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ORIGINAL PAPER

Weighted superposition operators on Fock spaces

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Abstract
We characterize all pairs of entire functions (u, ψ) for which the induced weighted superposition operator $S_{(u,\psi)}$ transforms one Fock space into another Fock space. Further analytical structures like boundedness and Lipschitz continuity of $S_{(u,\psi)}$ are described. We, in particular, show the Fock spaces support no compact weighted superposition operator.

Keywords Fock space · Bounded · Compact · Weighted superposition · Lipschitz continuous · Type and order · Zero set

Mathematics Subject Classification Primary: 47B32, 30H20; Secondary: 46E22, 46E20, 47B33

1 Introduction

The theory of superposition operator has a long history in the context of real valued functions [2]. In contrast, there have been only some studies on spaces of analytic functions which includes Hardy spaces [6], Bergman spaces [7], Dirichlet type spaces [4], Bloch type spaces [1, 5], and some weighted Banach spaces over the disc [3]. The goal of this note is to study the operator on the Fock spaces $\mathcal{F}_p$. We recall that $\mathcal{F}_p$ is the space of entire functions f for which

$$\|f\|_p = \begin{cases} \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z)\right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty, & p = \infty, \end{cases}$$

where dA is the usual Lebesgue area measure on the complex plane $\mathbb{C}$. The space $\mathcal{F}_2$ is a reproducing kernel Hilbert space with kernel function $K_w(z) = e^{\overline{w}z}$. For each $w \in \mathbb{C}$, a calculation shows the function $k_w = \|K_w\|_2^{-1} K_w \in \mathcal{F}_p$ and $\|k_w\|_p = 1$ for all p. By [10,p.

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37], for each entire function f and $p \neq \infty$

$$|f(z)| \leq e^{\frac{|z|^2}{2}} \Big(\int_{D(z,1)} |f(w)|^p e^{-\frac{p|w|^2}{2}} dA(w) \Big)^{1/p} \leq \Big(\frac{2\pi}{p} \Big)^{\frac{1}{p}} e^{\frac{|z|^2}{2}} \|f\|_p, \tag{1.1}$$

where $D(z, 1)$ is the disc with center z and radius 1. The same conclusion with the factor $\big(2\pi/p\big)^{1/p}$ replaced by 1 holds whenever $p = \infty$.

In this note, we study the weighted superposition operator on Fock spaces where the operator S_ψ is covered as a particular case. Let (u, ψ) be a pair of holomorphic functions on $\mathbb{C}$. For two metric spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the weighted superposition operator $S_{(u,\psi)} : \mathcal{H}_1 \to \mathcal{H}_2$ is defined by $S_{(u,\psi)} f = M_u S_\psi(f)$ where $S_\psi f = \psi(f)$ and $M_u f = uf$ are respectively the superposition and multiplication operators. Recently, $S_{(u,\psi)}$ was studied on the Bergman and Bloch spaces [9].

The first main question now is to identify which pairs of analytic symbols (u, ψ) define weighted superposition operators $S_{(u,\psi)}$ from $\mathcal{F}_p$ into $\mathcal{F}_q$. The questions for $S_{(u,\psi)}$ are in general technically more difficult than the corresponding questions for S_ψ since the presence of the multiplier u can complicate proofs and arguments. We may first begin with a simple example that illustrates the problem at hand. Let $f = \alpha$ be a constant and $g(z) = z$. Then, for $0 < q \leq \infty$ and a non-zero ψ,

$$\|S_{(u,\psi)} f\|_q = \|\psi(\alpha)u\|_q = |\psi(\alpha)|\|u\|_q \text{ and } \|S_{(u,\psi)} g\|_q = \|u\psi\|_q.$$

