Exploring students' metacognition in relation to an integral-area evaluation task

Farzad Radmehr[1,2] and Michael Drake[3]

[1]University of Agder, Norway; [2]Western Norway University of Applied Science, Norway;

[3]Victoria University of Wellington, New Zealand

Several studies have been conducted to explore students' understanding of the integral-area relationships, however, only a few focused on students' metacognition in relation to this topic. In this study, students' metacognitive knowledge, skills, and experiences in relation to an integral-area evaluation task are explored using semi-structured interviews and think-aloud protocol. The results show that several students developed monitoring strategies in relation to integral-area relationships; however, they do not use these strategies when solving integral-area tasks. The findings suggest that teachers and lecturers could use monitoring strategies more often when solving mathematical questions in class, and encourage students to use these strategies when solving problems.

Keywords: Metacognition, metacognitive knowledge, metacognitive experiences, metacognitive skills, integral-area relationships.

Introduction

Integral calculus is part of the upper secondary school and undergraduate university mathematics curriculum in many countries as a wide range of real-world problems require an understanding of integral calculus, including a range of contexts in physics and engineering (e.g., Thompson & Silverman, 2008). Many undergraduate and graduate courses in mathematics and the engineering sciences also rely heavily on parts of this topic (e.g., differential equations) (Czocher, Tague, & Baker, 2013). Several studies have been conducted to explore students' mathematical understanding of concepts within integral calculus such as the integral-area relationships (e.g., Jones, 2013; Sealey, 2014; Thomas & Hong, 1996) and the Fundamental Theorem of Calculus (FTC) (e.g., Thompson, 1994; Thompson, & Silverman, 2008). However, only a few studies (e.g., Radmehr & Drake, 2017, 2019, 2020) have explored students' metacognition in relation to integral calculus. Metacognition is "meta-level knowledge and mental action used to steer cognitive processes" (Jacobse & Harskamp, 2012, p. 133). Previous studies (e.g., Radmehr & Drake, 2017, 2019; Dallas, 2014) have highlighted that metacognition "is a driving force in mathematical problem solving" (Czocher, 2018, p. 140); however, many students have not developed well their metacognition in relation to mathematical problem solving (e.g., Radmehr & Drake, 2017, 2019; Jacobse & Harskamp, 2012). In this study, we explore students' metacognition in relation to an evaluation task related to integral-area relationships. We also investigate the differences that might exist between the metacognition of upper secondary and tertiary students in this regard. The research question explored here is: What metacognition do upper secondary and tertiary students reveal in relation to an integral-area evaluation task?

Facets of metacognition

Three facets of metacognition have been recognized in previous studies (Efklides, 2006, 2008): *metacognitive knowledge*, *metacognitive skills*, and *metacognitive experiences*. In this section, we describe these facets to frame the study. *Metacognitive knowledge* "is declarative knowledge stored in memory" (Efklides, 2008, p. 278) about factors (i.e., *persons*, *tasks*, *goals*, and *strategies*) that influence cognitive activities (Efklides, 2006, 2008). The *persons* factor relates to how individuals complete or feel about different tasks. For example, you are more confident in finding an enclosed area with respect to the x-axis rather than the y-axis when both ways can be used for finding the area using integral calculus. The *tasks* factor relates to categories, relationships, features, and how different tasks work. For instance, your knowledge about how a question related to integral-area can be checked. The *goals* factor refers to the goals individuals pursue when engaging in different tasks (e.g., you solve your integral assignment questions to have a better understanding of the topic). Finally, the *strategies* factor consists of strategies that are used for problem-solving, and how, why, and when such strategies should be used (e.g., knowing in the questions related to integral-area, sketching the related graphs of functions will help to determine which method should be used) (Efklides, 2006, 2008).

Metacognitive skills are activities that are performed deliberately to help individuals control their cognitive activities (Efklides, 2006, 2008). These activities consist of task orientating, planning, monitoring, regulating, and evaluating (Efklides, 2006, 2008). For example, making a visualization when solving a mathematical problem can help with identifying how the problem can be solved. We need to highlight the difference between metacognitive knowledge and skills. Students might have developed their metacognitive knowledge in different ways; however, they are not using them when engaging in problem solving, indicating lack of their metacognitive skills. For instance, many students know drawing a graph will help in solving mathematical problems, indicating the presence of metacognitive knowledge; however, they might not use this strategy when solving mathematical problems, indicating lack of metacognitive skills.

