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Resolvent growth condition for composition operators on the Fock space

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Abstract

For each analytic map ψ on the complex plane $\mathbb{C}$, we study the Ritt's resolvent growth condition for the composition operator $C_\psi : f \to f \circ \psi$ on the Fock space $\mathcal{F}_2$. We show that C_ψ satisfies such a condition if and only if it is either compact or reduces to the identity operator. As a consequence, it is shown that the Ritt's resolvent condition and the unconditional Ritt's condition for C_ψ are equivalent.

Keywords Fock space · Composition operators · Ritt's resolvent condition · Spectrum · Bounded · Compact · Unconditional Ritt's condition

Mathematics Subject Classification 47B32 · 30H20 · 46E22 · 46E20 · 47B33

1 Introduction

During the last decades there has been huge interest in the study of composition operators on Banach spaces of holomorphic functions. Relying on rich theory of analytic function, it has been possible to develop and characterize several structures of the operators. The interest to study the operators originates partly from its connection to the famous invariant subspace problem for Hilbert spaces. Namely that, the invariant subspace problem for separable Hilbert spaces can be reduced to the study of the invariant subspace structure of certain composition operators [14]. The theory of composition operators lies at the interface of analytic function theory and operator theory, and provides relevant connection among other areas of mathematics; this is another reason making it continue as an active area of research. There is an extensive literature about the operator C_ψ on various settings. We list some

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of them on Fock spaces which are related to our work here; on booundedness and compactness properties [2, 9], on spectrum [10], on power boundedness and mean ergodicity [16], and on various forms of cyclicity [11].

Though several basic properties of C_ψ on various settings are well understood, there are still lots of questions that require further investigations. In this note we take the study further and answer another basic question, namely that when a composition operator satisfies the Ritt's resolvent growth condition on the Fock space $\mathcal{F}_2$. We recall that the space $\mathcal{F}_2$ consists of all entire functions f for which $\|f\| = \sqrt{\langle f, f\rangle} < \infty$ where the inner product is defined by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}dA(z)$$

and dA denotes the Lebesgue area measure on the complex plane $\mathbb{C}$. The space is a reproducing kernel Hilbert space with kernel function $K_w(z) = e^{\overline{w}z}$.

We also note in passing that for any bounded operator T on a Banach space $\mathcal{X}$, identifying the relation between the size of the resolvent operator $(T-\lambda I)^{-1}$ when λ close to the spectrum of the operator, and the asymptotic properties of $T^n x : x \in \mathcal{X}$ has been one of the classical problems in operator related function theory. Here T^n denotes the notion that the operator T is composed to itself n times, and T^0 is the identity map I.

Our result shows that the composition operator C_ψ satisfies the Ritt's resolvent growth condition if and only if when it is compact or reduces to the identity operator on the space. As a result, it turns out that the operator satisfies in fact the stronger unconditional Ritt's condition as well.

2 The Ritt's resolvent condition for C_Ω

As noted above the power bounded and mean ergodic properties of C_ψ on Fock spaces have been identified recently in [16]. The notion of the resolvent growth conditions is another dynamical structure closely related to power boundedness. We recall that the resolvent set $\varrho(T)$ for a bounded operator T on a Banach space $\mathcal{X}$ is the complement of the spectrum $\sigma(T)$ and hence open. For each $\lambda \in \varrho(T)$, the resolvent function $R(\lambda, T) = (\lambda I - T)^{-1}$, $\lambda \in \varrho(T)$ is operator valued and analytic. Thus, for each $|\lambda| > \sup\{|w|; w \in \sigma(T)\}$, it has the expansion

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}. \tag{1}$$

If the operator T is power bounded, that is there is a positive constant M for which $\|T^n\| \le M$, for all $n \in \mathbb{N}$, then (1) immediately implies

$$\|R(\lambda, T)\| \le \sum_{n=0}^{\infty} \frac{\|T^n\|}{|\lambda|^{n+1}} = \frac{c}{|\lambda|-1}, \qquad c := \sup_{n\in\mathbb{N}_0} \|T^n\| < \infty \tag{2}$$

for all $|\lambda| > 1$. This shows that every power bounded operator satisfies the so called Kreiss's resolvent condition. Conversely, by the Kreiss Matrix Theorem, this condition implies power boundedness only when the space $\mathcal{X}$ is finite dimensional. For the infinite dimensional case, the condition only gives that $\|T^n\| = O(n)$ as $n \to \infty$; see for example [17].