This shows that if $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$, then both u and $u\psi$ belong to $\mathcal{F}_q$. On the other hand, set for example $q = 2$ and consider the function $h(z) = \sin(\frac{z^2}{2})/z^2$. Then h is an entire function which belongs to $\mathcal{F}_2$. To see this, observe that when $|z| = r$ gets larger, then $|h(z)|^2 \simeq e^{r^2}/r^4$ and

$$\int_0^{2\pi} \int_0^\infty |h(re^{it})|^2 r e^{-r^2} dt dr < \infty.$$

However, the function zh is not in $\mathcal{F}_2$ since $|zh(z)|^2 r e^{-r^2} \simeq 1/r$ for larger $|z|$. It follows that $S_{(z,z)}$ fails to map $\mathcal{F}_2$ into itself while it is easy to see that S_z does. This exhibits the existence of some degree of interplay between u and ψ in defining $S_{(u,\psi)}$ on Fock spaces. Our next main result provides their precise interplay.

Theorem 1.1 *Let ψ and u be nonzero entire functions on $\mathbb{C}$, and $0 < p, q \leq \infty$.*

(i) If $p \leq q$, then the following statements are equivalent.

(a) $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$;
(b) Either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and u is a constant or ψ is a constant and $u \in \mathcal{F}_q$. If u is in addition non-vanishing, then

$$u(z) = u(0)e^{a_1 z + a_2 z^2}, \quad a_1, a_2 \in \mathbb{C} \text{ and } |a_2| < 1/2; \tag{1.2}$$

(c) $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;
(d) $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is globally Lipschitz continuous.

(ii) If $p > q$, then the following statements are equivalent.

(a) $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$;
(b) ψ is a constant and $u \in \mathcal{F}_q$. If u is in addition non-vanishing, then the representation in (1.2) holds;
(c) $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;

(d) $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is globally Lipschitz continuous.*

(iii) The map $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *cannot be compact for any pair of* p *and* q.

The proof of Theorem 1.1 follows once we prove Lemma 2.2, Theorem 2.3, and Theorem 2.4 in the next section.

The operator $S_{(u,\psi)}$ reduces to S_ψ and M_u when $u = 1$ and $\psi(z) = z$ respectively. Consequently, we record the following special cases of Theorem 1.1.

Corollary 1.2 *Let* ψ *be a nonzero entire function on* $\mathbb{C}$ *and* $0 < p, q \leq \infty$.

(i) If $p \leq q$, *then the statements:* $S_\psi(\mathcal{F}_p) \subseteq \mathcal{F}_q$, $\psi(z) = az + b$ *for some* $a, b \in \mathbb{C}$, $S_\psi : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded and* $S_\psi : \mathcal{F}_p \to \mathcal{F}_q$ *is globally Lipschitz continuous, are all equivalent.*

(ii) If $p > q$, *then the statements:* $S_\psi(\mathcal{F}_p) \subseteq \mathcal{F}_q$, $\psi =$ *constant,* $S_\psi : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded, and* $S_\psi : \mathcal{F}_p \to \mathcal{F}_q$ *is globally Lipschitz continuous, are all equivalent.*

(iii) $S_\psi : \mathcal{F}_p \to \mathcal{F}_q$ *cannot be compact for any pair of* p *and* q.

Given the fact that $\mathcal{F}_p$ properly contains the space $\mathcal{F}_q$ when $q < p$ [10,Theorem 2.10], it should be clear that a superposition from the former to the latter is possible only via constant functions.

Corollary 1.3 *Let* u *be a nonzero entire function on* $\mathbb{C}$ *and* $0 < p, q \leq \infty$. *Then*

(i) if $p \leq q$, *then the statements:* $M_u(\mathcal{F}_p) \subseteq \mathcal{F}_q$, $u =$ *constant,* $M_u : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded, and* $M_u : \mathcal{F}_p \to \mathcal{F}_q$ *is globally Lipschitz continuous, are all equivalent.*

(ii) if $p > q$, *then* M_u *fails to map* $\mathcal{F}_p$ *into* $\mathcal{F}_q$.

