Spontaneous symmetry breaking in the S_3-symmetric scalar sector

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ABSTRACT: We present a detailed study of the vacua of the S_3-symmetric three-Higgs-doublet potential, specifying the region of parameters where these minimisation solutions occur. We work with a CP conserving scalar potential and analyse the possible real and complex vacua with emphasis on the cases in which the CP symmetry can be spontaneously broken. Results are presented both in the reducible-representation framework of Derman, and in the irreducible-representation framework. Mappings between these are given. Some of these implementations can in principle accommodate dark matter and for that purpose it is important to identify the residual symmetries of the potential after spontaneous symmetry breakdown. We are also concerned with constraints from vacuum stability.

Contents

1 Introduction

A possible direction in which to look for new physics beyond the standard model emerges by enlarging the scalar sector, e.g., by adding one or more scalar doublets. Models with two Higgs doublets [1] have received a lot of attention (for a recent review, see [2]). Attractive features of such extensions are additional sources of CP violation [3–14], and a way to accommodate dark matter [15, 16].

Scalar sectors with three doublets have been considered, sometimes inspired by the existence of three generations of fermions. The general case involves a large number of free parameters [17], and these parameters are only constrained by general principles, like positivity, and a viable spectrum. Several authors, starting in 1977 with Pakvasa and Sugawara [18], have imposed an S_3 permutation symmetry, in part motivated by a desire to model the fermion generations. As compared to the most general three-Higgs-doublet model, it is also attractive since it reduces the number of parameters.

Pakvasa and Sugawara exploited the reduction of S_3 to its irreducible doublet and singlet. However, their potential, which has seven quartic terms, was later [19] shown not to be the most general one, the term here referred to as the λ_4-term, was missing. In the sequel we show that this term plays a very important rôle. An alternative formulation in terms of the three Higgs doublets in the reducible representation transforming under the six three-by-three matrices of permutation (to be referred to in the following as the reducible triplet) was presented by Derman [20] and further studied in a paper with Tsao [21].

The descriptions of the potential in terms of the reducible-triplet and the irreducible frameworks are equivalent. But other sectors of the theory, in particular the Yukawa sector, would differentiate these frameworks and lead to different physics.

The vacua of the S_3-symmetric potential have been classified in terms of their residual symmetries by Ivanov and Nishi [22]. Here, we shall present another classification, namely in terms of constraints on the potential. For real vacua, it was known that the condition $\lambda_4 = 0$ (see below) is relevant for the classification of the different vacua. We find that this parameter is also important for the complex vacua. Furthermore, there are several other constraints that are useful in this classification. Such constraints are very important for model-building purposes. Therefore, we give a complete list of all possible vacua, starting from a scalar potential with real coefficients, and analyse their properties, giving the constraints on the parameter space which allow for each solution.

The paper is organised as follows. In section 2 we set up some notation and discuss the potential. Section 3 is devoted to a general discussion of how the different vacua constrain the potential, whereas sections 4 and 5 present our results for the real and complex vacua. In section 6 we discuss connections between complex and real vacua, and in section 7

we briefly comment on the special case of $\lambda_4 = 0$, when the potential has an additional, continuous symmetry. Section 8 is devoted to a detailed discussion of spontaneous CP violation, and in section 9 we comment on dark-matter scenarios. Section 10 contains some concluding remarks. Technical issues are delegated to appendices A (mappings), B (positivity) and C (stationarity conditions).

2 The S_3-symmetric potential

2.1 Field notations

We consider S_3-symmetric models with three $\mathrm{SU}(2) \times \mathrm{U}(1)$ reducible-triplet fields:

$$\phi_1, \quad \phi_2, \quad \phi_3. \tag{2.1}$$

Allowing for complex vacuum expectation values, each field is decomposed as

$$\phi_i = \begin{pmatrix} \varphi_i^+ \\ (\rho_i + \eta_i + i\chi_i)/\sqrt{2} \end{pmatrix}, \quad i = 1, 2, 3, \tag{2.2}$$

where ρ_i is in general complex, whereas the fields η_i and χ_i are real.

The reducible-triplet fields can alternatively be replaced by an S_3 doublet:

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_1 - \phi_2) \\ \frac{1}{\sqrt{6}}(\phi_1 + \phi_2 - 2\phi_3) \end{pmatrix}, \tag{2.3}$$

and an S_3 singlet

$$h_S = \frac{1}{\sqrt{3}}(\phi_1 + \phi_2 + \phi_3), \tag{2.4}$$

decomposed as

$$h_i = \begin{pmatrix} h_i^+ \\ (w_i + \tilde{\eta}_i + i\tilde{\chi}_i)/\sqrt{2} \end{pmatrix}, \quad i = 1, 2, \quad h_S = \begin{pmatrix} h_S^+ \\ (w_S + \tilde{\eta}_S + i\tilde{\chi}_S)/\sqrt{2} \end{pmatrix}, \tag{2.5}$$

where again w_i and w_S can be complex. Finally, the reducible-triplet fields may be replaced by a doublet and a pseudosinglet, denoted h_A, in which case the potential will take a slightly different form.

The potential, which has a quadratic and a quartic part,

$$V = V_2 + V_4 \tag{2.6}$$

can be expressed either in terms of the reducible-triplet fields ϕ_i, in terms of h_1, h_2, and h_S, or in terms of h_1, h_2, and h_A. The first two formulations are equivalent.

