper focuses on *identifying how teachers reflect on and facilitate students' awareness and understanding of the formatting power of mathematics*. The data material in this paper consists of the introductory-lesson, from the process when the quiz for an exhibition was made, and meta-reflections from the partnership meetings, when a fieldwork activity was planned.

The teachers facilitated collaborative work in groups and held discussions on topics such as measuring-results, climate change related graphs, and the potential effects and results of climate change measures. In the introduction-lesson, they expressed goals such as introducing and clarifying basic concepts and inviting students to contribute with ideas. They also explicitly emphasised aspects of democracy and critical citizens to the students and related this to a critical assessment of quantitative information in climate change, for instance by highlighting the challenges of melting ice and shrinking habitats for polar bears. The facilitation of reflections and discussions resulted in, for instance, students' thoughts on the 2°C-target and questioning a continued Norwegian oil exploration. However, although the teachers emphasised critically assessing quantitive information, the students took for granted the mathematical argumentation in play and used it as a premise in their discussions.

During the partnership meetings, the teachers meta-reflected on climate change issues like how the average global temperature is measured and how advisors and mathematical models (or hidden mathematics) can influence politicians and others. Their reflections resulted in, among other things, a fieldwork-activity where the students should perform hands-on measures of sea-temperature in different ways and thereafter choose the most representative measurement based on that discussion regarding measurement issues and the average global temperature.

In the quiz, the teachers' mathematical choices could potentially format the students' understanding of climate change and the choice of topics, graphs, and distractors. For instance, the choice of questions, such as the one addressing small changes since 1998, can influence students' understanding of average global temperature. Also, graphs showing a skewed picture between measures and predicted measures can format students' understanding of climate change models as untrustworthy.

4.1.4 Findings from paper IV: Climate change and students' critical competencies: A Norwegian study

Inspired by the question of how a critical STEM-approach can facilitate students' empowerment as critical citizens, I focus in this paper on *identifying how students' critical mathematical competencies can appear in their argumentation*. This focus is addressed by examining students' utterances in discussing climate change.

The students worked collaboratively and communicated in groups in a dialogue game and following plenary debate on CO_2-emission, climate change, and road tolls concerning a local road-project. The students' argumentation involved a range of topics and concerns and was categorised in 13 categories, e.g. road tolls, public transportation, climate models, the use of mathematics, and ethical concerns (Figure 7). Their argumentation on road tolls concerned perspectives such as economic consequences, energy-sources and potential environmental benefits from electric vehicles, and the inclusion of full-lifecycles of products. Argumentation regarding the use of mathematics involved an awareness or competencies associated with identifying the role that mathematics can play in strengthening an argument. The students also reflected on the role of researchers in mathematical models (climate models) and made ethical argumentation involving social-justice concerns for future citizens. The students discussed a graph on CO_2-emission and critically reflected on inquiry-based dialogues, with features like inviting other peers to join the dialogue, acknowledging their perspectives, thinking aloud and challenging each other.

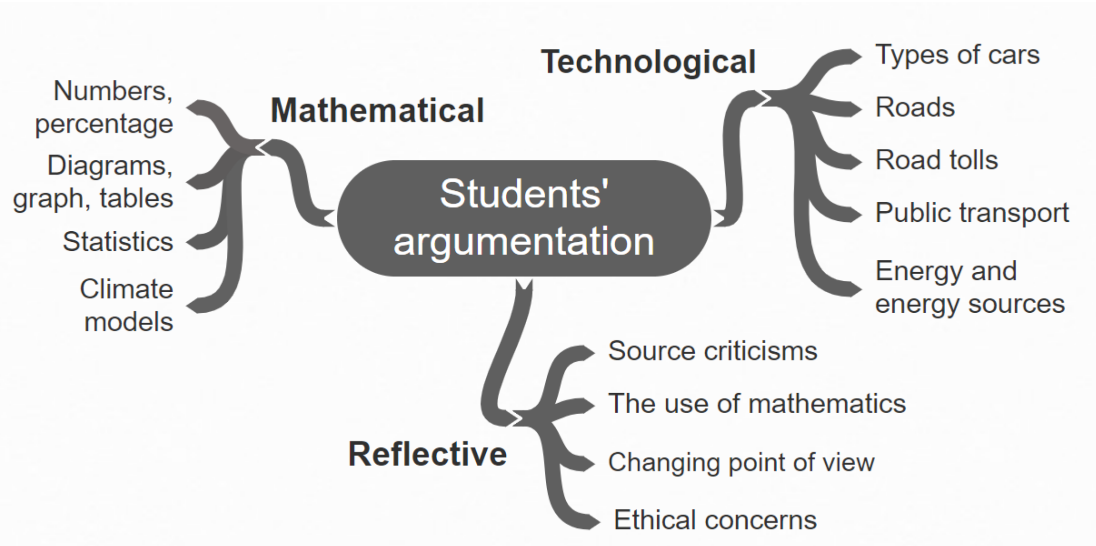

Figure 7 Students' utterances from dialogues and debates resulted in 13 categories and three main themes

4.2 Discussion of findings

In this discussion of findings, I take a meta-perspective on the findings from the PhD-study. In the process of synthesising the findings, I first identify what stood out as relevant from the three empirical papers (the literature overview in paper I is indirectly included since it influenced the focuses in the three other papers). Next, I thematically categorise these into themes identified from the data. This discussion is structured according to these themes: lived democracy and critical citizens; the mathematical formatting of climate change; critique and critical reflections; mathematical literacy and three kinds of knowing; controversies and values; and student-centred and dialogic learning. Some themes have a stronger emphasis on teachers' facilitation while others focus on students' critical mathematical competencies.

4.2.1 Lived democracy and critical citizens

An essential part of the teachers' facilitation of students' critical mathematical competencies was the aspects of critical citizens and lived democracy. An example was discussed in paper III when Max explicitly emphasised to the students that an active democracy requires students with critical mathematical competencies and implicitly connected democracy and climate change. She encouraged the students to reflect individually and collectively on challenges within climate change. By doing so, Max drew the students' attention to the relation between democracy, mathematics, and urgent societal issues. Another example was discussed in paper II, where Kim's explicit argued that students' critical mathematical competencies[12] are the most crucial thing students learn in school. He emphasised that students should know something about modelling to be critical and highlighted climate change as an excellent opportunity to enable students to think critically and make up their minds.