Metacognitive experiences are one's awareness and feelings when engaging in a task and processing its information (Efklides, 2008). Metacognitive experiences include feelings of knowing, familiarity, confidence, satisfaction, and difficulty in relation to different tasks (e.g., I think I can solve this mathematical problem because I saw similar tasks before). It also comprises judgment of learning and the correctness of solutions, and estimating effort and time needed to spend on tasks (e.g., I think I can solve this mathematical problem in 15 minutes or I think I solved this question correctly) (Efklides, 2006, 2008).

Integral-area relationships and definite integrals

Several studies have focused on students' understanding of integral-area relationships (e.g., Kiat, 2005; Mahir, 2009; Sealey, 2014). These studies have reported that many students have difficulties in finding area using integral, for example, when the function is below the x-axis (e.g., Radmehr & Drake, 2019); or the graph of the integrand is not given to students (e.g., Kiat, 2005). For instance, Kiat (2005) reported that 55% of students could not set up the integrals correctly to find a shaded area in a task in which one of the curves was below and one was above the x-axis (Kiat, 2005). Furthermore, students' understanding of why integral techniques should be used for finding enclosed area is limited (e.g., Thomas & Hong, 1996). Riemann sums and definite integrals, $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$, entail a number of important concepts (i.e., functions, limits, rate of change, and multiplication) (Sealey, 2014). Previous studies have reported understanding the definite integral as the limit of a sum is a difficult task for many students (e.g., Sealey, 2014). To develop a better understanding how students construct the concept of the Riemann integral, Sealey (2014) has designed a framework to characterise students' understanding of Riemann sums and the definite integral. Sealey (2014) has identified that understanding the product of $f(x)$ and Δx is the most complex part for students.

Methods

A multiple case study was used to explore students' cognition and metacognition in relation to the integral calculus. Two cases were selected composed of a sample of students that were interviewed in 2014-2015 academic year. Case 1 is one of the top five universities in New Zealand and Case 2 is one of 11 colleges (upper secondary schools) in Wellington city, New Zealand. Nine students from Case 1 and eight students from Case 2 voluntarily participated in the study. All the college students were enrolled in a Year 13 calculus course, and the university students were enrolled in a single variable calculus course (designed for students of the Faculty of Science) when the data collection started. The students of Case 1 were enrolled in a mathematics, statistics, or physics major.

The students participated in a one to one interview with the first author that lasted between 70 to 150 minutes to explore their metacognition and mathematical understanding. Some students completed the interview in one session and some students in two sessions. In the interviews, nine tasks related to integral-area relationships and the FTC were given to students, and they also responded to 14 questions related to their metacognitive knowledge. Students' performance and their metacognitive experiences and skills in relation to seven of these tasks have been reported in Radmehr and Drake (2017, 2019), and their metacognitive knowledge in Radmehr and Drake (2020). Here, we report the results of the following evaluation task that have not described in those studies.

Are these examples solved correctly? Please justify your answer.

Example 1: Find if possible, the area between the curve $y = x^2 - 4x$ and the x-axis from $x = 0$ to $x = 5$. $\int_0^5 (x^2 - 4x)dx = [\frac{x^3}{3} - \frac{4x^2}{2}]_{x=0}^{x=5} = \left[\frac{5^3}{3} - \frac{4(5)^2}{2}\right] - \left[\frac{(0)^3}{3} - \frac{4(0)^2}{2}\right] = \frac{-25}{3}$

Example 2: Find if possible, the area enclosed between the curve $y = \frac{1}{x^2}$ and the x-axis from $x = -1$ to $x = 1$. $\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{1} x^{-2} dx = [\frac{(x)^{-1}}{(-1)} = \frac{-1}{x}]_{x=-1}^{x=1} = \frac{-1}{1} - \frac{(-1)}{(-1)} = -2.$

Additionally, we have not explored how much of the metacognitive knowledge students possess about a mathematical topic, they have actually used when solving mathematical problems. In other words, this is our first attempt to compare students' metacognitive knowledge with their metacognitive skills and experiences. The first example is designed based on a task from Kiat's study (2005). Students may find an incorrect area for both examples if they do not sketch the graph of the curves. In addition, students who only focus on integration techniques and do not pay enough attention to the integral-area relationship may also make mistakes.