It is useful to note that the (complex) vevs are related:

$$w_1 = \frac{1}{\sqrt{2}}(\rho_1 - \rho_2), \tag{2.7a}$$

$$w_2 = \frac{1}{\sqrt{6}}(\rho_1 + \rho_2 - 2\rho_3), \tag{2.7b}$$

$$w_S = \frac{1}{\sqrt{3}}(\rho_1 + \rho_2 + \rho_3), \tag{2.7c}$$

with the inversion

$$\rho_1 = \frac{1}{\sqrt{3}}w_S + \frac{1}{\sqrt{2}}w_1 + \frac{1}{\sqrt{6}}w_2, \tag{2.8a}$$

$$\rho_2 = \frac{1}{\sqrt{3}}w_S - \frac{1}{\sqrt{2}}w_1 + \frac{1}{\sqrt{6}}w_2, \tag{2.8b}$$

$$\rho_3 = \frac{1}{\sqrt{3}}w_S - \frac{\sqrt{2}}{\sqrt{3}}w_2. \tag{2.8c}$$

Whereas the formulation in terms of reducible-triplet fields is symmetric in ϕ_1, ϕ_2, ϕ_3, the singlet-doublet representation is not. The decomposition into the doublet and singlet representations singles out a direction in terms of the ϕ fields. Any permutation of ϕ_i fields in eq. (2.3) would lead to an equally good definition for the components of the doublet. This is a trivial fact. However, this is the reason why in the tables of possible vacuum states that follow, some cases that are equivalent in terms of vacuum states of the reducible-triplet representation, given by (2.8), have to be split into different cases in terms of those of the irreducible framework (2.7) corresponding to different consistency conditions in terms of the minimisation of the potential.

2.2 The potential in terms of reducible-triplet fields

In terms of the reducible-triplet fields, the potential was written by Derman [20] as:

$$V_2 = -\lambda \sum_i \phi_i^\dagger \phi_i + \frac{1}{2}\gamma \sum_{i<j}\left[\phi_i^\dagger \phi_j + \text{h.c.}\right], \tag{2.9a}$$

$$V_4 = A \sum_i (\phi_i^\dagger \phi_i)^2 + \sum_{i<j}\left\{ C(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \overline{C}(\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i) + \frac{1}{2}D\left[(\phi_i^\dagger \phi_j)^2 + \text{h.c.}\right]\right\}$$

$$+ \frac{1}{2}E_1 \sum_{i\neq j}\left[(\phi_i^\dagger \phi_i)(\phi_i^\dagger \phi_j) + \text{h.c.}\right] + \sum_{i\neq j\neq k\neq i, j<k}\left\{\frac{1}{2}E_2\left[(\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_i) + \text{h.c.}\right]\right.$$

$$\left. + \frac{1}{2}E_3\left[(\phi_i^\dagger \phi_i)(\phi_k^\dagger \phi_j) + \text{h.c.}\right] + \frac{1}{2}E_4\left[(\phi_i^\dagger \phi_j)(\phi_i^\dagger \phi_k) + \text{h.c.}\right]\right\}. \tag{2.9b}$$

There are ten different coefficients in these equations.

2.3 The potential in terms of S_3 singlet and doublet fields

In terms of the S_3 singlet and doublet fields, the potential can be written as [19, 23, 24]:

$$V_2 = \mu_0^2 h_S^\dagger h_S + \mu_1^2(h_1^\dagger h_1 + h_2^\dagger h_2), \tag{2.10a}$$

$$V_4 = \lambda_1(h_1^\dagger h_1 + h_2^\dagger h_2)^2 + \lambda_2(h_1^\dagger h_2 - h_2^\dagger h_1)^2 + \lambda_3\left[(h_1^\dagger h_1 - h_2^\dagger h_2)^2 + (h_1^\dagger h_2 + h_2^\dagger h_1)^2\right]$$

$$+ \lambda_4\left[(h_S^\dagger h_1)(h_1^\dagger h_2 + h_2^\dagger h_1) + (h_S^\dagger h_2)(h_1^\dagger h_1 - h_2^\dagger h_2) + \text{h.c.}\right] + \lambda_5(h_S^\dagger h_S)(h_1^\dagger h_1 + h_2^\dagger h_2)$$

$$+ \lambda_6\left[(h_S^\dagger h_1)(h_1^\dagger h_S) + (h_S^\dagger h_2)(h_2^\dagger h_S)\right] + \lambda_7\left[(h_S^\dagger h_1)(h_S^\dagger h_1) + (h_S^\dagger h_2)(h_S^\dagger h_2) + \text{h.c.}\right]$$

$$+ \lambda_8(h_S^\dagger h_S)^2. \tag{2.10b}$$

(Teshima [23] uses $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \leftrightarrow (C, D, G, E, B, F, F', A)$.) In appendix A we give the translation between the parametrisations of the potential in terms of reducible-triplet fields and the one in terms of singlet and doublet fields.

Once again there are ten independent parameters. There are only four terms in this potential that are sensitive to the relative phase of different doublets, those in λ_2, λ_3, λ_4 and those in λ_7. In terms of the reducible-triplet fields, in eqs. (2.9) the number of such terms is higher since here we have γ, D, E_1, E_2, E_3, and E_4.