(iii) $M_u : \mathcal{F}_p \to \mathcal{F}_q$ *cannot be compact for any pair of* p *and* q.

We close this section with a word on notation. The notion $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2 Proof of the results

Let us first consider the following necessity condition about the superposition operator S_ψ. The proof of the lemma demonstrates an important test function that will be used in the proof of the main result.

Lemma 2.1 *Let* ψ *be nonzero entire function on* $\mathbb{C}$ *and* $0 < p, q \leq \infty$. *If* S_ψ *maps* $\mathcal{F}_p$ *into* $\mathcal{F}_q$, *then* $\psi = az + b$ *for some* $a, b \in \mathbb{C}$.

Proof To prove the assertion, we need to choose test function from $\mathcal{F}_p$ for any fixed p. Thus, for c with $\frac{1}{4} < c < \frac{1}{2}$, consider the function

$$f_c(z) = e^{cz^2}$$

In view of the estimate,

$$\left| f(z) e^{-|z|^2/2} \right| \leq e^{(c-\frac{1}{2})|z|^2},$$

the functions $f_c \in \mathcal{F}_p$. Moreover, the Fock space norm is invariant under rotations so if $|\lambda| = |\mu| = 1$, then all functions

$$f_{\lambda,\mu,c}(z) = \lambda e^{c\mu z^2} \in \mathcal{F}_p$$

and have the same norm as f_c which can be seen by making change of variables $\tau = \sqrt{\mu}z$ in the integration for $p \neq \infty$ and in estimating the supremum norm for $p = \infty$. By the same reasoning, we also observe that

$$\|S_\psi f_{\lambda,\mu,c}\|_q = \|S_\psi f_c\|_q.$$

Now for any $w \neq 0$, we can find $z, \lambda, \mu \in \mathbb{C}$ with $|\lambda| = |\mu| = 1$ and

$$w = \lambda e^{c\mu z^2}$$

and also that $c\mu z^2$ is real and positive which implies $w/\lambda = \overline{\lambda} w$. Clearly, for large w, we have

$$|z|^2 = \frac{1}{c}\log(\overline{\lambda} w) \simeq \frac{\log|w|}{c}.$$

Therefore, using the estimate above and in (1.1)

$$|\psi(w)| = |\psi\big(\lambda e^{c\mu z^2}\big)| \leq e^{|z|^2/2}\|S_\psi f_{\lambda,\mu,c}\|_q = e^{|z|^2/2}\|S_\psi f_c\|_q = \|S_\psi f_c\|_q |w|^{1/2c}$$

for sufficiently large w.

Since $\|S_\psi f_c\|_q$ is a constant value and $1/2c < 2$, the standard Cauchy estimates imply that ψ is a polynomial of degree at most one as desired. □

Next, we recall the notion of order and type of an entire function and prove one more important lemma. Let f be an entire function, and $M(r, f) = \max_{|z|=r} |f(z)|$. The order of f is

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

If $0 < \rho < \infty$, then the type of f is given by

$$\tau(f) = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\rho(f)}}.$$

By definition, if a function f has order ρ, then for every $\epsilon > 0$

$$M(f, r) = O(e^{\rho+\epsilon})$$

when $r \to \infty$. This clearly shows that any function of order less than 2 belongs to all the Fock spaces $\mathcal{F}_p$. Conversely, by [10,Theorem 2.12], every function in $\mathcal{F}_p$ has order at most 2, and if its order is exactly 2, it must be of type less than or equal to $1/2$. In the next lemma, we prove that if the function has no zeros, then its type can not be $1/2$.