On the other hand, by differentiating the resolvent function $n-1$ times we observe that power boundedness of T implies the following stronger iterated resolvent condition:

$$\|R(\lambda, T)^n\| \leq \frac{c}{(|\lambda| - 1)^n}.$$

This condition does not conversely ensure power boundedness either as it only implies $\|T^n\| = O(\sqrt{n})$ as $n \to \infty$ [6].

In this note, we are interested in a more stronger condition that requires an extremal decay rate from the asymptotic behaviours of the iterates of the operator, namely, the Ritt's resolvent condition. An operator T satisfies such a condition, which is also called Tadmor–Ritt resolvent condition, if there exists a positive constant M such that

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda - 1|} \tag{3}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Conditions of this type, (2) and power boundedness play important roll in numerical analysis specially in the study of numerical stability of difference of equations. For more details see [1, 13] and the references therein. Operators which satisfy (3) are simply discrete-time analogues of the generators of bounded holomorphic semigroups. In particular, they are known to generate bounded discrete-time semigroups (T^n), and the resolvent condition in this case plays vital roll in the study of the stability of such semigroups.

We now stat the main result.

Theorem 1 *Let the composition operator C_ψ be bounded on $\mathcal{F}_2$. Then C_ψ satisfies the Ritt's resolvent condition on $\mathcal{F}_2$ if and only if it is either compact or reduces to the identity operator.*

Except in the trivial case where $\psi(z) = z$, the main result asserts that the Ritt's resolvent condition for C_ψ is in fact equivalent to its compactness. It is interesting now to put into context our result by comparing and contrasting it with some other recently obtained results on the dynamical properties of the operators on Fock spaces. By [16, Theorem 2], all bounded composition operators are power bounded but fail to satisfy Ritt's resolvent growth condition. The symbol $\psi(z) = iz$ induces a counter example here. From the same result, it also follows that every composition operator satisfying the Ritt's resolvent condition is uniformly mean ergodic, that is the sequence of the operators

$$\left(\frac{1}{n} \sum_{m=1}^{n} C_\psi^m \right)_{n \in \mathbb{N}}$$

converges in the operator norm. But the converse fails to hold again with $\psi(z) = iz$ still being a counter example.

2.1 Proof of Theorem 1

We now give the proof of the main result. It is known that the operator C_ψ is bounded on $\mathcal{F}_2$ if and only if $\psi(z) = az + b$, $|a| \leq 1$ and $b = 0$ whenever $|a| = 1$. Compactness is described by the strict inequality $|a| < 1$; see [2, 9] for more details. Now, if $|a| = 1$, then the sufficiency of the condition follows easily as the identity map satisfies the Ritt's resolvent condition. Thus, let us verify the necessity of the condition in this case. By [10, Theorem 2.6], the spectrum of the operator C_ψ is given by

$$\sigma(C_\psi) = \overline{\{a^n, n = 0, 1, 2, 3, \dots\}}. \tag{4}$$

Furthermore, from [7] and [13, Theorem 4.5.4] if C_ψ satisfies the Ritt's condition, then its spectrum is contained in a stolz type domain $\overline{\beta_\theta}$ where β_θ is the interior of convex hull of the set $\{1\}$ and the disc $\{z \in \mathbb{D} : |z| \leq \sin\theta\}$ and $\theta = \arccos\frac{1}{M}$ where M is the best possible constant in (3) and $\theta \in [0, \frac{\pi}{2})$. In particular, we have that $\sigma(C_\psi) \cap \mathbb{T} \subset \{1\}$ where $\mathbb{T}$ denotes the unit circle.