In this formulation it is clear that the potential has an extra Z_2 symmetry of the form $h_1 \to -h_1$. In terms of the equivalent doublet representation:

$$\begin{pmatrix} \hat{\chi}_1 \\ \hat{\chi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{2.11}$$

which has also been used in the literature [25, 26], the above symmetry translates into a symmetry for the interchange of the fields $\hat{\chi}_1$ and $\hat{\chi}_2$.

Another interesting feature is the fact that the choice $\lambda_4 = 0$ leads to a continuous SO(2) symmetry defined by:

$$\begin{pmatrix} h'_1 \\ h'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{2.12}$$

This shows that the term with λ_4 plays a special rôle in the potential.

2.4 The potential in terms of S_3 pseudosinglet and doublet fields

Instead of choosing the three Higgs doublets as being the singlet and the doublet irreducible representations of S_3 we may choose them to be a pseudosinglet, and the doublet. These are also irreducible representations. Under S_3 the pseudosinglet, h_A, transforms into $(-h_A)$. In this case there is no direct translation of these fields into the defining reducible representation.

In terms of the S_3 pseudosinglet and doublet fields, the potential can be written as

$$V_2 = \mu_0^2 h_A^\dagger h_A + \mu_1^2 (h_1^\dagger h_1 + h_2^\dagger h_2), \tag{2.13a}$$

$$\begin{aligned} V_4 &= \lambda_1 (h_1^\dagger h_1 + h_2^\dagger h_2)^2 + \lambda_2 (h_1^\dagger h_2 - h_2^\dagger h_1)^2 + \lambda_3 \left[(h_1^\dagger h_1 - h_2^\dagger h_2)^2 + (h_1^\dagger h_2 + h_2^\dagger h_1)^2 \right] \\ &\quad + \lambda_4 \left[(h_A^\dagger h_2)(h_1^\dagger h_2 + h_2^\dagger h_1) - (h_A^\dagger h_1)(h_1^\dagger h_1 - h_2^\dagger h_2) + \text{h.c.} \right] + \lambda_5 (h_A^\dagger h_A)(h_1^\dagger h_1 + h_2^\dagger h_2) \\ &\quad + \lambda_6 \left[(h_A^\dagger h_1)(h_1^\dagger h_A) + (h_A^\dagger h_2)(h_2^\dagger h_A) \right] + \lambda_7 \left[(h_A^\dagger h_1)(h_A^\dagger h_1) + (h_A^\dagger h_2)(h_A^\dagger h_2) + \text{h.c.} \right] \\ &\quad + \lambda_8 (h_A^\dagger h_A)^2. \end{aligned} \tag{2.13b}$$

Apart from the "trivial" substitution $h_S \leftrightarrow h_A$, the two formulations (2.10b) and (2.13b) differ in the λ_4-term, the two doublet fields are interchanged: $h_1 \leftrightarrow h_2$. Within the constraint of renormalizability (only quadratic and quartic terms) this scalar potential is equivalent to the previous one. However, this choice of representations will obviously have implications for the Yukawa sector. We do not examine these implications in the present work.

In the discussion of vacua, all results obtained for the irreducible framework in terms of the S_3 singlet and doublet can be trivially translated into this case. Therefore, our discussion will only refer to two different frameworks.

2.5 Positivity

Das and Dey have given *necessary* conditions for positivity [24]. For the general potential, the *sufficient* conditions are rather involved. However, in the case of $\lambda_4 = 0$, they can be expressed quite explicitly, and are given in appendix B.

3 The vacua — generalities

Since we are interested in CP violation, we will in general allow some vacuum expectation values (vevs) to be complex. However, due to the U(1) invariance of the potential, one vev can always be chosen real. This holds in both frameworks.

The vacua can be determined from the conditions that derivatives of the potential with respect to the three independent fields must vanish. These derivatives are linear in the coefficients of the potential, but cubic in terms of the (complex) vacuum expectation values. One approach would be to take the potential parameters as input, and solve these cubic equations for the vevs. In this section we shall follow another approach, which is to take the vevs as input, and use the derivatives to constrain the potential. The quartic potential will also be constrained by positivity and an imposed particle spectrum.

We shall start this discussion by first quoting the minimisation conditions in the two frameworks. Clearly, one and the same vacuum will be phrased differently in the two frameworks. But one framework may give a simpler description than the other.

After writing out these derivatives in the next subsections, we shall first discuss how these conditions constrain the potential. Then (in section 4), we review the real case (no CP violation), followed (in section 5) by a discussion of the complex case, which may accommodate spontaneous CP violation.