To explore students' metacognition in relation to each facet of metacognition, different approaches have been proposed (Radmehr & Drake, 2018). To explore students' metacognitive knowledge, a set of questions could be asked from students about what they would do when they face specific mathematics-related situations (Radmehr & Drake, 2018). The results in relation to one of these questions are presented in this paper: *How do you check your answers when solving problems involving finding the area enclosed between curves?*

Concerning students' metacognitive skills, students can be asked to verbalize their thought when they are solving mathematical problems using a think-aloud protocol (Radmehr & Drake, 2018; Dallas, 2014; Jacobse & Harskamp, 2012). Then, their verbal thought can be analyzed using the rubrics designed for analyzing students' metacognitive skills (see Jacobse & Harskamp, 2012; Veenman, Kerseboom, & Imthorn, 2000). For example, Veenman et al. (2000) suggested checking calculation and answer, reflecting on the final answer, and the learning experience as some of the activities that can indicate the presence of metacognitive skills. In this study, from different components of metacognitive skills, students' monitoring strategies and making drawing in relation to the task were analyzed (when students were solving the evaluation task). These two are among the metacognitive activities important for successful mathematical problem solving (Jacobse & Harskamp, 2012).

To explore students' metacognitive experiences, two items from the Visualization and Accuracy instrument (Jacobse & Harskamp, 2012) were used. First, students read the task, and before solving it, were encouraged to answer the following question: *How well do you think you can solve this problem?* They could choose one of three following: *I am sure I will solve this problem*; *I am not sure whether I will solve this problem correctly or incorrectly*; or *I am sure I cannot solve this problem*. After selecting one of the choices, they were asked to provide a reason(s) for their choice. After working on the task, a similar question was asked: *Rate your confidence for having found the correct answer*. Students had similar choices (i.e., *I am sure I solved this question correctly*; *I am not sure whether I solved this question correctly or incorrectly*; *I am sure I solved this question incorrectly*), and they were encouraged to provide a reason(s) for their choice.

Results

In this section, first, students' monitoring strategies in response to the metacognitive knowledge question are provided. Then, students' pre-judgments, mathematical performances, metacognitive skills, and post-judgments in relation to the evaluation task are described. The university students are indicated by T and the college students with S.

Students' metacognitive knowledge in relation to integral-area relationships

Students in Cases 1 and 2 mentioned different strategies for checking their answers in integral-area problems. Three strategies were domain-specific, including approximating the enclosed area using geometric shapes to find out whether the answer makes sense (T: 1, 3, 4, 6, 8, 9; S: 7, 8); checking the area is positive (T: 3, 9; S: 1, 5, 6); and checking the antiderivative by differentiating it (T: 3, 9; S: 1, 5, 6). The other monitoring strategies mentioned by students were general and can be used in different settings. The most common strategy for checking answers was going over calculations (T: 1, 3, 4, 5, 6, 8; S: 2, 4, 5, 7). The other monitoring strategies mentioned by at least two students were: Using Wolfram alpha website to check answers (T: 3, 4, 5, 6, 7); checking answers with classmates (T: 4, 6; S: 6, 7); Using the answers at the end of the textbook (S: 2, 3, 4, 6); Using assignment solutions (T: 2, 3); and using a calculator to check answers/graph of curves (S: 3, 6). These findings show that approximating area using geometric shapes and also using online resources (e.g., Wolfram alpha) were more common within university students for checking answers. In contrast, college students relied more on solutions at the end of their textbook and standard calculators.

Students' metacognitive experiences (pre-judgments) in relation to the evaluation task

In relation to the students' pre-judgments, all the students thought they were able to do this task. However, only T5 identified that the first example was solved incorrectly and only two students (T: 5, 9) identified the second had been solved incorrectly. Students made their judgments based on the familiarity of the examples (T: 1, 2, 4; S: 3, 5, 6), saying the integrals are "simple" and "straightforward to integrate" (T: 2, 8; S: 7), and they knew how to solve these questions (T: 9; S: 2, 4, 8). T5 made that judgment because she thought only one curve was involved in each example: "sure, in both questions, they are only one equation involved so it is not as tricky as the ones with more equations". S1 made that judgment because he thought he did not need to graph the curves: "Also, I do not need to work out the formula from the graph". Two students' responses (T: 6, 7) related to the fact that the task involved evaluating. T7 said, "…easier to find someone wrong than to prove me right", and T6 mentioned, "I think I can find any wrong steps" [in the solutions]. Finally, T3 thought he was able to complete the task as "nothing looks immediately wrong".