Lemma 2.2 *Let $0 < p \leq \infty$ and u is a non-vanishing function in $\mathcal{F}_p$. Then*

$$u(z) = u(0)e^{a_1 z + a_2 z^2}, \quad a_1, a_2 \in \mathbb{C} \text{ and } |a_2| < 1/2. \tag{2.1}$$

Proof We argue as follows. Since $u \in \mathcal{F}_p$, by [10,Theorem 2.12], $\rho(u) \leq 2$ and if $\rho(u) = 2$, it must be of type less than or equal to $1/2$. On the other hand, since u is non-vanishing, it follows from the Hadamard Factorisation Theorem that

$$u(z) = e^{a_0 + a_1 z + a_2 z^2} = u(0)e^{a_1 z + a_2 z^2} \tag{2.2}$$

where $a_0, a_1, a_2 \in \mathbb{C}$ and $|a_2| \leq 1/2$. It remains to show that $|a_2| = 1/2$ can not happen. Assuming to the contrary, we may simply set $a_2 = 1/2$ (if not since the Fock space norm is invariant under rotation, we can find a μ with $|\mu| = 1$ such that $\mu a_2 = 1/2$). Then for $p < \infty$

$$\begin{aligned}\int_{\mathbb{C}} |u(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) &= |u(0)|^p \int_{\mathbb{C}} e^{p\left(\Re(a_1 z)+1/2\Re(z^2)\right)-\frac{p}{2}|z|^2} dA(z)\\ &= |u(0)|^p \int_{\mathbb{R}}\int_{\mathbb{R}} e^{p\Re(a_1)x - p(\Im(a_1)y+y^2)} dx dy\\ &= |u(0)|^p \left(\int_{\mathbb{R}} e^{p\Re(a_1)x} dx\right)\left(\int_{\mathbb{R}} e^{-p(\Im(a_1)y+y^2)} dy\right) = \infty\end{aligned}$$

as the first integral with respect to x diverges while the second integral with respect to y converges which contradicts the assumption that $u \in \mathcal{F}_p$.

If $p = \infty$, we replace the above integral argument with supremum to arrive at the same conclusion. □

For the sake of better exposition, we split Theorem 1.1 into two theorems below.

Theorem 2.3 *Let ψ and u be nonzero entire functions on $\mathbb{C}$ and $0 < p < q \leq \infty$. Then if*

(i) $p \leq q$, then $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$ if and only if either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and u is a constant or ψ is a constant and $u \in \mathcal{F}_q$. In the case when u is non-vanishing, it has the form

$$u(z) = u(0)e^{a_1 z + a_2 z^2}, \quad a_1, a_2 \in \mathbb{C} \text{ and } |a_2| < 1/2. \tag{2.3}$$

(ii) $p > q$, then $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$ if and only if ψ is a constant and $u \in \mathcal{F}_q$. In the case when u is non-vanishing, it has the form in (2.3).

Proof of part (i)

The sufficiency of the condition is easy to verify. It is the necessity that may require a new techniques which we present below. We shall first prove that if $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$. The assumption implies $u \in \mathcal{F}_q$ as already seen before. To this end, if u is non-vanishing, by Lemma 2.2

$$u(z) = u(0)e^{a_1 z + a_2 z^2}, \ a_1, a_2 \in \mathbb{C} \ \text{ and } |a_2| < 1/2. \tag{2.4}$$

Suppose for the purpose of contradiction that ψ is not linear. Then there exists a sequence $w_n \in \mathbb{C}$ such that $|w_n| \to \infty$ as $n \to \infty$ and $|\psi(w_n)| \geq n|w_n|^2$ for $n \in \mathbb{N}$. Arguing as in the proof of Lemma 2.1, for each $w_n \neq 0$, we can find $z_n, \lambda_n, \mu_n \in \mathbb{C}$ with $|\lambda_n| = |\mu_n| = 1$ and

$$w_n = \lambda_n e^{c\mu_n z_n^2} \tag{2.5}$$

and also that $c\mu_n z_n^2$ is real and positive. We may further pick a sparse subsequence of z_n such that the discs $D(z_n, 1)$ are mutually disjoint. Now, for $q < \infty$, applying the operator to the sequence

