



Convex-Cyclic Weighted Composition Operators and Their Adjoints

Tesfa Mengestie

Abstract. We characterize the convex-cyclic weighted composition operators $W_{(u,\psi)}$ and their adjoints on the Fock space in terms of the derivative powers of ψ and the location of the eigenvalues of the operators on the complex plane. Such a description is also equivalent to identifying the operators or their adjoints for which their invariant closed convex sets are all invariant subspaces. We further show that the space supports no supercyclic weighted composition operators with respect to the pointwise convergence topology and, hence, with the weak and strong topologies, and answers a question raised by T. Carrol and C. Gilmore in [5].

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1. Introduction

The study of the weighted composition operator $W_{(u,\psi)} : f \mapsto u \cdot f(\psi)$ with symbol ψ and multiplier u acting on various spaces of holomorphic functions traces back to works related to isometries on the Hardy spaces [11, 14] and commutants of Toeplitz operators [8, 9]. Since then, the operator has attracted much research interest and there exists now rich body of literatures dealing with many of its properties in various settings; see for example [2, 7, 16, 18] and the references therein.

In this note, we are interested in linear dynamical properties of the operators and their adjoints on the Fock space \mathcal{F}_2 which consists of square integrable analytic functions in \mathbb{C} with respect to the Gaussian measure $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dA(z)$, where dA is the Lebesgue measure in \mathbb{C} . The space is a reproducing kernel Hilbert space endowed with the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z) \overline{g(z)} d\mu(z),$$

norm $\|f\|_2 := \sqrt{\langle f, f \rangle}$ and kernel function $K_w(z) = e^{\langle z, w \rangle}$.

A bounded linear operator T on a separable Banach space \mathcal{H} is said to be cyclic if there exists a vector f in \mathcal{H} for which the span of the orbit

$$\text{Orb}(T, f) = \{f, Tf, T^2f, T^3f, \dots\}$$

is dense in \mathcal{H} . Such an f is called a cyclic vector for T . The operator is hypercyclic if the orbit itself is dense, and supercyclic with vector f if the projective orbit,

$$\text{Projorb}(T, f) = \{\lambda T^n f, \lambda \in \mathbb{C}, n = 0, 1, 2, \dots\},$$

is dense. These dynamical properties of T depend on the behavior of its iterates

$$T^n = \underbrace{T \circ T \circ T \circ \dots \circ T}_{n\text{-times}}.$$

We may note that identifying cyclic and hypercyclic operators have been a subject of high interest partly because they play central rolls in the study of other operators. More specifically, it is known that every bounded linear operator on an infinite-dimensional complex separable Hilbert space is the sum of two hypercyclic operators [1, p. 50]. This result holds true with the summands being cyclic operators as well [24].

1.1. Cyclic Weighted Composition Operators and Their Adjoints

The bounded and compact properties of weighted composition operators on \mathcal{F}_2 are identified in [16, 18], and $W_{(u,\psi)}$ is bounded if and only if $u \in \mathcal{F}_2$ and

$$\sup_{z \in \mathbb{C}} |u(z)| e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} < \infty. \tag{1.1}$$

Furthermore, it was proved that condition (1.1) implies $\psi(z) = az + b, |a| \leq 1$ and when $a \neq 0$ the operator norm is estimated by

$$\sup_{z \in \mathbb{C}} |u(z)| e^{\frac{1}{2}(|az+b|^2 - |z|^2)} \leq \|W_{(u,\psi)}\| \leq |a|^{-1} \sup_{z \in \mathbb{C}} |u(z)| e^{\frac{1}{2}(|az+b|^2 - |z|^2)}. \tag{1.2}$$