Thus, if $|a| = 1$, and $a \neq 1$, then it is easy to see from (4) that the set $\sigma(C_\psi) \cap \mathbb{T}$ contains more than one element, and hence we must have $a = 1$.

Assume now that $|a| < 1$. Nagy and Zemanek [12] proved that a bounded operator T in a complex Banach space satisfies the Ritt's resolvent condition if and only if it is power bounded and the difference of the consecutive powers satisfy the decaying behaviour

$$\sup_{n\in\mathbb{N}} n\|T^{n+1} - T^n\| < \infty. \tag{5}$$

Note that condition (5) alone does not imply power boundedness of the operator T in general [4]. But by [16, Theorem 1.2], all bounded composition operators on Fock spaces are power bounded. Thus, our task will be to show that condition (5) is equivalent to the Ritt's resolvent condition for C_ψ. First observe that for each $n \in \mathbb{N}$, the operator C_ψ^n itself is a composition operator induced by the symbol ψ^n and hence $C_\psi^n = C_{\psi^n}$. It might be of interest to point that for $\psi(z) = az$, $|a| < 1$, the conclusion follows easily since for each integer $n \geq 0$, the operator is $C_{\psi^{n+1}} - C_{\psi^n}$ is diagonal with respect to the standard basis of monomials $\{e_k(z) = z^k/\sqrt{k!} : k \geq 0\}$. The eigenvalue corresponding to e_k is $a^{k(n+1)} - a^{nk}$. Consequently

$$\left\|C_{\psi^{n+1}} - C_{\psi^n}\right\| = \sup_{k\geq 0}\left|a^{k(n+1)} - a^{nk}\right| \leq 2|a|^n.$$

It follows that

$$\sup_{n\geq 0} n\left\|C_{\psi^{n+1}} - C_{\psi^n}\right\| \leq \sup_{n\geq 0} 2n|a|^n < \infty$$

which is the required condition (5).

Since b can be nonzero, we give the following general argument. Let δ_n be a positive number such that $a^{n+1}z + \frac{b(1-a^{n+1})}{1-a} \in D\Big(a^n z + \frac{b(1-a^n)}{1-a}, \delta_n\Big)$ where by $D(\tau, r)$ we denote a disc of radius r and center τ. An explicit expression for δ_n will be given latter. Then for any $f \in \mathcal{F}_2$, by the Mean Value Theorem

$$\begin{aligned}\Big|C_{\psi^{n+1}}f(z) - C_{\psi^n}f(z)\Big|^2 &\leq \left|a^{n+1}z + \frac{b(1-a^{n+1})}{1-a} - a^n z - \frac{b(1-a^n)}{1-a}\right|^2 \sup_{w\in D\left(a^n z+\frac{b(1-a^n)}{1-a},\beta_n\right)} |f'(w)|^2 \\ &= |a^n z(a-1) + ba^n|^2 \sup_{w\in D\left(a^n z+\frac{b(1-a^n)}{1-a},\beta_n\right)} |f'(w)|^2\end{aligned} \tag{6}$$

for some $\beta_n < \delta_n$.

Using the reproducing property,

$$f(w) = \langle f, K_w\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)e^{w\bar{z}}e^{-|z|^2}dA(z)$$

and hence

$$\begin{aligned}|f'(w)| &\leq \frac{1}{\pi}\int_{\mathbb{C}} |z||f(z)|e^{\Re(w\bar{z})}e^{-|z|^2}dA(z) \\ &\leq \frac{1}{\pi}\left(\sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{|z|^2}{2}}\right)\int_{\mathbb{C}} |z|e^{\Re(w\bar{z})}e^{-\frac{|z|^2}{2}}dA(z) \\ &\leq \frac{1}{\pi}\|f\|_2\int_{\mathbb{C}} |z|e^{\Re(w\bar{z})}e^{-\frac{|z|^2}{2}}dA(z) \\ &\leq \frac{1}{\pi}e^{\frac{|w|^2}{2}}\|f\|_2\int_{\mathbb{C}} (1+|z|)e^{-\frac{|z-w|^2}{2}}dA(z).\end{aligned}$$