$$f_{c,\lambda_n,\mu_n}(z) = \lambda_n e^{c\mu_n z^2}$$

and eventually invoking the estimate in (1.1),

$$\begin{aligned}\int_{\mathbb{C}} |S_{u,\psi} f_{c,\lambda_n,\mu_n}(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) &= \int_{\mathbb{C}} |u(z)|^q |\psi(\lambda_n e^{c\mu_n z^2}|^q e^{-\frac{q}{2}|z|^2} dA(z)\\ &\geq \sum_{n=1}^{\infty} \int_{D(z_n,1)} |u(z)|^q |\psi(w_n)|^q e^{-\frac{q}{2}|z|^2} dA(z)\\ &\gtrsim \sum_{n=1}^{\infty} |u(z_n)|^q |\psi(w_n)|^q e^{-\frac{q}{2}|z_n|^2} \geq \sum_{n=1}^{\infty} n^q |u(z_n)|^q e^{2cq\mu_n z_n^2 - \frac{q}{2}|z_n|^2}. \end{aligned} \tag{2.6}$$

We consider two separate cases depending on whether the multiplier function u has zeros or not in $\mathbb{C}$. If u is non-vanishing, by (2.4) and (2.6)

$$\int_{\mathbb{C}} |S_{u,\psi} f_{c,\lambda_n,\mu_n}(z)|^q n^q e^{-\frac{q}{2}|z|^2} dA(z) \gtrsim \sum_{n=1}^{\infty} |e^{a_0 + a_1 z_n + a_2 z_n^2}|^q |e^{2cq\mu_n z_n^2 - \frac{q}{2}|z_n|^2}. \tag{2.7}$$

Now, if $a_2 = 0$, then we may chose c in the interval $\left(\frac{3}{8}, \frac{1}{2}\right)$ such that $2c - \frac{1}{2} > 0$. Thus, the last sum in (2.6) diverges. On the other hand, if $0 < |a_2|$, then we may chose c in $\left(\frac{1}{4} + \frac{|a_2|}{2}, \frac{1}{2}\right)$ such that

$$2c - \frac{1}{2} > 2\left(\frac{1}{4} + \frac{|a_2|}{2}\right) - \frac{1}{2} = |a_2| > 0$$

and hence the sum in (2.6) still diverges. This is a contradiction and hence $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ in this case.
Next, assume that u has zeros and analyze the case when the sequence z_n in the sum (2.6) could belong to the zero set of u. Let us for example assume that u is a polynomial. Then there exist positive constants C and R such that for $|u(z)| \geq C$ for $|z| \geq R$. It follows that

$$\begin{aligned}\|S_{(u,\psi)} f\|_q^q &= \frac{q}{2\pi} \int_{\mathbb{C}} \left|u(z) S_\psi f(z)\right|^q e^{-\frac{q}{2}|z|^2} dA(z)\\ &\geq C^q \frac{q}{2\pi} \int_{\{z\in\mathbb{C}:|z|\geq R\}} \left|S_\psi f(z)\right|^q e^{-\frac{q}{2}|z|^2} dA(z).\end{aligned}$$

By Lemma 2.1, the last integral above is finite only when $\psi(z) = az + b$.

Now if infinitely many $z_n' s$ belong to the zero set of u such that last sum in (2.6) converges, then we keep changing our choice of the constant c in the interval $(1/4, 1/2)$ above until the convergence is not possible. We explain how this can done below. From (2.5) and since c has uncountably many possible values in the said interval, we collect all these values to see that

$$\left\{\frac{1}{\sqrt{c}} \frac{\log(w_n/\lambda_n)}{\mu_n} : (c, n) \in (1/4, 1/2) \times \mathbb{N}\right\}$$

is uncountable. As known, an entire function can not have uncountable zero set. Taking this into account, we can choice the sequence z_n in such a way that it does not belong to the zero set of u when $n \to \infty$.