The ideas and thoughts from Max and Kim in the two examples go beyond the focus of merely calculating and mastering mathematical procedures and techniques. Their emphasis is more in line with ideas from CME, as described by Skovsmose (1994, 2008) and Ernest (2002, 2019). It can involve examples of mathematics teachers reclaiming schooling as an emancipatory project, as highlighted by Giroux (2011), where critical thought and agency are essential to students as critical citizens in a

[12] In one of the partnership meetings, we clarified relevant concepts, and the terms critical mathematical competencies and critical democratic competencies were used interchangeably.

lived democracy. A mathematics education does not necessarily explicitly embrace the ideas and thoughts on critical mathematical competencies and democracy, as expressed by Max and Kim. It can, therefore, be relevant to reflect on how this aspect should be included in the mathematics classroom. When including this topic in the classroom, the teachers engaged the students in matters of social justice, as highlighted by Chomsky (2003), Dewey (2011), and Lund and Carr (2008). For instance, the discussions in the classroom involved examples of how climate change can be socially unjust both for present and future citizens, or for rich and poor citizens, and students took standpoints in these issues.

When the teachers included climate change in their teaching, this involved authentic and controversial socio-political problems relevant for the participation of lived democracy (Atweh, 2012; Breivega et al., 2019; Gutstein, 2006; Hess, 2009). Their facilitation involved a variety of forms, such as investigation, exploring claims, discussions, debates, and listening to peers' argumentation and dialogues. The teachers used student-centred, dialogical and critical approaches. These approaches can be valuable in regard to students' competencies and inquiry habits of mind (Artigue & Blomhøj, 2013; Maaß, 2018). Students who become used to critical inquires in school can potentially continue with this also outside school, especially if the context is problems from the real-world. The teachers combined these approaches with an explicit emphasis on critical mathematical competencies and democracy, focusing on enabling students as critical citizens, rather than mastering basic mathematics competencies.

By inviting the students into discussions about complex, real-world problems, the teachers could both develop them as critical citizens and treat them as having already achieved this status. Treating them as citizens is in contrast to seeing the students as a part of citizens-in-the making (Biesta et al., 2009; Marshall, 1950), and recognising both these aspects is relevant when teachers facilitate students' critical mathematics competencies. As discussed in paper IV, their facilitation helped the students to critically interpret mathematics-based information, make critical judgments, and show critical awareness of mathematics. Such qualities were emphasised as crucial for empowering students to become critical citizens (Ernest, 2002).

Although the recent school strikes can be considered an example of how students can make critical judgments, and act as critical citizens, in this PhD-study, other forms of action were observed. In the classroom, the students' action took a variety of forms, such as their mathematics-based argumentation to peers, or in their individual critical reflections when discussing implications of a graph for society. An example of student's critical judgments was described in paper II[13] and concerned argumentation when examining a graph on CO_2-emission. A student was building up a moral claim and stated that she would act for bettering the future for generations to come. This example (and other related examples) started with the student interpreting mathematics-based information. She then continued by using this information to make critical judgments and ended up expressing her views to peers: that she would act. Enabling students to take action for making a difference in society is important for schooling democracy (Giroux, 1989). Based on observations in these four classrooms, along with the research literature presented, mathematics teachers can enable students as critical citizens in the classroom and for a lived democracy.

4.2.2 The mathematical formatting of climate change

The mathematical formatting of climate change, as discussed in paper III and described by Barwell (2013, 2018), was present in the research partnership as a focus for the discussions as well as an influential aspect of our talks. In this sub-chapter, I highlight five aspects of how the formatting power of mathematics was involved concerning teachers' facilitation of students' awareness for the formatting power of mathematics and students' identification and understanding of the formatting power. One aspect was about how the teachers explicitly expressed that they consider it vital for students as critical citizens to identify, understand and reflect on how mathematics can influence how we perceive climate change issues. It is important to become aware of how mathematics is intertwined in human activities (Greer, 2009; Skovsmose, 1994), which in this case concerns issues regarding climate change. Teachers' concerns in regard to time management – the large number of curricula aims, preparations for tests and exams, as expressed by Kim, Tim and Max – can make teachers choose to exclude a focus on students' awareness of the formatting powers of mathematics. However, in the partnership meetings, the teachers

[13] The utterance is also reproduced at page 64.

emphasised that they considered the formatting power of mathematics[14] to be an essential part of critical mathematical competencies. Without awareness of how mathematics can shape socio-political issues, processes such as critiquing or making choices as critical citizens can be challenging and impeded.

While the first aspect was about what the teachers said, the second aspect concerns what they did regarding facilitating students' awareness of the mathematical formatting of climate change. An example of such facilitation was when Max explicitly asked students to identify numbers (and the use of mathematics) in climate change and asked them to reflect on these (discussed in paper II). This explicit focus made the students actively look for numbers in climate change issues. Another example was when they combined mathematical-based information with different claims and asked the students to discuss them. In both cases, the teachers' facilitation was not "show and tell"; instead, they allowed the students to investigate, explore, and reflect themselves. The teachers provided a variety of contexts concerning how mathematics appeared in climate change. These contexts resulted in, for instance, two girls discussing different countries' renewable energy and making value-judgments (paper II), or three boys calculating their carbon footprint and discussing potential actions (Steffensen & Kacerja, accepted). These two first aspects show consistency between what teachers say they do and what they actually do.

A third aspect concerns how teachers' deliberate or undeliberate choices can influence students or others. An example of this was discussed in papers II and III and involved a quiz. In this quiz, the teachers' choice of topic, graphs, and any distractors could potentially format the understanding of quiz-participants and students. For instance, as discussed in paper III, one of the graphs included in the quiz was used in a testimony to the U.S. Senate to support the claim that climate models are not trustworthy, potentially impacting decision-makers and politicians. When including controversial topics in the classroom, teachers risk being accused of trying to influence or indoctrinate students (Hess, 2009). Teachers' choices and focuses can be more or less deliberate, but regardless if they are deliberate or not, teachers must reflect on these choices.