When $|a| = 1$, the multiplier function has the special form $u = u(0)K_{-\bar{a}b}$ and the relation in (1.2) simplifies to

$$\|W_{(u,\psi)}\| = |u(0)| e^{\frac{|b|^2}{2}}. \tag{1.3}$$

In [20], we reported that there exists no supercyclic composition operator on Fock spaces. On the other hand, the orbit of any vector f under $W_{(u,\psi)}$ has elements of the form

$$W_{(u,\psi)}^n f = f(\psi^n) \prod_{j=0}^{n-1} u(\psi^j) \tag{1.4}$$

for all nonnegative integers n and ψ^0 is the identity map. The relation in (1.4) shows that the power of the weighted composition operators is another

weighted composition operator with symbol (u_n, ψ^n) , where

$$u_n = \prod_{j=0}^{n-1} u(\psi^j). \tag{1.5}$$

In [22], we continued the work in [20] and investigated whether the relation in (1.4) induces an interplay between the symbol ψ and the multiplier u and result in supercyclic weighted composition operators. It turns out that such an interplay fails to make any projective orbit dense enough in \mathcal{F}_2 . See also [5] for a different approach. Recently, the author [19] proved that the Fock space support no supercyclic adjoint weighted composition operators either. Recall that, the adjoint $W_{(u,\psi)}^*$ of a bounded weighted composition operator $W_{(u,\psi)}$ on \mathcal{F}_2 is the operator which satisfies the relation

$$\langle W_{(u,\psi)}f, g \rangle = \langle f, W_{(u,\psi)}^*g \rangle$$

for all $f, g \in \mathcal{F}_2$. We note that the adjoint of a weighted composition operator on \mathcal{F}_2 is not necessarily a weighted composition operator. Zhao and Pang [25] proved that for pairs of entire functions u_1, ψ_1 and u_2, ψ_2 from \mathcal{F}_2 , the relation $W_{(u_1,\psi_1)}^* = W_{(u_2,\psi_2)}$ holds if and only if

$$\psi_1(z) = az + b, u_1(z) = dK_c(z), \psi_2(z) = \bar{a}z + c, \quad \text{and} \quad u_2(z) = \bar{d}K_c(z),$$

where a, b, c and d are constants such that $d \neq 0$ and either $|a| < 1$ or $|a| = 1$ and $c + \bar{a}b = 0$. Thus, an operator and its adjoint can have quite different dynamical structures; see, for example, [1, p.26] about the hypercyclic structure of the multiplication operator on Hardy spaces.

Having observed the absence of supercyclic weighted composition operators and their adjoints on the Fock space, we considered the cyclicity problem in [19] and proved the following interesting result.

Theorem 1.1. *Let u and ψ be entire functions on \mathbb{C} such that $W_{(u,\psi)}$ is bounded on \mathcal{F}_2 and hence $\psi(z) = az + b, |a| \leq 1$. Then, the following statements are equivalent.*

- (i) $W_{(u,\psi)}$ is cyclic on \mathcal{F}_2 ;
- (ii) u is non-vanishing and $a^k \neq a$ for all positive integer $k \geq 2$;
- (iii) $W_{(u,\psi)}^*$ is cyclic on \mathcal{F}_2 .

In this note, we plan to answer two other basic questions related to the dynamics of the operators. The first deals with identifying weighted composition operators which admit convex-cyclicity property in the space. As will be seen latter, convex-cyclicity is stronger than the cyclicity property and may require conditions stronger than part (ii) Theorem 1.1. The second takes up the question whether weakening the topology of the space results in weakly supercyclic weighted composition operators.

1.2. Convex-Cyclic Weighted Composition Operators and Their Adjoints

Another dynamical concept related to the iterates of an operator is convex-cyclicity. A bounded operator T on a Banach space \mathcal{H} is said to be convex-cyclic if there is a vector f in \mathcal{H} such that the convex hull of $\text{Orb}(T, f)$

is dense in \mathcal{H} . Recall that the convex hull of a set is the set of all convex combinations of its elements, that is, all finite linear combinations of its elements where the coefficients are non-negative and their sum is one. Thus, studying the convex-cyclicity of an operator requires a good understanding of the convex combinations of its powers. The notion of convex-cyclicity is a relatively young subject of study which was introduced by Rezaei [23] in 2013. A few more studies were made recently in [4, 10, 17].