3.1 The reducible-triplet framework

Within the reducible-triplet framework, three complex derivatives must vanish:

$$\frac{\partial V}{\partial \rho_i^*} = 0, \quad i = 1, 2, 3,$$ (3.1)

where

$$\begin{aligned}
\frac{\partial V}{\partial \rho_1^*} &= \frac{-1}{2}\rho_1\lambda + \frac{1}{4}(\rho_2 + \rho_3)\gamma + \frac{1}{2}\rho_1^*\rho_1^2 A + \frac{1}{4}\rho_1(|\rho_2|^2 + |\rho_3|^2)(C + \overline{C}) \\
&\quad + \frac{1}{4}\rho_1^*(\rho_2^2 + \rho_3^2)D + \frac{1}{8}[2|\rho_1|^2(\rho_2 + \rho_3) + \rho_1^2(\rho_2^* + \rho_3^*) + \rho_2^*\rho_2^2 + \rho_3^*\rho_3^2]E_1 \\
&\quad + \frac{1}{8}[\rho_1(\rho_2^*\rho_3 + \rho_2\rho_3^*) + |\rho_2|^2\rho_3 + \rho_2|\rho_3|^2](E_2 + E_3) \\
&\quad + \frac{1}{8}(2\rho_1^*\rho_2\rho_3 + \rho_2^*\rho_3^2 + \rho_2^2\rho_3^*)E_4,
\end{aligned}$$ (3.2)

and $\partial V/\partial\rho_2^*$ and $\partial V/\partial\rho_3^*$ can be obtained by cyclic permutations.

We note that these derivatives do not depend on C and $\overline{C}$ separately, only on the sum, $C + \overline{C}$. Likewise, they only depend on E_2 and E_3 via their sum. This means that the vacuum conditions are independent of the space spanned by the two parameters orthogonal to these, namely $C - \overline{C}$ and $E_2 - E_3$. However, the spectrum will depend also on these parameters.

3.2 The irreducible framework: singlet and doublet fields

The three relevant derivatives that must vanish are now

$$\frac{\partial V}{\partial w_S^*} = \frac{1}{2} w_S \mu_0^2 + \frac{1}{4}[2|w_1|^2 w_2 + w_2^*(w_1^2 - w_2^2)]\lambda_4$$
$$+ \frac{1}{4} w_S(|w_1|^2 + |w_2|^2)(\lambda_5 + \lambda_6) + \frac{1}{2} w_S^*(w_1^2 + w_2^2)\lambda_7 + \frac{1}{2} w_S^* w_S^2 \lambda_8 = 0, \qquad (3.3)$$

$$\frac{\partial V}{\partial w_1^*} = \frac{1}{2} w_1 \mu_1^2 + \frac{1}{2} w_1(|w_1|^2 + |w_2|^2)\lambda_1 + \frac{1}{2} w_2(w_1^* w_2 - w_1 w_2^*)\lambda_2$$
$$+ \frac{1}{2} w_1^*(w_1^2 + w_2^2)\lambda_3 + \frac{1}{2}(w_1^* w_2 w_S + w_1 w_2^* w_S + w_1 w_2 w_S^*)\lambda_4$$
$$+ \frac{1}{4} w_1 |w_S|^2(\lambda_5 + \lambda_6) + \frac{1}{2} w_1^* w_S^2 \lambda_7 = 0, \qquad (3.4)$$

$$\frac{\partial V}{\partial w_2^*} = \frac{1}{2} w_2 \mu_1^2 + \frac{1}{2} w_2(|w_1|^2 + |w_2|^2)\lambda_1 - \frac{1}{2} w_1(w_1^* w_2 - w_1 w_2^*)\lambda_2$$
$$+ \frac{1}{2} w_2^*(w_1^2 + w_2^2)\lambda_3 + \frac{1}{4}[2(|w_1|^2 - |w_2|^2)w_S + (w_1^2 - w_2^2)w_S^*]\lambda_4$$
$$+ \frac{1}{4} w_2 |w_S|^2(\lambda_5 + \lambda_6) + \frac{1}{2} w_2^* w_S^2 \lambda_7 = 0. \qquad (3.5)$$

We note that these derivatives do not depend on λ_5 and λ_6 separately, only on the sum, $\lambda_5 + \lambda_6$. Likewise, they do not depend on λ_1, λ_2 and λ_3 separately, only on two combinations orthogonal to $\lambda_1 + \lambda_2 - 2\lambda_3 = 0$.

3.3 Constraining the potential by the vevs

We are interested in the possibility of having spontaneous CP violation, therefore we impose that all the parameters of the potential should be real. Let us now consider the vevs as given a priori and solve the above minimisation conditions in terms of parameters of the potential. Our basic discussion will be in the reducible triplet framework. In this case the three vevs can be denoted as:

$$\rho_i = v_i e^{i\tau_i}, \qquad (3.6)$$

and we can write six minimisation conditions by computing the derivatives of V with respect to each of the v_i's and of the τ_i's. It is clear from eqs. (2.9) that, concerning phases, the potential is only sensitive to phase differences. In particular, we could choose without loss of generality a phase convention where one of these phases is rotated away, however, in this case we would loose symmetry among these equations. The explicit forms of these equations are given in appendix C.