[14] The teachers did not explicitly used the concept "formatting power", but instead used phrases such as "understand how mathematics influence" or "identify how mathematics can be hidden".

A fourth concern involves how the teachers meta-reflected on the mathematical formatting of climate change in the partnership meetings. For instance, they reflected on how the mathematical formatting could influence politicians and decision-makers in climate change issues. Kim pointed out that researchers and experts act according to numbers and climate models, and in turn, politicians and decision-makers act according to experts. As described by Yasukawa et al. (2012), mathematical thinking affects how we understand and deal with our environment. It follows that it is relevant to have a critical awareness of the use of mathematics in society, as well as how mathematics is intertwined in many aspects of problems such as climate change. The teachers also challenged each other during the meetings. For instance, Kim challenged Max by asking whether she deliberately tried to influence the students and participants of the quiz by using mathematics. As discussed in paper III and Steffensen, Herheim, and Rangnes (2018), Max admitted that she deliberately excluded the Antarctic in the numbers she used to direct the focus towards the alarming ice melting of Artic. In the partnership meetings, the teachers had the opportunity to reflect, discuss and challenge each other on a variety of issues concerning both practical and more epistemic questions. The communication between them involved similar qualities to the ones described in the students' inquiry-based dialogues (Alrø & Johnsen-Høines, 2010, 2012). In controversial topics, where the formatting powers can also involve teachers' choices (as described in papers II and III), these opportunities to communicate were essential in the teachers' facilitation of students' (and their own) awareness of the mathematical formatting of climate change.

A fifth aspect concerns the students' identification and understanding of the formatting power of mathematics. As discussed in paper IV, one student underlined how politicians could use mathematics to strengthen a claim regarding electric vehicles. Another student reflected on the role of researchers in mathematical models (climate models). In these situations, it can be argued that the students show the competency needed to identify the role that mathematics can play in strengthening an argument or potentially influencing people's standpoint. Furthermore, by this they express an awareness of the mathematical formatting of climate change. These are crucial competencies for critical citizens in a lived democracy. However, it is relevant to emphasise that this awareness occurs in a context where the teachers facilitated the appearance of such views. Examples of situations in which such identification and awareness did not take place are also emphasised. For instance, as discussed in paper III, students took for granted the mathematical premise provided (in this case, the 2°C target). Although the teachers explicitly stated that the students should critically assess quantitative information, it was not documented that the students showed an awareness or were able to identify how mathematics potentially formatted their argumentation and discussion in this situation. With these examples in mind, facilitating students' awareness of the formatting powers of mathematics is not a straightforward task and cannot be taken lightly. If students do not have the competencies needed to identify, understand or critique how mathematics is an essential construct in climate change or other socio-political issues, they risk being misled and make their standpoints based on interest-driven argumentation that uses mathematics to influence deliberately.

4.2.3 Mathematical literacy and kinds of knowing

The teachers' emphasis on students' reflections and discussions about climate change issues generated students' critical engagement in highly relevant socio-political problems, which is underlined as relevant for mathemacy by Skovsmose (1994). A striking aspect of students' mathematical literacy observed in the research partnership was how this became an integrated part of how they dealt with this socio-political context. For instance, as discussed in paper IV, their argumentation on road tolls involved a range of perspectives: economic consequences, energy-sources, potential environmental benefits for electric vehicles, and the inclusion of full-lifecycles of products. They interpreted mathematics, reasoned, and formulated argumentation mathematically to reflect on real-world problems (OECD, 2018b).

The real-world context of climate change issues brought in perspectives different from the perspectives from pizza-cutting tasks (Simic-Muller et al., 2015) or tasks with reference to pure mathematics and semi-reality (Skovsmose, 2001, 2011). As discussed in paper IV, when four students started to interpret and make sense of a graph and a claim regarding the CO_2-emission of a car, they started by identifying typical mathematical characteristics of the graph (e.g. quadrant, parabola, linear equations). Apparently, they neglected the real-world aspects, an approach sometimes used when students are dealing with real-world problems (Busse, 2005; Christiansen, 2001; Maaß, 2006). However, they became stuck and frustrated. Then a student named Pete brought a new perspective based on the real-world aspects and his knowledge about cars and engines, although he seemingly neglected the mathematical aspects (Busse, 2005; Christiansen, 2001; Maaß, 2006). Pete's argument became the turning point in the students' discussions. When they continued, their utterances involved a more intertwined mathematical, technological, and reflective argumentation, arguably more in line with Skovsmose's (1992, 1994) three forms of knowing and a transdisciplinary approach described by English (2016b). As mathematics teachers, it is sometimes necessary to narrow the students' focus so that is concerns a specific aspect of mathematics, e.g. to learn about characteristics of graphs. However, if these (narrow) aspects are all the students learn, and they do not receive the opportunity to reflect on how these graphs relate to reality, then students might miss out on opportunities to recognise the relevance of mathematics in real-life. They can also miss out the social aspects of mathematical literacy, as described by Jablonka (2003, 2015), for instance, by not seeing climate change issues as transformable and reading the world as open for changes (Skovsmose, 1994). By facilitating discussions on socio-political problems, the students can develop their mathematical literacy to make well-founded judgments. Mathematical practices can empower students as critical citizens (Freire & Macedo, 2005).

In the research partnership, students' mathematical literacy involved critical thinking, communication, collaboration, problem-solving, ethical concerns, and global and environmental literacy. These are relevant 21st-century skills (Care et al., 2018; Maaß et al., 2019; OECD, 2018b). In times when different parts of the world are growing closer due to, e.g. the Internet and problems like climate change that affect the whole world, it is important to think globally. In this research partnership, the teachers had both local and global perspectives. The OECD (2018a, 2018c) highlights competencies to examine local and global issues and understand and act for sustainable development. Local perspectives in the research partnership involved, for instance, an excursion to a near-by location, students investigating how climate change affected their local communities in the posters, and students communicating in dialogues and debates about climate change and a road-toll on a local bridge. Global perspectives involved exploring data such as renewable energy worldwide, global temperatures, CO_2-emission, and climate modelling. However, to think globally is to act locally. A lived democracy has no borders, and to be able to see the world as interconnected is relevant for critical citizens when socio-political issues are included in the classroom.