Since the convex hull of a set lies between the set and its linear span, every hypercyclic operator is convex-cyclic while every convex-cyclic operator is cyclic, and obviously, every convex-cyclic vector for an operator is a cyclic vector. On a finite dimensional Banach space, the notions of cyclicity and convex-cyclicity are equivalent [10, Theorem 1.1]. Supercyclicity is another structure that stands between hypercyclic and cyclic operators and one may wonder its location in reference to convex-cyclicity. By [23, Proposition 3.2], the norm of every convex-cyclic operator is bigger than one. Hence, if a bounded operator T is supercyclic, then for positive parameters α , all the operators $1/(\alpha + \|T\|)T$ are supercyclic but not convex-cyclic. On the other hand, as shown in Theorem 1.2, there exists convex-cyclic weighted composition operators on \mathcal{F}_2 which are not supercyclic; see the diagram in the last section for a good illustration of the relations among the various forms of cyclicities.

Then, it naturally follows to ask which cyclic weighted composition operators are convex-cyclic on \mathcal{F}_2 . This note aims to answer this question. Our first result, Theorem 1.2, completely describes the convex-cyclic weighted composition operators and their adjoints in terms of cyclicity and the location of the eigenvalues of the operators on the complex plane. The second main result, Theorem 1.7, shows that the space fails to support supercyclic weighted composition operators with respect to the pointwise topology and hence with the weak topology.

Theorem 1.2. *Let u and ψ be analytic maps on \mathbb{C} such that $W_{(u,\psi)}$ is bounded on \mathcal{F}_2 , and hence $\psi(z) = az + b$, $|a| \leq 1$. Then, the following statements are equivalent.*

- (i) $W_{(u,\psi)}$ is convex-cyclic on \mathcal{F}_2 ;
- (ii) $W_{(u,\psi)}^*$ is convex-cyclic on \mathcal{F}_2 ;
- (iii) $W_{(u,\psi)}$ has the property that all of its invariant closed convex-sets are invariant subspaces;
- (iv) $W_{(u,\psi)}^*$ has the property that all of its invariant closed convex-sets are invariant subspaces;
- (v) $W_{(u,\psi)}$ is cyclic on \mathcal{F}_2 , $|a| = 1$, $|u(z_0)| > 1$, and $\Im(u(z_0)a^m) \neq 0$ for all $m \in \mathbb{N}_0$ where $\Im(z)$ refers to the imaginary part of a complex number z and $z_0 := b/(1 - a)$.

The theorem describes the convex-cyclic weighted composition operators and their adjoints by simple to check requirements. The condition $|a| = 1$ restricts that a non-normal weighted composition operator can not be convex-cyclic on \mathcal{F}_2 , while the condition $\Im(u(z_0)a^m) \neq 0$ requires all the eigenvalues of the operator and its adjoint not to be located on the real line.

The proof of the theorem follows from Lemmas 1.3, 1.4 and 1.6 where their proofs are mainly based on the relation between the dynamical and spectral properties of the operators. The relationship between the density of orbits and the spectral properties of an operator plays a vital roll in the study of dynamical structures of operators. Thus, identifying the location of the eigenvalues of the operator has been a fundamental tool for studying convex-cyclic operators. We plan to use this tool to prove our main results.

Lemma 1.3. *Let u and ψ be analytic maps on \mathbb{C} such that $W_{(u,\psi)}$ is bounded on \mathcal{F}_2 , and hence $\psi(z) = az + b$, $|a| \leq 1$. Then $W_{(u,\psi)}$ is convex-cyclic if and only if the following holds.*

- (i) $W_{(u,\psi)}$ is cyclic on \mathcal{F}_2 ;
- (ii) $|a| = 1$, $|u(z_0)| > 1$ and $\Im(u(z_0)a^m) \neq 0$ for all $m \in \mathbb{N}_0$.