As mentioned above, the pair of coefficients C and $\overline{C}$ as well as the pair E_2 and E_3 occur in each equation with a common factor, and therefore we are left with eight independent

combinations of coefficients and five independent real equations which should be chosen as the three equations obtained from $\partial V/\partial v_i = 0$ and any pair of those from $\partial V/\partial \tau_i = 0$. We could in principle solve these equations for any set of five of the eight independent parameters of the potential. These equations take the form:

$$
\begin{aligned}
a_{11}P_1 + a_{12}P_2 + a_{13}P_3 + a_{14}P_4 + a_{15}P_5 &= b_1, \\
a_{21}P_1 + a_{22}P_2 + a_{23}P_3 + a_{24}P_4 + a_{25}P_5 &= b_2, \\
a_{31}P_1 + a_{32}P_2 + a_{33}P_3 + a_{34}P_4 + a_{35}P_5 &= b_3, \\
a_{41}P_1 + a_{42}P_2 + a_{43}P_3 + a_{44}P_4 + a_{45}P_5 &= b_4, \\
a_{51}P_1 + a_{52}P_2 + a_{53}P_3 + a_{54}P_4 + a_{55}P_5 &= b_5,
\end{aligned}
\tag{3.7}
$$

where the P_i denote different parameters of the potential. However not all of the possible $\binom{8}{5} = 56$ combinations will lead to five independent equations.

These five equations define five hyperplanes in the parameter space. In the case of the reducible-representation framework, since C and $\overline{C}$ appear together, as do E_2 and E_3, we have effectively an 8-dimensional parameter space. Where the 5 hyperplanes intersect, we then have an $8 - 5 = 3$-dimensional parameter space, over which the vacuum is the same.

The requirement for the five equations to be independent is that the determinant of the matrix $\mathcal{A}$ defined by:

$$
\mathcal{A} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{pmatrix},
\tag{3.8}
$$

should be different from zero. It can readily be verified that the coefficients of the three parameters λ, A and $(C + \overline{C})$ are not independent and therefore these equations cannot be solved simultaneously for these three parameters. The terms with these coefficients are not sensitive to the relative phases and therefore they do not appear in the equations obtained from differentiating with respect to the phases. As a result, in order to check this point it suffices to compute the 3×3 determinant involving the coefficients obtained from the first three minimisation conditions. This determinant is zero.

In the case of no spontaneous CP violation, the relative phases of the ρ_i are zero and the corresponding minimisation condition, obtained from eq. (C.3) and cyclic permutations, reduce to $0 = 0$ since each term in these equations is proportional to the sine of relative phases. We are then left with only three independent equations and we can solve at most for three parameters of the potential.

Returning to the complex case, we are now ready to classify the vacua, according to how many independent equations we have. In order for the five equations to be independent, it is sufficient that one of these 56 determinants be non-zero. Conversely, in order for at most four of the equations to be independent, all 56 possible such 5×5 determinants must vanish.

For arbitrary vevs,

$$
v_1, \quad v_2 e^{i\tau_2}, \quad v_3 e^{i\tau_3},
\tag{3.9}
$$

we find that 16 out of the 56 possible 5×5 determinants vanish identically, whereas the remaining 40 are non-zero. The five equations (3.7) can for any of these choices be solved in terms of the five parameters $P_1, \ldots, P_5$, with the exception of 5-parameter sets containing $(C, \overline{C})$, (E_2, E_3), (λ, A, C) or $(\lambda, \gamma, E_1, E_2)$. The complements of these account for 14 out of the 16 vanishing ones. The remaining two are (γ, A, C, E_1, E_2) and (γ, C, D, E_2, E_4). In these sets, C could be replaced by $\overline{C}$, and E_2 by E_3.

The remaining 40 determinants factorise, and vanish when either

$$\rho_i = 0, \quad i = 1, 2, 3 \quad \text{or} \tag{3.10}$$

$$\rho_i = \rho_j, \quad j \neq i. \tag{3.11}$$

In these cases we can have at most 4 independent equations among the set (3.7), and must investigate the corresponding 4×4 sub-determinants.

In the irreducible-representation framework, since λ_5 and λ_6 only appear as a sum in the minimisation conditions, we have effectively 9 parameters. Thus, we could have $\binom{9}{5} = 126$ different 5×5 determinants. However, only 19 of these are non-vanishing. In this sense, this framework is more "compact". Here, the following parameter sets can not appear among the 5: (μ_0^2, λ_8), (μ_1^2, λ_1), (λ_5, λ_6), $(\mu_0^2, \mu_1^2, \lambda_5)$, $(\mu_0^2, \lambda_1, \lambda_5)$, $(\mu_1^2, \lambda_2, \lambda_3)$, $(\mu_1^2, \lambda_5, \lambda_8)$, $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1, \lambda_5, \lambda_8)$, $(\mu_0^2, \lambda_2, \lambda_3, \lambda_5)$, $(\lambda_2, \lambda_3, \lambda_5, \lambda_8)$, and $(\lambda_3, \lambda_4, \lambda_5, \lambda_8)$, as well as sets where in the above list λ_5 is replaced by λ_6. Among these, the sets (λ_5, λ_6), and $(\lambda_1, \lambda_2, \lambda_3)$ correspond to $(C, \overline{C})$ and (E_2, E_3) in the reducible-representation framework.

We shall distinguish the real and complex cases.

4   Real vacua

For a real vacuum, the five equations (3.7) discussed above reduce to a set of three. Again, they are not necessarily all independent. If we, for example, try to solve for λ, γ and A, the 3×3 determinant corresponding to (3.8) is particularly simple:

$$\det \mathcal{A}_{3 \times 3} = -(\rho_1 + \rho_2 + \rho_3)(\rho_2 - \rho_1)(\rho_3 - \rho_2)(\rho_1 - \rho_3). \tag{4.1}$$

Thus, when this quantity is non-zero, we can solve for λ, γ and A. Conversely, when $\det \mathcal{A}_{3 \times 3} = 0$ (meaning the sum of the vevs is zero, or two are equal), then we have at most two independent equations, and can for example only solve for λ and γ.