4.2.4 Critique and critical reflections

Different forms of critique and critical reflections were observed in the research partnership. They involved and ranged from critical reflections on the mathematics, e.g. graphs, to discussing the potential implications of graphs. Here I highlight how the teachers facilitated different forms of critical reflections and what type of reflections the students made. The teachers facilitated critique and reflections as a logical endeavour, critiquing argumentation, as highlighted by Skovsmose and Greer (2012a). One example is when the teachers encouraged the students to examine critically arguments raised in the dialogue game. The teachers also facilitated critical reflections as a political endeavour by critiquing real-world activities of climate change. An example of the latter is discussed in paper III, where the students' critical reflections result in argumentation involving questioning continued Norwegian oil exploration when taking into consideration the 2°C-target. The public debate in Norway about whether to proceed with oil exploration is highly relevant. It involves a need to analyse and evaluate related claims and argumentation, as well as reaching a judgment about “what is good”. Ethical considerations are an essential part of the critique, as underlined by Atweh (2012). Ethical aspects were also discussed and related to critical judgments in paper II, and both the teachers and students’ value-perspectives came into play. When discussing climate change, whether it is in the mathematics classroom or in real-life, ethical considerations are often embedded, and these reflections are essential parts of critique.

In one of the partnership meetings, Max expressed that in addition to students’ critical reflections, she wanted the students to act and make conscious choices about the consumption of meat, clothes and so forth (discussed in paper II). Although the teachers did not directly promote actions (e.g. engage in strikes or projects involving eating less meat), it could be argued that their facilitation of students’ critical reflections in dialogues, discussions, and debates involved an action element[15]. The students' argumentation often involved standpoints about climate change issues and

[15] In Steffensen and Kacerja (accepted), the students’ critical agency was related to actions on eating less meat during their discussions in the poster-lesson, on the students own initiative.

was observed in their dialogues and when they analysed graphs, numbers and claims. Skovsmose and Greer (2012a) emphasise that a critical agency can involve actions, both regarding the reflective aspects, but also the agency to make changes. In paper II and paper IV, I discuss students' critical agency concerning environmental awareness. One example concerns a student who argued for driving a bit slower because of the environment and concerns for future citizens. Another example involves argumentation about why it is vital to take care of our atmosphere and our earth. These type of arguments involve ethical concerns, with a normative aspect and an indication that action is needed. These aspects are relevant parts of critical reflection (Atweh, 2012; Brookfield, 2009; Gardiner, 2008, 2011; Wardekker et al., 2009). The teachers' facilitation of critical reflections in, for instance, the dialogues, provided multiple opportunities for the students to include such aspects. Hence, facilitating students' dialogues, discussions, and debates, through controversial and real-world-problems, is a form of enabling students' actions and critical agency.

The teachers expressed that the students needed to learn to reflect critically on the scientific processes and gain some insights into how researchers construct climate models. As discussed in paper III, the teachers chose to include practical, hands-on experiences including parameters and units such as CO_2, CO_2e, O_2. These parameters are relevant in climate change, but perhaps not so familiar to students. They expressed concerns about students' awareness of the uncertainties of real-world measures and sources of error. They posed questions to the students aiming to generate critical reflections on their findings regarding the excursion location (physical measures), any associated uncertainties, their use of these numbers in the field report (mathematical computation), and how they communicated about these findings. Uncertainties are an essential part of the scientific processes and relevant to include when discussing climate change. Uncertainty in measuring-results or climate models does not necessarily need to imply that the result or models are not trustworthy and should be rejected. Sometimes they need adjustment, or sometimes one must just learn to live with uncertainty. As discussed in paper IV, if scientific uncertainties are wrongfully used to deliberately hinder action towards climate change, this can become a problem in a lived democracy. Therefore, teachers can enable students to learn about different types of uncertainties (Hauge & Barwell, 2017; Steffensen & Hansen, 2019) to develop as critical citizens.

Skovsmose's (1992) six steps of reflections helps identify the different levels of the students' critical reflections. As discussed in paper III, when the students measured the seawater temperature with both digital and analogue methods and calculated the average, they had to consider whether they used the algorithm correctly and if the appropriate algorithm was applied. These types of critical reflections are in line with Skovsmose's (1992) two first steps that focus on "pure" mathematics. However, in addition to focusing on merely asking the students to calculate the average, the teachers asked questions generating critical reflections concerning the real-world measurements, the mathematical method, uncertainties, as well as relating their measurements to how the global average temperature of the Earth is measured. These latter questions contributed to an emphasis on critical reflections as not "purely" mathematical but related to the connections between mathematics and the real-world. Critical reflections on such problems guided the students' attention towards the other reflection steps of Skovsmose (1992). An example of a situation generating critical reflections within the fourth reflection step – "could we do without formal calculations" – was discussed in paper IV. Pete disagrees with a claim without doing any formal mathematical calculation or evaluation. He just uses common sense. Another aspect of Pete's disagreement with the claim is that it can be considered a critique of the presuppositions on which the beliefs are built. Such a critique is underlined by Mezirow (1990) as necessary for a reflection to be critical. When Pete disagrees, he evaluates and questions the claim. He does not take the explanation for granted and displays independent thinking. These qualities are emphasised as crucial for being critical by Ernest (2002).