Now, if u has order less than 2, then every choice of c in the interval $(1/4, 1/2)$ gives that the sum in (2.6) diverges. On the other hand, if $\rho(u) = 2$ and its type, $\tau(u)$, is less than $1/2$, then we can still choice c in the interval $(\frac{\tau(u)}{2} + \frac{1}{4}, \frac{1}{2})$ such that sum in (2.6) diverges. Thus, it remains to consider the extremal case when

$$|u(z_n)| \simeq e^{-\frac{1}{2}|z_n|^2} \tag{2.8}$$

as $n \to \infty$. To this end, setting the expression in (2.8) in (2.6)

$$\int_{\mathbb{C}} |S_{u,\psi} f_{c,\lambda_n,\mu_n}(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \gtrsim \sum_{n=1}^{\infty} n^q |u(z_n)|^q e^{2cq\mu_n z_n^2 - \frac{q}{2}|z_n|^2}$$
$$\simeq \sum_{n=1}^{\infty} n^q |u(z_n)|^q e^{2cq\mu_n z_n^2 - q|z_n|^2} \tag{2.9}$$

holds for all c in $(1/4, 1/2)$. Letting $c \to \frac{1}{2}$, we note that the sum in (2.9) converges only when the sequence $\{n^q :\in \mathbb{N}\}$ is summable, which is a contradiction again, and hence $\psi(z) = az+b$ for some constants $a, b \in \mathbb{C}$.

Next, we set $\psi(z) = az + b$ and show that u is necessarily a constant function whenever $a \neq 0$. Aiming to argue in the contrary, assume that u is not a constant. Then we can pick a sparse sequence w_n such that $|u(w_n)| \to \infty$ as $n \to \infty$ and that the discs $D(w_n, 1)$ are mutually disjoint. For each positive ϵ, observe that the function $g_\epsilon(z) = e^{\frac{z^2}{2+\epsilon}} \in \mathcal{F}_p$ for all p. Then we can find a sequence μ_n such that $|\mu_n| = 1$ and $\mu_n w_n^2$ are real and positive for sufficiently large n. Setting $h_{\epsilon,\mu_n}(z) = e^{\frac{\mu_n z^2}{2+\epsilon}} - \frac{b}{a}$ and eventually applying (1.1)

$$\begin{aligned}
\|S_{(u,\psi)} h_{\epsilon,\mu_n}\|_q^q &= \frac{q}{2\pi} \int_{\mathbb{C}} \left|S_{(u,\psi)} h_{\epsilon,\mu_n}(z)\right|^q e^{-\frac{q}{2}|z|^2} dA(z)\\
&= \frac{q}{2\pi}|a|^q \int_{\mathbb{C}} \left|u(z) h_{\epsilon,\mu_n}(z)\right|^q e^{-\frac{q}{2}|z|^2} dA(z)\\
&\geq \frac{q}{2\pi}|a|^q \sum_{n=1}^{\infty} \int_{D(w_n,1)} \left|u(z) h_{\epsilon,\mu_n}(z)\right|^q e^{-\frac{q}{2}|z|^2} dA(z)\\
&\geq \frac{q}{2\pi}|a|^q \sum_{n=1}^{\infty} |u(w_n)|^q e^{\left(\frac{q}{2+\epsilon} - \frac{q}{2}\right) w_n^2}
\end{aligned}$$

for all $\epsilon > 0$. Letting $\epsilon \to 0$, we observe that the sum diverges, which is again a contradiction.

For $q = \infty$, we may simply replace the integral above by the supremum norm and argue to arrive at the same conclusion. This completes the proof of part (i).

Proof of part (ii)

The sufficiency of the condition is clear again. Let $p > q$. Then as already proved above in the first part, $\psi(z) = az + b$ is a necessary condition for $S_{(u,\psi)}$ to map $\mathcal{F}_p$ into $\mathcal{F}_q$ independent of the size of p and q. Assume that $a \neq 0$ and for any $f \in \mathcal{F}_p$, consider the function $f_b = f - \frac{b}{a} \in \mathcal{F}_p$ and observe

$$\left\| S_{(u,\psi)} f_b \right\|_q = |a||\alpha| \|f\|_q$$

Then, our conclusion follows from the fact that $\mathcal{F}_p \backslash \mathcal{F}_q$ is non-empty [10,Theorem 2.10] and $|a||\alpha| \neq 0$ and completes the proof of Theorem 2.3.