Proof. Let us first prove the sufficiency. By Theorem 1.1, the cyclicity condition implies that $a^m \neq a$ for all $m \geq 2$. By [19, Lemma 2], the numbers $a^m u(z_0)$ constitutes a sequence of distinct eigenvalues for the operator with corresponding eigenvectors

$$f_m(z) = (z - z_0)^m e^{\frac{a\bar{b}}{a-1}z} \tag{1.6}$$

for all $m \in \mathbb{N}_0$. Furthermore, since the sequence of the polynomials is dense in \mathcal{F}_2 and $e^{\frac{a\bar{b}}{a-1}z}$ is a non-vanishing function in \mathcal{F}_2 , by a result of Izuchi [15], the sequence of the eigenvectors $(f_m), m \in \mathbb{N}_0$ is also dense in \mathcal{F}_2 . This along with condition (ii) of the theorem and Theorem 6.2 in [4] ensure that the operator is convex-cyclic, and indeed has a dense set of convex-cyclic vectors.

Conversely, suppose now that the operator is convex-cyclic. Since every convex-cyclic operator is cyclic, condition (i) follows and hence $a \neq 0$ and $a \neq 1$. Then, a simple modification, like changing the corresponding kernel function, of the arguments in the proof of Lemma 3 of [12] gives that the adjoint operator $W_{(u,\psi)}^*$ has the following set of eigenvalues:

$$\{\overline{u(z_0)a^m} : m \in \mathbb{N}_0\}.$$

By [23, Proposition 3.3], it follows that $\overline{u(z_0)a^m} \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R})$. Therefore, $\Im(u(z_0)a^m) \neq 0$ and $|a^m u(z_0)| = |a|^m |u(z_0)| > 1$ for all $m \in \mathbb{N}_0$, where the later holds only when $|a| = 1$ and $|u(z_0)| > 1$, and completes the proof. \square

Lemma 1.4. *Let u and ψ be analytic maps on \mathbb{C} such that $W_{(u,\psi)}$ is bounded on \mathcal{F}_2 , and hence $\psi(z) = az + b$, $|a| \leq 1$. Then, $W_{(u,\psi)}^*$ is convex-cyclic if and only if the following holds.*

- (i) $W_{(u,\psi)}^*$ is cyclic on \mathcal{F}_2 ;
- (ii) $|a| = 1$, $|u(z_0)| > 1$ and $\Im(u(z_0)a^m) \neq 0$ for all $m \in \mathbb{N}_0$.

Proof. We first assume that conditions (i) and (ii) hold, and proceed to prove the sufficiency. The condition $|a| = 1$ implies the operator $W_{(u,\psi)}$ is normal [16] and hence, $W_{(u,\psi)}^*$ has the same sequence of eigenvectors (f_m) in (1.6) as $W_{(u,\psi)}$ with corresponding distinct eigenvalues $\overline{u(z_0)a^m}$. Then, following the same argument as in the proof of Lemma 1.3 and applying [4, Theorem 6.2], the operator is convex-cyclic, and has a dense set of convex-cyclic vectors.

Conversely, assume $W_{(u,\psi)}^*$ is convex-cyclic. Then, it is obviously cyclic. Moreover, the operator $W_{(u,\psi)}$, which is the adjoint of $W_{(u,\psi)}^*$, has eigenvalues $u(z_0)a^m$ as referred above. Thus, by [23, Proposition 3.3], it follows that $u(z_0)a^m \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R})$ for all $m \in \mathbb{N}_0$ from which the remaining necessity conditions follow and completes the proof. \square

If $u = 1$, then $W_{(u,\psi)}$ is just the composition map $C_\psi : f \mapsto f(\psi)$. On the other hand, if ψ is the identity map, then $W_{(u,\psi)}$ reduces to the multiplication operator $M_u : f \mapsto u \cdot f$. Thus, $W_{(u,\psi)}$ generalizes the two operators and can be written as a product $W_{(u,\psi)} = M_u C_\psi$. The following statement is an immediate consequence of Theorem 1.2 about the two factor operators in the product.