4.2.5 Values and controversies

Controversial topics and contexts involving students' values can bring about different challenges compared to those typically seen in the mathematics classroom. For instance, rather than making or interpreting graphs, the students discuss the implications of graphs. The controversies and the value-aspect of climate change can lead teachers to avoid including the topic (Hess, 2009; Hess & McAvoy, 2014; Simic-Muller et al., 2015; Steffensen & Hansen, 2019) or stick to conveying factual, scientific information (Hicks, 2014; Monroe et al., 2017). Contrary to this view, the teachers emphasised the controversies in climate change. Examples of such emphasis were given by Kim, who explicitly expressed in a research partnership meeting that "I emphasise all the controversies" (paper II, p.10). This emphasis was also observed in the lessons, e.g. in the design of the poster and the quiz. Kim used a cartoon, graphs and numbers to deliberately create discussions, debates, and reflections (paper II). The inclusion and emphasis of controversies in the mathematics classroom are in line with the views from Gutstein (2006), Hess (2009), and Atweh (2012). All of them emphasise that education should include controversies. Some of the reasons for why Kim chose to emphasise controversies was because he wanted the students to develop their own opinions, learn to think, become critical, and gain critical mathematical competencies (as described in paper II).

When including controversial socio-political topics such as climate change in the mathematics classroom, teachers' values and political views come into play to a greater extent than if the topic, for instance, is teaching the algorithm of subtraction. The controversies in climate change made the teachers choose differently. While Kim emphasised the relatively small temperature change and chose the graph accordingly, Max underlined the big ice-melting in the Arctic and used a number that stood out from the other alternatives. Paper II discusses these choices and how the teachers used numbers and graphs to highlight particular features of climate change, and it is concluded that controversies and values could influence teachers' facilitation of CME. The examples illustrate how teachers can become moral agents and value educators, as described by Seah (2016, p. 2), and how controversies and values can enter the mathematics classroom. Teachers also risk being criticised by parents, the local community, and the school administration when dealing with controversial issues, if parents think that the students are somehow being influenced. Teachers may also consider it a challenge to be objective and neutral (Steffensen & Hansen, 2019; Steffensen & Rangnes, 2019). When the context is climate change, the aspect of neutrality becomes more apparent in the classroom. However, in all teaching, whether it is the teaching of the algorithm of subtraction, or the choice to include or avoid climate change, underlying values are present. For instance, the choice of not including important socio-political issues can imply that mathematics teachers do not consider mathematics education to be a socially responsible discipline, as suggested by Ernest (2009), Atweh and Brady (2009) and Abtahi et al. (2017). Therefore, it is vital to make a conscious choice of what to include in the classroom, and have an awareness of how values may influence the teaching of mathematics.

The controversies and values also came in to play among students. Conflicting perspectives were present, as well as perspectives of righteousness, justice and moral duty to prevent climate change, as described by Johansen et al. (2000) and Fløttum et al. (2016). The students were given opportunities to make arguments and discuss ethical actions on climate change issues, in line with the problem-posing approach suggested by Freire (1998, 2007). They were considered as conscious, thinking beings, and not as empty containers. Their discussions concerned how best to deal with the impacts of climate change and actions on preventing further climate change; no examples of controversies concerning whether anthropogenic climate change exists (Cook et al., 2013; Oreskes, 2018) were observed. As discussed in papers II and IV, the students' argumentation involved numbers and graphs as a basis for building up moral claims, value-judgements, and ethical and social-justice concerns for future citizens. This type of argumentation differs slightly from those based on, e.g. economics or science. When students' argumentation included ethical aspects in addition to mathematical or scientific argumentation, this is considered a move towards a more sophisticated type of argumentation. The complexity of climate change calls for a multiplicity of concerns. The controversies and values in climate change served as a foundation for the students' argumentation and were essential in the students' critical mathematical competencies.

4.2.6 A student-centred and dialogic learning environment

In this research partnership, there are very few examples of traditional mathematics classrooms as described by Vavik et al. (2010), Bergem et al. (2016), and Echazarra et al. (2016), where teachers instruct students with little student interaction. In this sub-chapter, I highlight six aspects that characterise the learning environment in this research partnership: student-centred approach, type of understanding, choice of mathematic tasks, knowledge conflicts, the content of dialogues, and qualities of dialogues.

The first aspect addressed is the student-centred approach. In the initial phase of the research, Max expressed that she wanted to expand her ways of teaching mathematics. She regarded herself as a traditional teacher who followed the mathematics textbooks and allowed students work with tasks. This approach can be associated with the task discourse, exercise paradigm, and teacher-centred transmission-based teaching (Maaß, 2018; Mellin-Olsen, 1991; Skovsmose, 2001, 2011). However, during the 42 lessons, none of the teachers used textbooks. They said that the textbook did not have relevant tasks, and they wanted to point out relevant and current issues. All the lessons concerned real-world situations through the focus on climate change. They were organised so that students interacted, collaborated, and participated in dialogue games, group discussions, and panel-debates, and the teachers emphasised students' argumentation and ways of thinking. The students formulated questions, explored explanations, and carried out hands-on experiments. Their references came from their real-world, and their answers did not only involve one correct answer. These characteristics relate to Skovsmose's (2001, 2011) landscapes of investigation with real-life references, what Maaß and Dorier (2012), Artigue and Blomhøj (2013), Maaß et al. (2017), and Maaß (2018) refer to as inquiry-based learning.

Another aspect involves types of understanding. As discussed in paper III, the students not only calculated the average-temperature mathematically (what to do), they also considered chosen methods, as well as discussed the different measures and how to decide on which numbers to use (to know why). This focus can be regarded as a relational understanding described by Skemp (2006) as an way to know what to do and why. Also, they discussed their results and related mathematical knowledge to situations outside the classroom, as highlighted by Mellin-Olsen (1981). During these discussions, a student asked why they needed to discuss the answer since it was already there. If students are used to thinking that mathematical tasks are completed when an answer is provided, they might miss out on essential features of mathematical competencies such as evaluating their answer from a real-world perspective. In this research partnership, the teachers emphasised that the students should become aware of how mathematics does not always provide a single answer. Rather, it involves human choices regarding aspects such as the chosen location of measurements and scientific and mathematical methods. In these learning environments, collaboration, communication, and the inclusion of mathematics related to real-world issues outside the classroom are crucial.