By triangle inequality

$$\frac{1+|z|}{1+|w|} \leq \frac{1+|w-z|+|w|}{1+|w|} \leq 1+|w-z|$$

and hence

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{C}}(1+|z|)e^{-\frac{|z-w|^2}{2}}dA(z) &\leq (1+|w|)\frac{1}{\pi}\int_{\mathbb{C}}|w-z|e^{-\frac{|z-w|^2}{2}}dA(z \\ &= (1+|w|)2^{\frac{3}{2}}\Gamma\Big(1+\frac{1}{2}\Big) = \sqrt{2\pi}(1+|w|)\end{aligned}$$

where Γ refers to the gamma function. Therefore,

$$|f'(w)| \leq \sqrt{2\pi}(1+|w|)e^{\frac{|w|^2}{2}}\|f\|.$$

On the other hand, since $w \in D\Big(a^n z + \frac{b(1-a^n)}{1-a}, \beta_n\Big)$ we have

$$|w| \leq \left|w - a^n z + \frac{b(1-a^n)}{1-a}\right| + \left|a^n z + \frac{b(1-a^n)}{1-a}\right|$$
$$\leq \beta_n + \left|a^n z + \frac{b(1-a^n)}{1-a}\right| \leq \beta_n + |a|^n|z| + \frac{2|b|}{|1-a|} =: \gamma_n(z)$$

Now taking all these estimates in (6)

$$\frac{1}{\pi}\int_{\mathbb{C}} \left|C_{\psi^{n+1}}f(z) - C_{\psi^n}f(z)\right|^2 e^{-|z|^2} dA(z)$$
$$\leq 2|a|^{2n}\|f\|^2 \int_{\mathbb{C}} |(a-1)z+b|^2|1+\gamma_n(z)|^2 e^{\gamma_n^2(z)-|z|^2} dA(z) \leq 32|a|^{2n}\|f\|^2 S_n$$

where

$$S_n = \int_{\mathbb{C}} \left(|z|^2 + |z|^2\gamma_n^2(z) + |b|^2 + |b|^2\gamma_n^2(z)\right) e^{\gamma_n^2(z)-|z|^2} dA(z). \qquad (7)$$

Next, we show that the integral above is uniformly bounded by a positive number independent of $n \in \mathbb{N}$. To this end, observe that since $a^{n+1}z + \frac{b(1-a^{n+1})}{1-a} \in D\left(a^n z + \frac{b(1-a^n)}{1-a}, \delta_n\right)$ and hence

$$\left|a^{n+1}z + \frac{b(1-a^{n+1})}{1-a} - a^n z - \frac{b(1-a^n)}{1-a}\right| = |a^n z(a-1) + ba^n|$$
$$\leq |a|^n|a-1||z| + |a|^n|b|,$$

δ_n can be taken to be $|a|^n|a-1||z| + |a|^n|b|$. Thus, β_n can also be chosen in such away that $\beta_n = \alpha_n|z| + |a|^n|b|$ for some $0 < \alpha_n < |a|^n|a-1|$ whenever $z \neq 0$. Then

$$\gamma_n^2(z) < \left(|a|^n(|a-1|+1)|z| + |a|^n|b| + \frac{2|b|}{|1-a|}\right)^2$$
$$= |a|^{2n}(|a-1|+1)^2|z|^2 + 2|a|^n|z|(|a-1|+1)\left(|a|^n|b| + \frac{2|b|}{|1-a|}\right) \qquad (8)$$
$$+ \left(|a|^n|b| + \frac{2|b|}{|1-a|}\right)^2.$$