Corollary 1.5. (i) *Let C_ψ be a bounded composition operator on \mathcal{F}_2 . Then neither C_ψ nor its adjoint C_ψ^* is convex-cyclic on \mathcal{F}_2 .*
 (ii) *Let M_u be a bounded multiplication operator on \mathcal{F}_2 . Then, M_u cannot be convex-cyclic on \mathcal{F}_2 .*

The proof for the composition operator follows immediately from Theorem 1.2 since $u(z_0) \neq 1$. On the other hand, as proved in [6, Lemma 2], for $\psi = az + b$, the adjoint of C_ψ is the weighted composition operator $C_\psi^* = W_{(K_b,\phi)}$, where $\phi(z) = \bar{a}z$. Then $z_0 = 0$ in this case and $|u(z_0)| = |K_b(0)| = 1$. Then, the claim follows from condition (ii) of Theorem 1.2 again.

By Theorem 1.1, the multiplication operator M_u is not cyclic and cannot be convex-cyclic either.

We note in passing that there exists an interesting interplay between u and ψ such that $W_{(u,\psi)} = M_u C_\psi$ is bounded (compact) on \mathcal{F}_2 while both the factors C_ψ and u fail to be. For example, one can set $u_0(z) = e^{-z}$, $\psi_0(z) = z + 1$, and observe that $W_{(u_0,\psi_0)}$ is bounded while both the factors remain unbounded. Now, Theorem 1.2 and Corollary 1.5 provide another interplay between u and ψ for which the weighted composition can be convex-cyclic while both the factors fail to be.

1.3. Invariant Convex Sets for Weighted Composition Operators and Their Adjoints

Let T be a bounded operator on a Banach space \mathcal{H} and M be a subset of \mathcal{H} . We say M is invariant under T if $T(M) \subseteq M$. We now study when the invariant closed convex-sets of the weighted composition operators and their adjoints are all invariant subspaces. As shown below, this happens if and only if the operators are convex-cyclic.

Lemma 1.6. *Let u and ψ be analytic maps on \mathbb{C} such that $W_{(u,\psi)}$ is bounded on \mathcal{F}_2 , and hence $\psi(z) = az + b$, $|a| \leq 1$. Then,*

- (i) *$W_{(u,\psi)}$ has the property that all of its invariant closed convex-sets are invariant subspaces if and only if*
 - (a) *$W_{(u,\psi)}$ is cyclic on \mathcal{F}_2 ;*
 - (b) *$|a| = 1$, $|u(z_0)| > 1$ and $\Im(u(z_0)a^m) \neq 0$ for all $m \in \mathbb{N}_0$.*

- (ii) $W_{(u,\psi)}^*$ has the property that all of its invariant closed convex-sets are invariant subspaces if and only if $W_{(u,\psi)}$ is cyclic on \mathcal{F}_2 and $|a| = 1$, $|u(z_0)| > 1$ and $\Im(u(z_0)a^m) \neq 0$ for all $m \in \mathbb{N}_0$.

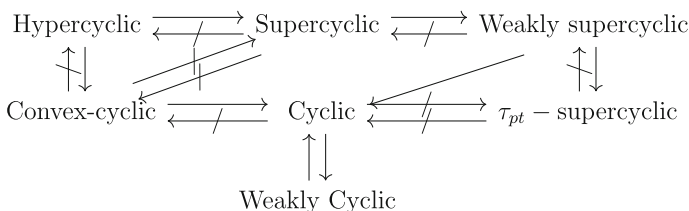
Proof. (i) By [10, Proposition 8.11], the operator $W_{(u,\psi)}$ has the property that all of its invariant closed convex-sets are invariant subspaces if and only if for every closed invariant subspace M of $W_{(u,\psi)}$, the operator $W_{(u,\psi)}|_M$ is convex-cyclic and the convex-cyclic vectors for $W_{(u,\psi)}|_M$ are the same as the cyclic vectors for $W_{(u,\psi)}|_M$ whenever it is cyclic. Here, by $W_{(u,\psi)}|_M$, we mean the operator obtained by restricting $W_{(u,\psi)}$ to a closed subset M of the space \mathcal{F}_2 . In particular, if we set $M = \mathcal{F}_2$ and apply Lemma 1.3 above, we observe that the necessary conditions in (a) and (b) follow.