A third aspect concerns the choice of mathematics tasks. The tasks concerned real-world issues (Jurdak, 2016a, 2016b); they were wicked problems (Levin et al., 2012; Rittel & Webber, 1973); and they involved controversies as described by Hess (2009). As discussed in papers II and IV, the teachers designed the tasks themselves. They posed questions that promoted critical reflections about the field-report, and they developed a dialogue game that also promoted the students' critical reflections. They designed a task where the students should create a poster that required them to discuss numbers and climate change, and they tweaked and adjusted a mathematics exam task so that students should discuss aspects of the graph rather than just solving it. None of the tasks had one correct answer. Choosing mathematics tasks is one of several choices teachers make that impacts how students learn. The tasks in this research partnership are inquiry-based and student-centred, two qualities underlined as important by Maaß and Dorier (2012), Artigue and Blomhøj (2013), Maaß et al. (2017), and Maaß (2018). Such tasks can contribute to a more relational learning and understanding of mathematics.

A fourth aspect involves how the teachers deliberately search for knowledge conflicts. As discussed in paper III, during the partnerships meetings, Kim expressed an interest in the uncertainties of climate change measurements. He questioned the location and number of measurement stations, and what techniques are most relevant to use, pointing out that no absolute method exists. By organising students to measure with different instruments and methods, the teachers deliberately emphasized the competing knowledge-claims on human choices of such as statistical methods. Knowledge conflicts were also observed in the claims mad in the dialogue game, where the students discussed these. Knowledge conflicts are highlighted by Hauge and Barwell (2017) and Skovsmose (1994) as relevant for CME. Students' reflection, critique, and negotiation of competing knowledge-claims are easily recognisable in the public debate on climate change issues (discussed in paper III).

The fifth aspect concerns the content of the dialogues. All seven themes in the research partnership involved dialogues, although the context varied. A crucial aspect of these dialogues was that the teachers gave priority to dialogues involving not just mathematical concerns but also the reflective knowing of a graph's potential implications. Vithal (2003) highlights that conflicts in reflective knowing are often given a lower status in the mathematics classroom. It is not given that teachers include dialogues concerning reflective knowing in the classroom. Teachers might be stressed due to an extensive curriculum or time management and therefore choose tasks that prepare students for the exam rather than a life as critical citizens. Reflective knowing dialogues can provide opportunities for student-centred learning in ways that dialogues involving pure mathematics perhaps may not. In the dialogues, Kim, Tim, and Max orchestrated; instead of being considered the expert and the one with all the correct answers, they became more a support who raised questions and helped students make connections between their thoughts and ideas, as underlined by Maaß and Dorier (2012).

The sixth aspect involves qualities in the students' dialogues. As discussed in paper IV, certain qualities characterised the students' dialogues, for instance, inviting peers to join the dialogue, asking open questions, thinking aloud, collective wondering, making hypotheses, listening to peers' argumentation and questions, acknowledging peers' perspectives, making inquiries without posing questions, and challenging each other's perspectives. The students appeared to want to understand more. Such communication was described as inquiry-based dialogue by Alrø and Johnsen-Høines (2010, 2012). An example of these qualities involved how students acknowledged other perspectives and sometimes adjusted their own as well, even when controversial issues were involved. For instance, in the dialogue, the teachers made a point that the students should listen, without interrupting, to peers' argumentation. They could change or adjust their standpoint as a result of other group members' argumentation. In paper IV, an example was discussed where one student explicitly expressed that she adjusted her standpoint because of her peers' argumentation. To facilitate so that students respect genuine views and differences and acknowledge peers' standpoints is vital (Gutstein, 2006; Hess, 2009; Hess & McAvoy, 2014). Acknowledging peers' standpoints is perhaps even more relevant when controversies are present, as was the case in these dialogues. Another example concerns the quality of the dialogues during the 42 lessons. Not one example of IRF-dialogues (Sinclair & Coulthard, 1975) was observed. Instead, those student dialogues that were involved justified stances based on mathematics-based argumentation, emphasised as relevant in inquiry-based dialogues by Herheim and Rangnes (2016).

4.3 Limitations of chosen boundaries

Some of the limitations and delimitations of this PhD-study were discussed in the four papers and in the methodology chapter of this thesis. These involved dropouts from the sample, non-representative samples, researcher bias (e.g. partly one coder), chosen boundaries concerning the research focus, context, sampling, exclusion-criteria, and methodological and theoretical framework. Here I emphasise the limitations and delimitations of the chosen boundaries, and how this could potentially have an impact on the findings.

One limitation concerns the choice of methods. In this study, I chose a qualitative approach to research how teachers can facilitate students' critical competencies. I could also have chosen a quantitative approach or a mixed-methods approach. This choice led to something being excluded. For instance, a quantitative approach could have provided an overview of more and other ways that teachers can deal with climate change in their mathematics classroom. However, by choosing a qualitative approach, I saw opportunities to provide a more in-depth approach to understand the complexity of how teachers can facilitate students' critical mathematical competencies.

Another limitation concerns the strategical sampling of the teachers. They were chosen for several reasons: I know them (we used to work together), they are experienced natural science and mathematics teachers, and they expressed an interest in both climate change and CME. The focus of the PhD-study was to explore the potential of how teachers can facilitate students' critical mathematical competencies in the context of climate change. Before starting the research partnership, I did not regard the teachers as representatives for particular views such as the teacher-centred or student-centred approaches, climate change activism or climate change denial. Two of the teachers who declined to participate in the research partnership did so because they thought they did not have sufficient content knowledge in either mathematics or natural science (amongst other reasons). On the one hand, these two teachers could have contributed to bringing up other perspectives than those mentioned by Kim, Tim, and Max. On the other hand, the educational background of the three teachers who did take part made them particularly relevant as participants in this research, as scientific relevance is so present.