Given the estimate in (8) and the exponential integrating weight in (7), we claim that

$$|a|^{2n}(|a-1|+1)^2 - 1 \leq 0 \qquad (9)$$

for all n. Setting $a = |a|e^{i\theta}$ we write

$$|a|^{2n}(|a-1|+1)^2 - 1 = |a|^{2n}\left(|a|\sqrt{(\cos\theta - 1)^2 + (\sin\theta)^2} + 1\right)^2 - 1.$$

Thus, it is enough to show that $|a|^2(2|a|+1)^2 - 1 \leq 0$. But this obviously holds since $r = |a| < 1$ and the values of the function $g(r) = r^2(2r+1)^2 - 1$ lies in the interval $[-1, 0]$. Now, integrating with polar coordinates shows that the integral in (7) is uniformly bounded by positive number C independent of n. Therefore,

$$\left\| C_{\psi^{n+1}} - C_{\psi^n} \right\|^2 \leq C|a|^{2n} \tag{10}$$

from which and since $|a| < 1$, the relation in (5) holds, and completes the proof.

2.2 The unconditional Ritt's condition

Once we have completely identified the composition operators which satisfy the Ritt's resolvent condition, we may proceed to identify further those operators which satisfy the unconditional Ritt's condition. We recall that an operator T on a Banach space $\mathcal{X}$ satisfies the unconditional Ritt's condition if there exists a nonnegative constant K such that

$$\left\| \sum_{k=1} a_k \left(T^k - T^{k-1} \right) \right\| \leq K \sup_k \{ |a_k| \} \tag{11}$$

for any finite sequence (a_k) of complex numbers. We note that the notion of the unconditional Ritt's condition is the discrete analogue of the H^∞ calculus for sectorial operators [8]. The condition plays a roll in identifying operators which have ℓ_1 -maximal regularity property: see [3] for more details.

Kalton and Portal [3] proved that the unconditional Ritt's condition implies the Ritt's resolvent condition in general, but not conversely. However, for the composition operator on the Fock space, it turns out that these two conditions are evidently equivalent.

Corollary 1 *Let the composition operator C_ψ be bounded on $\mathcal{F}_2$. Then C_ψ satisfies the Ritt's resolvent condition on $\mathcal{F}_2$ if and only if it satisfies the unconditional Ritt's condition.*

Proof We only need to prove one side of the implication. The other side has already been disclosed above. As in the proof above, set $\psi(z) = az + b$ and assume that C_ψ satisfy condition (3). We need to show that (11) holds too. Then by Theorem 1, either $a = 1$ or $|a| < 1$. For the first where C_ψ reduces to the identity map, the required conclusion follows trivially. Thus, assume that $\psi(z) = az + b$ with $|a| < 1$. In this case by (10), for any finite sequence (a_k) of complex numbers

$$\begin{aligned} \left\| \sum_{k=1} a_k \left(C\psi^k - C_\psi^{k-1} \right) \right\| &\leq \sqrt{C} \sum_{k=1}^{\infty} |a_k||a|^k \leq \sqrt{C} \sup_k \{ |a_k| \} \sum_{k=1}^{\infty} |a|^k \\ &= \sqrt{C} \sup_k \{ |a_k| \} \frac{1}{1 - |a|} < \infty. \end{aligned}$$

We finish this note with the following observation on functional calculus of the composition operators. Once the operators which satisfy the Ritt's condition are identified, the next natural question is to study their polynomial calculus. In [5], Le Merdy proved that any operator T satisfying the Ritt's condition on a Hilbert space is polynomially bounded, that is

$$\sup\{\|p(T)\| \,:\, p \ \text{polynomial}, \ \|p\|_\infty \leq 1\} < \infty,$$

if and only if it is similar to a contraction. We note in passing that there exists operator on Hilbert spaces which satisfy the Ritt's condition but not polynomially bounded; see [5] for counter examples. □

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