Conversely, assume now that conditions (a) and (b) are satisfied. Let M be a closed invariant subspace for $W_{(u,\psi)}$ and $W_{(u,\psi)}|_M$ is cyclic. We need to show that $W_{(u,\psi)}|_M$ is convex-cyclic. But this follows readily from Lemma 1.3 since all eigenvalues of $W_{(u,\psi)}|_M$ are also eigenvalues of $W_{(u,\psi)}$ over the whole space \mathcal{F}_2 .

The proof of part (ii) follows from a similar argument as part (i) and Lemma 1.4. □

1.4. Weak and τ_{pt} -Supercyclic Weighted Composition Operators

Having completely identified the cyclic and convex-cyclic weighted composition operators and knowing that the space supports no such supercyclic operators, it is natural to seek weakening the topology of the space and study the supercyclic structure with respect to the weak topology (weak supercyclicity) and the pointwise convergence topology τ_{pt} (τ_{pt} -supercyclic). The weak and τ_{pt} -supercyclicities are defined by simply replacing the norm topology by these respective topologies on the space. Clearly, weak supercyclicity is a stronger property than τ_{pt} -supercyclicity. The next diagram exhibit the relations among the various forms of cyclicity for bounded operators.



We note that there exist other forms of weaker hypercyclicity and supercyclicity which are not listed in the diagram above. For example, between supercyclicity and weak supercyclicity, one can find weak l-sequentially supercyclic and weak sequentially supercyclic properties which the first implies the second. A weighted composition operator never satisfies any of these forms as it fails to satisfy the weakest form of supercyclicity with respect to the pointwise convergence topology as will be seen in Theorem 1.7. We also note that since the weak closure of the convex set, $\text{span Orb}(T, f)$ for any f in the given space, coincides with its norm closure, cyclicity in the norm topology is equivalent to cyclicity in the weak topology.

Let us now consider the weak and τ_{pt} -supercyclicity of the weighted composition operator. A simple observation in this regard is that if $W_{(u,\psi)}$ is weakly supercyclic on \mathcal{F}_2 , then by [4, Theorem C], the whole point spectrum of $W_{(u,\psi)}$ and its adjoint must belong to the open disc $(0, \|W_{(u,\psi)}\|)$. On the other hand, if the operator is convex-cyclic, by Theorem 1.2, $\psi(z) = az + b, |a| = 1$, then (1.3) implies $\|W_{(u,\psi)}\| = |u(0)|e^{|b|^2/2} = |u(z_0)|$. This together with the eigenvalues gives the necessary condition that

$$|a^m u(z_0)| = |a|^m |u(z_0)| = |u(z_0)| < \|W_{(u,\psi)}\| = |u(z_0)|$$

for all $m \in \mathbb{N}_0$ which is a contradiction. This assures that the concepts of weak supercyclicity and convex-cyclicity are not related either. In fact, we will prove that $W_{(u,\psi)}$ is never weakly supercyclic on \mathcal{F}_2 . The next result shows that the Fock space supports no τ_{pt} -supercyclic weighted composition operators either.

Theorem 1.7. *Let (u, ψ) be a pair of entire functions on \mathbb{C} which induce a bounded weighted composition operator $W_{(u,\psi)}$ on \mathcal{F}_2 . Then $W_{(u,\psi)}$ can not be supercyclic on \mathcal{F}_2 with respect to the pointwise convergence topology.*

As illustrated in the diagram, pointwise topology is weaker than the weak topology on \mathcal{F}_2 and hence the space supports no weakly supercyclic weighted composition operators. It is interesting to remark that convex-cyclicity is the strongest form of cyclicity for weighted composition operators supported on \mathcal{F}_2 .