A third limitation concerns the selected sample of students. The number of 120 students can seemingly represent a variety of standpoints and different backgrounds. However, all of them live in the same local community in Norway. They can have some shared values compared to students living in different parts of the world, and the findings in this research may, therefore, have limited transfer value to students in other countries. For instance, in Norway, a relatively large number of students say they like going to school (Wendelborg, Røe, Buland, & Hygen, 2019). They live in a democratic society where students are free to disagree with teachers and where teachers encourage students to take a stand and express their opinions (Huang & Biseth, 2016). They also have some shared perspectives on climate change (Fløttum et al., 2016). When these students discussed climate change issues such as CO_2-emission and oil production, it was in a Norwegian context. As emphasised by Corner et al. (2014) and Nurse and Grant (2019), values can act as a filter, and political views could influence how mathematical problems are solved. The students' utterances, as well as my reflections, should therefore be viewed in this (Norwegian) context and may represent a limitation of the findings.

4.4 Implications and future research

The findings of this PhD-study complements existing research in the socio-political turn in mathematics education research (Gutiérrez, 2013). It adds to the research on climate change and CME conducted by researchers like Barwell and Suurtamm (2011), Barwell (2013, 2018), Coles et al. (2013), and Hauge and Barwell (2017). In this sub-chapter, I discuss seven potential implications of this study that are relevant to the mathematics classroom and future research.

One area concerns *the challenging aspect of teaching critical mathematics*. In the startup of the research partnership, it was a concern for both teachers and the school administrator that teaching “outside the curricula” on topics not directly connected to the students’ final exam would be a challenge. Related challenges were also emphasised by researchers such as Felton-Koestler (2017) and Bartell (2013). For instance, they describe that teachers may have limited experiences with socio-political issues in the mathematics classroom and when investigating real-world issues. It is not a matter of method, they argue, but rather a process where teachers adapt to the context they find themselves in, and critical mathematics and social justice perspectives are often met with resistance. Gutstein (2018) describes his journey as a CME-teacher, where he “slowly and painstakingly learned to teach critical mathematics”, and was “challenged when supporting others to teach critical mathematics” (p. 134). These challenges with teaching CME, along with the limited empirical research on climate change in the mathematics classroom, suggest that insights into the processes of how to include CME in the classroom are relevant and valuable for mathematics researchers and educators, pre-and in-service teachers, teachers, and curriculum-designer and policymakers. Future research can address challenges located within the systems of the schools (e.g. structural and practical limitations) or within the teachers (e.g. lack of experience in teaching socio-political issues).

A second area is *the collaboration in the research partnership*. The partnership meetings brought the teachers and the researcher together and provided a space to collaborate and share ideas, discuss and reflect, as well as practical plan. Bjuland and Jaworski (2009) describe a research project within the field of mathematics education that concerns how teachers collaborate in learning communities. They explore how learning communities between mathematics teachers and mathematics teacher educators involved learning for all. This research was based on a *community of practice* approach (Lave & Wenger, 1991; Wenger, 1998). Relatedly, Chauraya and Brodie (2017) describe how professional learning communities, as a form of teacher development, can support changes for teachers' teaching. Kacerja and Herheim (2019) emphasise critical colleagueship based on Lord (1994), with an emphasis on a critical stance and seeing differences as a driving force. Teachers who engage in critical perspectives in the mathematics classroom may need different support than those who work on a pure calculus-related project. When wicked problems are included in the mathematics classroom, teachers can benefit from an arena where they can reflect on the complexity, uncertainty, and value perspectives. As discussed in paper II, the teachers used the partnership meetings to discuss ideas, dwell on the controversies, and challenge each other's perspectives. Such collaboration for teachers can be particularly relevant when controversial issues are involved, as underlined by Hess and McAvoy (2009; 2014). Within schools, such collaborative partnerships can be initiated by teachers or school administrators, and researchers could provide a collaborative learning environment for teachers involved in the research.

A third area concerns *the controversial aspect of socio-political problems*. Teachers might identify with how climate change controversies challenge Kim and Max in their use of numbers and graphs. Teachers who include wicked problems in their teaching can sometimes consider this a normal scientific problem, where it is enough to obtain adequate knowledge (Hicks, 2014). The last few decades of public confusion concerning how to deal with climate change have shown us that this is not the case. Teachers and researchers might, therefore, have to reconsider how they teach climate change in the classroom. Kim, Tim, and Max explored different ways of how to facilitate students' critical mathematical competencies in the context of a complex and controversial issue. Their choices and different approaches can provide thoughts and ideas for others who consider including controversial topics such as climate change in their mathematics classroom, either as teachers or as researchers. This PhD-thesis can provide insight into what types of problems students can learn to deal with, what role students can take as critical citizens in a lived democracy, aspects of competencies relevant when dealing with controversial and complex problems, and why and how critical citizens and lived democracy can be included in mathematics education.

A fourth area *involves students' attributes*. In this PhD-study, the focus was not on the attributes of the students, such as gender, abilities, motivations, or their religious, cultural, and socio-economic backgrounds. However, these concerns might be relevant to include in further research on climate change and education. The United Nations (2019) called for an awareness of how genders are affected differently by climate change, and that genders understand climate change differently. They identified a correlation between females in power and ratification of environmental treaties, and that females and males' attitudes and concerns differ. Similarly, McCright (2010) found that women are more informed and concerned about climate change than their male counterparts. In my data, I found that climate change concerns brought up by females differed from male students, and that male students' dominated the classroom discussion (Steffensen, accepted; Steffensen & Hauge, submitted). If students' critical mathematical competencies are facilitated through dialogues and discussions, as described in this PhD-study, what implications does it have if females participate to a lesser extent in these dialogues? Does it imply that male concerns are more likely to be heard? Does it imply that female voices are not heard in a lived democracy? Or perhaps they are just being expressed in different ways? Teachers and researchers addressing climate change in the classroom can consider how students' different attributes might influence how they deal with such problems.

A fifth area involves *external factors*. In this study, I did not focus on external factors such as the economy, ICT, numbers of students in the classroom, curricula and competence objectives, and evaluations (tests and exam). Although I have provided some background information about the curricula and competencies objectives, more research is needed on how curricula and evaluations in the form of grading and written and oral exams may impact teachers' facilitation of students' critical mathematical competencies. Also, the teachers' background, autonomy, expectations and hopes are relevant to further research. For instance, to what extent do mathematics teachers need to be scientifically knowledgeable to include topics like climate change, and will this influence how they incorporate this context?