In the irreducible-representation framework, the three vacuum conditions (3.3)–(3.5) can be solved to give μ_0^2 and μ_1^2 in terms of the quartic coefficients:[1]

$$\mu_0^2 = \frac{1}{2w_S} \left[\lambda_4(w_2^2 - 3w_1^2)w_2 - (\lambda_5 + \lambda_6 + 2\lambda_7)(w_1^2 + w_2^2)w_S - 2\lambda_8 w_S^3 \right], \tag{4.2a}$$

$$\mu_1^2 = -\frac{1}{2} \left[2(\lambda_1 + \lambda_3)(w_1^2 + w_2^2) + 6\lambda_4 w_2 w_S + (\lambda_5 + \lambda_6 + 2\lambda_7)w_S^2 \right], \tag{4.2b}$$

$$\mu_1^2 = -\frac{1}{2} \left[2(\lambda_1 + \lambda_3)(w_1^2 + w_2^2) - 3\lambda_4(w_2^2 - w_1^2)\frac{w_S}{w_2} + (\lambda_5 + \lambda_6 + 2\lambda_7)w_S^2 \right]. \tag{4.2c}$$

---

[1]There are misprints in the corresponding expressions given in ref. [24], their eq. (9): (i) a factor of $1/2$ is missing on the right-hand side of all three expressions, and (ii) in μ_0^2 (μ_3^2 in their notation) the coefficient of λ_4 should be $(v_2/2v_3)(v_2^2 - 3v_1^2)$. These misprints were corrected in the Erratum provided by the authors and included in ref. [24].

The two equations (4.2b) and (4.2c) are not valid when $w_1 = 0$ and $w_2 = 0$, since they were derived from (3.4) and (3.5) dividing by $w_1 = 0$ and $w_2 = 0$, respectively. Furthermore, they are not automatically consistent. Consistency requires

$$w_1 = 0, \qquad \text{or else} \tag{4.3a}$$

$$\lambda_4(3w_2^2 - w_1^2)w_S = 0. \tag{4.3b}$$

For $w_1 = 0$ the derivative of the potential with respect to w_1 is identically zero and therefore there is no clash in the determination of μ_1^2 from the derivative with respect to w_2. From equation (4.3b) we see that these two derivatives are consistent if either $\lambda_4 = 0$ or $w_1 = \pm\sqrt{3}w_2$ or else $w_S = 0$. The case $w_S = 0$ is special since if we now take into account the derivative of the potential with respect to w_S, which is given by eq. (3.3), we are left in the real case with

$$\lambda_4 w_2(3w_1^2 - w_2^2) = 0, \tag{4.4}$$

which is the only term in eq. (3.3) where w_S does not appear as a factor. As a result, solutions with $w_S = 0$ require in addition that $\lambda_4 = 0$ or $w_2 = \pm\sqrt{3}w_1$, or else $w_2 = 0$. See cases R-I-2 in table 1. These do not require $\lambda_4 = 0$, while case R-II-3 has $w_S = 0$ and requires $\lambda_4 = 0$.

The different solutions can be summarised as given in table 1, where the descriptions in terms of both the reducible- and irreducible-representation frameworks are given. For the purpose of making this table as well as the corresponding one for complex vacua more compact, we introduce the abbreviations

$$\lambda_a = \lambda_5 + \lambda_6 + 2\lambda_7, \tag{4.5a}$$

$$\lambda_b = \lambda_5 + \lambda_6 - 2\lambda_7. \tag{4.5b}$$

One should note that

- Vacuum R-I-1 is a special case of Vacuum R-II-1. In this case, the vacuum value x is determined by

$$\lambda - \gamma = x^2[A + C + \overline{C} + D + 2E_1 + E_2 + E_3 + E_4]. \tag{4.6}$$

- For Vacuum R-I-1, in the irreducible framework, we have

$$\mu_0^2 = -w_S^2\lambda_8, \tag{4.7}$$

which corresponds to eq. (4.6), with $w_S^2 = 3x^2$.

- In the Vacua R-I-2a, R-I-2b, R-I-2c, the vacuum value x is determined by:

$$2\lambda + \gamma = x^2[2A + C + \overline{C} + D - 2E_1]. \tag{4.8}$$

In the irreducible framework, μ_0^2 is not constrained by eq. (4.2a), whereas $\mu_1^2 = -\lambda - \frac{1}{2}\gamma$ is determined by the minimisation condition given above.