Theorem 2.4 *Let ψ and u be nonzero entire functions on $\mathbb{C}$ and $0 < p, q \leq \infty$. If $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$, then it is bounded and globally Lipschitz continuous. But $S_{(u,\psi)}$ cannot be compact.*

Proof Clearly, f ψ is a constant and $u \in \mathcal{F}_q$, then $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded. Thus, lets dispose the sufficiency of the condition when $\psi(z) = az + b$ and $u = \alpha$. Consider $f \in \mathcal{F}_p$ and compute

$$\begin{aligned}\|S_{(u,\psi)} f\|_q^q &= \frac{q}{2\pi}\int_{\mathbb{C}} \big|u(z)(af(z)+b)\big|^q e^{-\frac{q}{2}|z|^2} dA(z)\\ &\le \frac{q}{2\pi}|2\alpha|^q\Big(|a|^q\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) + \int_{\mathbb{C}} |b|^q e^{-\frac{q}{2}|z|^2} dA(z)\Big)\\ &= |2\alpha|^q|a|^q\|f\|_q^q + |2\alpha|^q|b|^q \le |2\alpha|^q|a|^q\|f\|_p^q + |2\alpha|^q|b|^q < \infty\end{aligned}$$

where the second inequality follows from [10,Theorem 2.10].

Next, let us show that the spaces support no compact weighted superposition operators. Aiming to arrive at a contradiction, let $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ be compact and hence $\psi(z) = az+b$ for some complex number a, b and $u = \alpha$ or $\psi = b$ and $u \in \mathcal{F}_q$. The sequence $k_w \in \mathcal{F}_p$ is bounded and converges to zero uniformly on compact subsets of $\mathbb{C}$ when $|w| \to \infty$. Applying the operator to k_w and eventually (1.1)

$$\big\|S_{(u,\psi)}k_w\big\|_q \ge |ak_w(w) + b||\alpha|e^{-|w|^2/2} = |a + be^{-|w|^2/2}||\alpha|$$

and $\big\|S_{(u,\psi)}k_w\big\|_q \to 0$ as $|w| \to \infty$ only when $a = 0$. It follows that

$$\big\|S_{(u,\psi)}k_w\big\|_q = |b||\alpha|\|1\|_q \to 0$$

as $|w| \to \infty$ only if $b = 0$ which contradicts that ψ is nonzero.

If $a = 0$ and $u \in \mathcal{F}_q$ is non-zero, the same conclusion follows easily.

It remains to show that $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is globally Lipschitz continuity. Since the case for $p > q$ is trivial, we assume $p \le q$. An application of Theorem 2.3 gives that $\psi(z) = az + b$ and $u = \alpha \neq 0$ or $\psi = b$ and $u \in \mathcal{F}_q$. If $a = 0$, then the conclusion follows easily. Thus, assume $a \neq 0$ and hence $u = \alpha \neq 0$. For $f, g \in \mathcal{F}_p$

$$\|S_{(u,\psi)}f - S_{(u,\psi)}g\|_q^q = \|auf - aug\|_q = |a\alpha|\|f - g\|_q \le |a\alpha|\|f - g\|_p$$

where the last inequality follows from the inclusion property. This shows that $S_{(u,\psi)}$ is globally Lipschitz continuous with constant $|a\alpha|$. □

2.1 Remark

In the remainder of this section, we present alternative proofs for the necessity in Theorem 2.3 when

(i) $q = \infty$
(ii) $2 = p < q < \infty$
(iii) $q < p < \infty$ and u is non-vanishing.