Students' mathematical performance on the evaluation task

As stated earlier, only T5 identified that the first example was solved incorrectly. For the second example, only two students (T: 5, 9) recognized it was solved incorrectly; however, three students (T: 5, 7, 9) showed they had the conceptual knowledge that the integrand should be continuous on the interval being integrated. T5 showed that the area is diverging by using the improper integral. T9 believed that if an integrand is not continuous at a given point, the calculation is not possible: "Why this work, I thought it will be something in the calculation that wouldn't work [sic]". T7 sketched the graph of $y = \frac{1}{x^2}$ and said "we do not have the division by zero problem here", indicating he did not notice that the function is undefined at zero. The remaining students only checked the integration and calculations, indicating procedural knowledge of the integral-area relationship. A lack of numerical proficiency was also observed whilst students were attempting to answer this question. For example, four students from Case 2 (S: 1, 4, 5, 6) used a calculator to check if the bounds were substituted correctly in the integrand, suggesting they were not confident with calculations such as 25×2 and $\frac{5^3}{3}$. Moreover, S1 simplified $\frac{5^3}{3} - 2 \times 5^2$ as $\frac{125}{3} - 20$, then asked to use a calculator to find the end result.

Students' metacognitive skills in relation to the evaluation task

Regarding making a drawing to solve the task, only two students (T: 5, 7) made a drawing for this task. However, Radmehr and Drake (2019) showed that for more typical integral-area questions (e.g., "Please calculate the area enclosed between the curve enclosed between the curve x=y^2 and $y = x - 2$ in two ways. Which way is better to use? Why?" (p. 91)) many students try to draw the enclosed area. T5 made a drawing for both parts of the question and solved the question correctly. T7 made a drawing for the second part, however, could not identify that the function was not defined at zero.

In relation to checking calculations and answers, five students (T: 3; S: 1, 4, 5, 6) undertook some form of check. College students used their calculators to check their calculations, as mentioned above. T3 checked the calculation after he had solved the question, without using a calculator, to explore whether he had substituted the bounds in the antiderivative correctly, because the answers were negative.

Students' metacognitive experiences (post-judgments) in relation to the evaluation task

In relation to the post-judgments, 15 students were sure they had solved the task correctly; however, only T5 judged the first example correctly, and only two students (T: 5, 9) judged the second example correctly. The other two students (T: 1, 2) were unsure if they had solved the task correctly because they obtained a negative area. In addition, T1 was unsure whether the antiderivative of $\frac{1}{x^2}$ was $\frac{-1}{x}$ or not, indicating a lack of procedural knowledge for finding antiderivatives. The 15 students were sure for several reasons. 13 students (T: 3, 4, 6, 7, 9; S: 1, 2, 3, 4, 5, 6, 7, 8) made that judgment because they had the same answer (for one or both examples). T8 was "confident" with the way he checked the solutions of the examples. Three students (T: 5; S: 1, 5) were sure because they had found a different answer to that written in the task; however, S1 had found a wrong answer through a wrong calculation and S5 had an error relating to the integral-area relationships.

Discussion and conclusion

The paper adds to the literature in mathematics education in several ways. First, a literature search revealed that this is one of the first attempts in regards to exploring how much of the metacognitive knowledge students possess about a mathematical topic is actually used by them when solving mathematical problems. Secondly, only a few studies have explored students' metacognitive knowledge, experiences, and skills in relation to upper secondary and tertiary mathematics (e.g., Radmehr & Drake, 2017, 2019), therefore, the findings could help mathematics education researchers to have a better understanding of students' metacognition in relation to learning mathematics at the upper secondary and tertiary levels.

In relation to students' metacognitive knowledge, students highlighted several useful monitoring strategies that can be used for checking answers of certain types of integral-area questions. However, it seems some of the students in the sample did not develop these monitoring strategies. For instance, nine students did not refer to the approximating area using geometric shapes, and thirteen students did not mention differentiating antiderivative as a monitoring strategy in response to the interview question. Secondly, in relation to students' metacognitive skills, when they engaged in the evaluation task, the students did not use these strategies, and therefore, most of them were unable to identify the mistakes in the examples. Only some of them checked their calculations without drawing the graphs. Teaching monitoring strategies could be considered as a key element of teaching mathematics, and ways in which students can check their solutions could be suggested to them (e.g., Goos, Galbraith, & Renshaw, 2002). The active role of the lecturers/teachers in posing questions like "how can we check this answer?", then, discussing and modelling the use of appropriate strategies, more students might learn, and motivate to use monitoring strategies during solving mathematical problems (e.g., Schoenfeld, 1987). In addition, if teachers and lecturers use monitoring strategies more often when solving questions in class, and encourage students to do so as a part of their problem solving, these strategies might be used more often by students. Designing mathematical problem-solving tasks with metacognitive questions at each problem-solving phase (e.g., the post-judgment item), could also support developing students' metacognition (Dallas, 2014). These activities could lead to fewer errors in solving mathematical problems, and students developing greater confidence in their mathematical understanding (Radmehr & Drake, 2019; Dallas, 2014).