Proof of Theorem 1.7. Since $W_{(u,\psi)}$ is bounded, we set $\psi(z) = az + b$, with $|a| \leq 1$ and consider two different cases.

Case 1 Let $|a| < 1$ or $|a| = 1$ and $a \neq 1$ or $a = 1$ and $b = 0$. For this case the map ψ fixes the point $z_0 = \frac{b}{1-a}$ for $a \neq 1$ and $z_0 = 0$ for the rest. Assume on the contrary that there exists a τ_{pt} -supercyclic vector f in \mathcal{F}_2 . First we claim that u is zero free on \mathbb{C} because if u vanishes at point w , then (1.4) implies that every element in the projective orbit of f vanishes at w which extends to the closure resulting a contradiction. Observe also that f cannot have zero in \mathbb{C} . if not, all the elements in the projective orbit will also vanish at a possible zero which extends to the closure and contradicts. Thus, by Proposition 4 of [3], for any two different numbers $z, w \in \mathbb{C}$,

$$\overline{\left\{ \frac{u_n(z)f(\psi^n(z))}{u_n(w)f(\psi^n(w))} \right\}} = \mathbb{C}. \tag{1.7}$$

Let $r > 0$ be given. Then, the set $K = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ is a compact neighborhood of z_0 which also contains $\psi(K)$ since for each $z \in K$

$$|\psi(z) - z_0| = |az + b - z_0| \leq |az - az_0| + |az_0 - z_0 + b| \leq |a|r \leq r.$$

Now, if we set $w = \psi(z), z \in K, z \neq z_0$ and consider the expression in (1.7)

$$\left| \frac{u_n(z)f(\psi^n(z))}{u_n(w)f(\psi^n(w))} \right| = \left| \frac{u(z)f(\psi^n(z))}{u(\psi^n(z))f(\psi^{n+1}(z))} \right| \leq M$$

for all $n \in \mathbb{N}$, where

$$M = \frac{\max_{z \in K} |u(z)| \cdot \max_{z \in K} |f(z)|}{\min_{z \in K} |u(z)| \cdot \min_{z \in K} |f(z)|}.$$

This obviously contradicts the relation in (1.7).

Case 2 It remains to show the case for $a = 1$ and $b \neq 0$. To this end, let $Q_b = \{z \in \mathbb{C} : |z - b| \leq 2|b|\}$. Then, Q_b is a compact neighborhood of b , and also contains $\psi(Q_b)$ since for each $z \in K_b$

$$|\psi(z) - b| = |z + b - b| \leq |z - b| + |b| \leq 2|b|.$$

Then, we arrive at the desired conclusion by simply replacing the compact set K by Q_b in the above argument and completes the proof.

The following corollary is now an immediate consequence of Theorem 1.7.

Corollary 1.8. (i) *Let C_ψ be a bounded composition operator on \mathcal{F}_2 . Then neither C_ψ nor its adjoint C_ψ^* can be τ_{pt} -supercyclic on \mathcal{F}_2 .*

(ii) *Let M_u be a bounded multiplication operator on \mathcal{F}_2 . Then M_u can not be τ_{pt} -supercyclic on \mathcal{F}_2 .*

Setting $u = 1$ in Theorem 1.1, we note that C_ψ is cyclic on \mathcal{F}_2 if and only if $a^k \neq a$ for all positive integers $k > 1$. The same conclusion can be also read in [13, 20].

We end this section with an important remark. We note that the above proof does not use Hilbert space property from \mathcal{F}_2 . Thus, the same result with the same proof is valid on other classical Fock spaces $\mathcal{F}_p, 1 \leq p < \infty$ which consists of all analytic functions f for which

$$\|f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) < \infty,$$

where dA is the usual Lebesgue area measure on the complex plane \mathbb{C} . This further answers a question raised by T. Carrol and C. Gilmore in [5].

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Tesfa Mengestie
Mathematics Section
Western Norway University of Applied Sciences
Klingenbergvegen 8
5414 Stord
Norway
e-mail: Tesfa.Mengestie@hvl.no

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