The proof is interest of its own as it shows how the zero sets and uniqueness sets in Fock spaces play important roles in the study of weighted superposition operators. Unfortunately, a complete characterization of such sets are still an open problem as far as we know. Some necessary and sufficient conditions can be read in [10,Chapter 5].

(i) We consider the square lattice in the complex plane

$$\Lambda = \big\{\omega_{mn} = \sqrt{\pi}(m + in) : m, n \in \mathbb{Z}\big\}$$

where $\mathbb{Z}$ denotes the set of all integers. As known the Weierstrass σ-function associated to Λ is defined by

$$\sigma(z) = z \prod_{\substack{(m,n)\neq(0,0)\\(m,n)\in\mathbb{Z}^2}} \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left(\frac{z}{\omega_{mn}} + \frac{z^2}{2\omega_{mn}^2}\right).$$

Furthermore, it is known that Λ is the zero set for σ and by [10,Lemma 5.6], $\sigma \in \mathcal{F}_\infty$ but not in any of the other Fock spaces $\mathcal{F}_p$. Aiming to argue in the contrary, suppose now that $u \neq 0$ and ψ is not a constant. Since $S_{(u,\psi)}$ maps $\mathcal{F}_\infty$ into $\mathcal{F}_q$, the function

$$F = S_{(u,\psi)}\sigma - \psi(0)u = u\psi(\sigma) - \psi(0)u \in \mathcal{F}_q.$$

Now since ψ is not a constant, the function F is non-zero and vanishes on Λ. On the other hand, by [10,Lemma 5.7], Λ is a uniqueness set for $\mathcal{F}_q$ for all $q \neq \infty$ which implies that $F = 0$. This is contradicts the fact that $F \neq 0$. Therefore, ψ is a constant.

(ii) We consider the set $\Omega = \Lambda - \{0\}$. As shown in [10,p.204], Ω is a uniqueness set for $\mathcal{F}_2$. Indeed, the function $g(z) = \sigma(z)/z$ belongs to $\mathcal{F}_p$ if and only if $p > 2$. Now to argue as above, assume ψ is not a constant and hence

$$G(z) = S_{(u,\psi)}g - \psi(0)u = u\psi(g) - \psi(0)u \in \mathcal{F}_2$$

is non-zero and vanishes on Ω, but Ω is a uniqueness set for $\mathcal{F}_2$. Thus, $G = 0$ resulting a contradiction again.

(iii) For a positive number R, we consider the following modified lattice

$$\Lambda_R = \left\{\omega_{mn} : m, n \in \mathbb{Z}\right\}$$

where

$$\omega_{mn} = \begin{cases} z_{mn}, & \text{if } n \neq 0 \text{ or } n = m = 0 \\ \sqrt{\pi}\left(m + \frac{Rm}{|m|}\right), & \text{if } n = 0 \text{ and } m \neq 0 \end{cases}$$

Then the modified Weierstrass function associated to Λ_R is given by

$$\sigma_R(z) = z \prod_{\substack{(m,n)\neq(0,0)\\(m,n)\in\mathbb{Z}^2}} \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left(\frac{z}{\omega_{mn}} + \frac{z^2}{2z_{mn}^2}\right).$$

Now if we choose R to be a number such that $\frac{1}{p} < R < \frac{1}{q}$, then as shown in the proof of [8,Theorem 1.1], the function $\sigma_R \in \mathcal{F}_p$ and σ_R constitutes a zero set for $\mathcal{F}_p$ while it fails to be a zero set for $\mathcal{F}_q$.

Now to argue as in the previous two cases, assume $u \in \mathcal{F}_q$ is non-vanishing and ψ not a constant. Then

$$H = S_{(u,\psi)}\sigma_{a,R} - \psi(0)u = u\psi(\sigma) - \psi(0)u \in \mathcal{F}_q$$

is non constant and vanishes only on the set Λ_R. It follows that Λ_R is a zero set for $\mathcal{F}_q$ which is a contradiction.

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