A sixth area concerns *transdisciplinary teaching*. In the real-world, problems are seldom subject-related because they are transdisciplinary by nature. Students who deal with problems such as critical citizenship in a lived democracy can benefit from transdisciplinary approaches in school, and not only through a subject-oriented classroom-approach. Initially, the teachers in the research partnership discussed the latter approach but ended up with a transdisciplinary approach by including competencies aims from natural science and mathematics (described in paper IV). Researchers such as Radakovic et al. (2018) emphasise that a transdisciplinary critical mathematics perspective is required for topics like sustainability and environmental education. Moreover, as described in papers I and IV, a transdisciplinary approach on climate change involves competencies from several fields (English, 2016b), a reflexive and meta-level thinking (Spangenberg, 2011), as well as unchaining concepts from the sometimes rigid framework of a subject (Colucci-Gray, Perazzone, Dodman, & Camino, 2013).

In the research partnership, the teachers knew I had a mathematics perspective, and they emphasised mathematics when planning the lessons. However, it is relevant to explore what role mathematics can or should have in transdisciplinary teaching regarding problems like climate change. For instance, should climate change be used as a context to learn how to interpret graphs? Or should mathematics be used to learn climate change? In addition, to conduct further research on the role of mathematics, one could also include questions on what subjects to include. For instance, the close affinity between mathematics and natural science can make these two subjects particularly favourable to combine. In this research partnership, Kim, Tim, and Max were natural science teachers as well as mathematics teachers. However, when engaging in transdisciplinary topics, different subjects can impact on how topics are dealt with in the classroom. For instance, religion was excluded from this study. However, as emphasised by Wardekker et al. (2009), in the US, the public debate on climate change is strongly influenced by religion. It is also relevant to research how transdisciplinary topics can be avoided by teachers because they do not regard themselves as competent enough. In this PhD-study, two of the teachers who choose not to participate said that they did not have the formal qualification (either as mathematics teachers or natural science teachers). An opportune question is thus the following: Do mathematics teachers need to have a scientific background to include topics such as climate change? Jorgensen and Larkin (2018) emphasise that teaching out of the field was perceived as a challenge, and that teacher confidence and content knowledge across the STEM spectrum relates to students' learning. Therefore, when including climate change in the mathematics classroom, future research should include how a transdisciplinary approach can explore the role of mathematics (and other subjects) and teachers' educational background.

The last area involves *action and hope*. During the research partnership, actions were explicitly addressed by the teachers, for instance when Max expressed that she wanted the students to act, or when they encouraged the students to take justified stands on climate change issues. In actions, there lies an element of hope. Ryan and Steffensen (accepted) discuss how these two concepts of action and hope are important when complex problems are included in CME. In the current public debate on climate change, some voices argue for protecting young people from the severeness of climate change (Ambrose, 2004). As described in Abtahi et al. (2017), Steffensen and Hansen (2019), and Steffensen and Rangnes (2019), teachers can have conflicting beliefs about bringing students' attention to serious challenges like climate change. However, students are already aware of the potential impacts of climate change, and schools need to address this. Otherwise, this is left to the students to deal with alone. Researchers such as Freire (1992) and Wals (2010) argued that when education includes problems that might be overwhelming to students, this can lead to hopelessness and paralysis. It is therefore important that education also includes a pedagogy of hope.

Future research on climate change in the mathematics classroom should address how to deal with the aspect of action and hope. Nairn (2019) describes how young people relate to hope and despair, in the prospect of a climate-altered future, and how they deal with "the end of the world" discourse to avoid a situation where they start with hope but end up disillusioned. Hicks (2014) asks how one could educate for hope in troubled times towards a post-carbon future. Stevenson and Peterson (2016) highlight that little research exists on whether hope could counteract despair. They did find that hopes and concerns regarding climate change positively related to behaviour, while climate change despair was negatively related to behaviour.

4.5 Concluding comments

The research question *how can focusing on climate change can facilitate students' critical mathematical competencies* has been answered by conducting empirical research in the classroom. In the discussion, I have emphasised six aspects relevant to the facilitation of students' critical mathematics competencies: lived democracy and critical citizens; the mathematical formatting of climate change; mathematical literacy and kinds of knowing; critique and critical reflections; values and controversies; and a student-centred and dialogic learning. These six aspects are neither exhaustive nor exclusive.

When I started on this PhD-study, I was of the opinion that mathematics education needs to engage students in dealing with complex socio-political problems by developing their critical mathematical competencies. That way students can become critical citizens in a lived democracy. This PhD-study provides valuable insight into how focusing on climate change in the mathematics classroom can contribute to enabling students as critical citizens. Wicked problems are challenging to deal with. Based on the findings, my views were strengthened, and I do consider it relevant to include socio-politic topics such as climate change in mathematics education. Moreover, facilitation of students' critical mathematics competencies should include controversies, values and uncertainty, not avoid them.

Observing the students and teachers' dedication for more than a year has left me with some deep impressions. Seeing the students' engagement, their variety of competencies, their willingness to explore and communicate in dialogues and discussions, and their consideration of others peer standpoints has been a joy. I will not forget how one student during the first lesson suddenly switched from showing no interest at all, to be actively engaged in argumentation on why electric vehicles are beneficial for reducing climate change. I will remember the students who became so passionate when studying different countries percentages of renewable energy, the student who was standing at the seashore making scientific and mathematical reflections about global measurements on sea-water temperature with seawater flooding around his legs, and lastly, the girl who straightforwardly said it was about time I arrived. So many of the students have made a permanent impression on me.

These observations have given me hope that students can deal with socio-political problems, even wicked ones, in the classroom in meaningful ways where learning takes place. Also, how the teachers worked as facilitators and their variety of approaches, and the enthusiasm they showed together with their students, provides a hope that mathematics education can mean something for the students' lives. This is something more than just passing the exam, something that enables them as critical citizens in a lived democracy.

5 Literature

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