Vacuum	ρ_1, ρ_2, ρ_3	w_1, w_2, w_S	Comment
R-0	$0, 0, 0$	$0, 0, 0$	Not interesting
R-I-1	x, x, x	$0, 0, w_S$	$\mu_0^2 = -\lambda_8 w_S^2$
R-I-2a	$x, -x, 0$	$w, 0, 0$	$\mu_1^2 = -\left(\lambda_1 + \lambda_3\right) w_1^2$
R-I-2b	$x, 0, -x$	$w, \sqrt{3}w, 0$	$\mu_1^2 = -\frac{4}{3}\left(\lambda_1 + \lambda_3\right) w_2^2$
R-I-2c	$0, x, -x$	$w, -\sqrt{3}w, 0$	$\mu_1^2 = -\frac{4}{3}\left(\lambda_1 + \lambda_3\right) w_2^2$
R-II-1a	x, x, y	$0, w, w_S$	$\mu_0^2 = \frac{1}{2}\lambda_4 \frac{w_2^3}{w_S} - \frac{1}{2}\lambda_a w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right) w_2^2 + \frac{3}{2}\lambda_4 w_2 w_S - \frac{1}{2}\lambda_a w_S^2$
R-II-1b	x, y, x	$w, -w/\sqrt{3}, w_S$	$\mu_0^2 = -4\lambda_4 \frac{w_2^3}{w_S} - 2\lambda_a w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -4\left(\lambda_1 + \lambda_3\right) w_2^2 - 3\lambda_4 w_2 w_S - \frac{1}{2}\lambda_a w_S^2$
R-II-1c	y, x, x	$w, w/\sqrt{3}, w_S$	$\mu_0^2 = -4\lambda_4 \frac{w_2^3}{w_S} - 2\lambda_a w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -4\left(\lambda_1 + \lambda_3\right) w_2^2 - 3\lambda_4 w_2 w_S - \frac{1}{2}\lambda_a w_S^2$
R-II-2	$x, x, -2x$	$0, w, 0$	$\mu_1^2 = -\left(\lambda_1 + \lambda_3\right) w_2^2, \ \lambda_4 = 0$
R-II-3	$x, y, -x-y$	$w_1, w_2, 0$	$\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\left(w_1^2 + w_2^2\right), \lambda_4 = 0$
R-III	ρ_1, ρ_2, ρ_3	w_1, w_2, w_S	$\mu_0^2 = -\frac{1}{2}\lambda_a(w_1^2 + w_2^2) - \lambda_8 w_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\left(w_1^2 + w_2^2\right) - \frac{1}{2}\lambda_a w_S^2,$ $\lambda_4 = 0$

Table 1. Possible real vacua (partly after Derman and Tsao [21]). The classification of vacua uses the notation R-X-y, where R means that the vacuum is real. The roman numeral X is the number of constraints on the parameters of the potential that arise from solving the stationary-point equations. The letter y is used for distinguishing different vev's that have the same X, and λ_a is defined in eq. (4.5).

Vacua R-I-2a, R-I-2b and R-I-2c, which correspond to $w_S = 0$, require special discussion. It is clear from eq. (4.2a) that μ_0^2 remains undetermined. According to eqs. (4.2b) and (4.2c), in these vacua the following relation must hold:

$$\mu_1^2 = -(w_1^2 + w_2^2)(\lambda_1 + \lambda_3). \tag{4.9}$$

(Special cases are given in table 1.) Using the translation given in appendix A, and taking $2x^2 = w_1^2 + w_2^2$, this constraint is seen to be equivalent to (4.8).

- There are also solutions with $(\rho_1, \rho_2, \rho_3) = (x, x, -2x)$, $(x, -2x, x)$, and $(-2x, x, x)$. These are reminiscent of vacua R-I-2a, R-I-2b and R-I-2c, with the interchange of w_1 and w_2.

- In the Vacua R-II-1 and R-III, the two coefficients of the bilinear potential, (μ_0^2 and μ_1^2) or (λ and γ), can be determined from chosen vacuum values, together with the quartic potential.

- Vacua R-II are characterised by *two* independent vevs, referred to as x and y in the reducible-triplet framework, and as w and w_S in the irreducible framework. In the framework of the reducible-triplet representation the three permutations presented as subcases a), b) and c) are trivial, however, it should be pointed out that in the irreducible framework different consistency conditions apply in each case, either $w_1 = 0$ or $w_1 = -\sqrt{3}w_2$ or $w_1 = \sqrt{3}w_2$.

- Vacuum R-III requires $\lambda_4 = 0$ or (in the reducible-triplet framework):

$$4A - 2(C + \overline{C} + D) - E_1 + E_2 + E_3 + E_4 = 0. \tag{4.10}$$

In this sense, only two of the three minimisation equations are independent. As a special case of this solution, we can also have $w_S = 0$. This is R-II-3, with μ_0^2 unconstrained.

Table 1 clearly illustrates the point we have made before about the translation from the reducible-representation framework (RRF) to the irreducible one (IRF). The splitting of the R-I case into three cases (a, b, c) would be meaningless due to the S_3 symmetry, if we were only considering ρ_1, ρ_2 and ρ_3. However, in terms of w_1, w_2 and w_S they appear as different cases. The consistency of the derivatives with respect to $\hat{w}_1$ and $\hat{w}_2$ in this framework is verified since they all have $w_S = 0$. However now, taking into account the derivative with respect to w_S these solutions must obey eq. (4.4) and each case fulfils this requirement in a different way. We have $w_2 = 0$, $w_2 = \sqrt{3}w_1$ and $w_2 = -\sqrt{3}w_1$ in the three cases. Another similar example is case R-II-1. Here the difference is that consistency of the derivatives with respect to $\hat{w}_1$ and $\hat{w}_2$ in the IRF is verified for $w_1 = 0$, $w_1 = -\sqrt{3}w_2$ and $w_1 = \sqrt{3}w_2$, respectively. Since in this case we do not impose $w_S = 0$ the constraint of eq. (4.4) does not apply.