In relation to the metacognitive experiences, the fact that the task was an evaluation task did not negatively affect the confidence of students as they were all confident they could solve the question. Students' pre-judgments show that most students based their judgment on familiarity with the integrand and knowing how to find antiderivatives, and not by focusing on the shape of the enclosed area. Most of the students did not make any drawing of the given curves, which resulted in most students being unable to find the mistake in the solution they were evaluating. This suggests that it might be useful for their lecturers and teachers to more strongly highlight the importance of drawing curves for integral-area problems and help them focus on the enclosed area's shape when trying to solve such questions. This might also help students to not over-simplify a problem. In this study, consistent with previous studies in relation to the integral-area relationships (e.g., Mahir, 2009), students' procedural knowledge was better developed compared to conceptual knowledge. For instance, the fact that the integrand should be continuous was ignored by most students when solving the task. Also, a lack of algebraic manipulation skills and prior knowledge were observed for several students. These findings, also highlighted by previous studies (e.g., Kiat, 2005), suggesting several students would benefit from improving their algebraic manipulation and/or graph sketching prior to starting integral (Kiat, 2005).

Acknowledgements

The authors would like to thank Mr Markos Dallas, PhD candidate in Mathematics Education at the University of Agder, for his feedback on this paper.

References

Czocher, J. A., Tague, J., & Baker, G. (2013). Where does the calculus go? An investigation of how calculus ideas are used in later coursework. *International Journal of Mathematical Education in Science and Technology, 44*(5), 673–684.

Czocher, J. A. (2018). How does validating activity contribute to the modeling process? *Educational Studies in Mathematics, 99*(2), 137–159.

Dallas, M. (2014). *Metacognitive strategies employed by third graders at primary school during the process of solving a math problem* (Master's thesis). National and Kapodistrian University of Athens. Retrieved from https://pergamos.lib.uoa.gr/uoa/dl/frontend/en/browse/1318919

Efklides, A. (2006). Metacognition and affect: What can metacognitive experiences tell us about the learning process? *Educational Research Review, 1*(1), 3–14.

Efklides, A. (2008). Metacognition: Defining its facets and levels of functioning in relation to self-regulation and co-regulation. *European Psychologist, 13*(4), 277–287.

Goos, M., Galbraith P. & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving, *Educational Studies in Mathematics, 49*(2), 193–223.

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology, 40*(2), 201–211.

Jacobse, A. E., & Harskamp, E. G. (2012). Towards efficient measurement of metacognition in mathematical problem solving. *Metacognition and Learning, 7*(2), 133–149.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122–141.

Kiat, S. E. (2005). Analysis of students' difficulties in solving integration problems. *The Mathematics Educator, 9*(1), 39–59.

Radmehr, F., & Drake, M. (2017). Exploring students' mathematical performance, metacognitive experiences and skills in relation to fundamental theorem of calculus. *International Journal of Mathematical Education in Science and Technology, 48*(7), 1043–1071.

Radmehr, F., & Drake, M. (2018). An assessment-based model for exploring the solving of mathematical problems: Utilizing revised bloom's taxonomy and facets of metacognition. *Studies in Educational Evaluation, 59*, 41–51.

Radmehr, F., & Drake, M. (2019). Students' mathematical performance, metacognitive experiences and metacognitive skills in relation to integral-area relationships. *Teaching Mathematics and its Applications: An International Journal of the IMA, 38*(2), 85–106.

Radmehr, F., & Drake, M. (2020). Exploring students' metacognitive knowledge: The case of integral calculus, *Education Sciences, 10*(3), 55. doi: 10.3390/educsci10030055

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale: Lawrence Erlbaum.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*, 230–245.

Thomas, M. O. J., & Hong, Y. Y. (1996). The Riemann integral in calculus: Students' processes and concepts. In *Proceedings of the 19th Annual Conference of the Mathematics Education Research Group of Australasia* (MERGA) (pp. 572–579). Melbourne: The Mathematics Education Research Group of Australasia.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2), 229–274.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43–52). Washington: Mathematical Association of America.

Veenman, M. V., Kerseboom, L., & Imthorn, C. (2000). Test anxiety and metacognitive skillfulness: Availability versus production deficiencies. *Anxiety, Stress and Coping, 13*(4), 391–412.