The high-scale validity of models based on two of these vacua, namely R-I-1 and R-II-1c, has recently been studied in ref. [27].

Special limits. Some of the vacua listed in table 1 can be seen as special limits[2] of another, more general case. These include

- R-I-1 is contained in R-II-1a, 1b, 1c for $w_2 = 0$ (or $x = y$).

- R-II-2 is contained in R-II-3 with $w_1 = 0$ (or $x = y$).

- R-II-1a is contained in R-III with $w_1 = 0$ (or $\rho_1 = \rho_2$).

- R-II-1a, 1b, 1c, with $\lambda_4 = 0$, are special cases of R-III.

- R-I-2a, 2b, 2c are contained in R-II-3 in the special limits of $w_2 = 0$, $w_2 = \sqrt{3}w_1$, and $w_2 = -\sqrt{3}w_1$, respectively.

---

[2]The terminology "special limits" is not perfect. While a vacuum specification R-X-y is obtained as a special limit of the specification R-X′-y′, the constraints defining R-X′-y′ may be a subset of those defining R-X-y. This is analogous to the discussion of real "origins" of complex vacua in section 6.

5 Complex vacua

5.1 The Irreducible-Representation Framework (IRF)

As a prelude to studying the three complex minimisation equations, we may start with a simpler, linear combination of the last two. If we in the irreducible-representation framework adopt a convention where w_S is real, and take

$$w_1 = \hat{w}_1 e^{i\sigma_1}, \quad w_2 = \hat{w}_2 e^{i\sigma_2}, \tag{5.1}$$

with the $\hat{w}_i$ real and non-negative, then we find the consistency condition (for $w_1 \neq 0$ and $w_2 \neq 0$)

$$
\begin{aligned}
\frac{\partial V}{w_1 \partial w_1^*} &- \frac{\partial V}{w_2 \partial w_2^*} \\
&= 2(\lambda_2 + \lambda_3)\hat{w}_1\hat{w}_2 \left[\hat{w}_1^2(e^{i(\sigma_1+\sigma_2)} - e^{i(3\sigma_1-\sigma_2)}) + \hat{w}_2^2(e^{i(-\sigma_1+3\sigma_2)} - e^{i(\sigma_1+\sigma_2)}) \right] \\
&+ \lambda_4 w_S \hat{w}_1 \left[-\hat{w}_1^2(e^{3i\sigma_1} + 2e^{e^{i\sigma_1}}) + \hat{w}_2^2(3e^{i(\sigma_1+2\sigma_2)} + 4e^{i\sigma_1} + 2e^{i(-\sigma_1+2\sigma_2)}) \right] \\
&+ 2\lambda_7 w_S^2 \hat{w}_1\hat{w}_2(e^{i(-\sigma_1+\sigma_2)} - e^{i(\sigma_1-\sigma_2)}) = 0.
\end{aligned}
\tag{5.2}
$$

This condition, which is simpler than any of the individual derivatives (3.3)–(3.5), is a *necessary*, but not sufficient condition for the vacuum.

If $w_1 = 0$, then equation (3.4) is identically zero, and as a result (3.4) and (3.5) are automatically consistent. The same does not apply to the case $w_2 = 0$ since this case requires $\lambda_4 = 0$ for eq. (3.5) to be satisfied.

The generalisation of equation (4.3b) to the complex case is that the right-hand side of equation (5.2) be zero. This condition defines a hypersurface in a multidimensional parameter space. Whereas $w_S = 0$ and $\lambda_4 = 0$ are possible solutions in the real case, they are not in the complex case, unless supplemented by additional conditions. For example, if $w_S = 0$, we must also have

$$(\lambda_2 + \lambda_3)\hat{w}_1\hat{w}_2[\hat{w}_1^2(1 - e^{2i(\sigma_1-\sigma_2)}) + \hat{w}_2^2(e^{-2i(\sigma_1-\sigma_2)} - 1)] = 0. \tag{5.3}$$

Splitting this equation into real and imaginary parts, we find

$$(\lambda_2 + \lambda_3)(\hat{w}_1 - \hat{w}_2)\{1 - \cos[2(\sigma_1 - \sigma_2)]\} = 0, \quad \text{and} \tag{5.4a}$$

$$(\lambda_2 + \lambda_3)\sin[2(\sigma_1 - \sigma_2)] = 0. \tag{5.4b}$$

The solutions are

$$\lambda_2 + \lambda_3 = 0, \quad \text{or} \tag{5.5a}$$

$$\sigma_2 = \sigma_1 \pm n\pi, \quad n \text{ integer.} \tag{5.5b}$$

Likewise, the condition $\lambda_4 = 0$ must be supplemented by

$$
\begin{aligned}
(\lambda_2 + \lambda_3)e^{i(\sigma_1+\sigma_2)}&[\hat{w}_1^2(1 - e^{2i(\sigma_1-\sigma_2)}) + \hat{w}_2^2(e^{-2i(\sigma_1-\sigma_2)} - 1)] \\
&+ \lambda_7 w_S^2(e^{-i(\sigma_1-\sigma_2)} - e^{i(\sigma_1-\sigma_2)}) = 0.
\end{aligned}
\tag{5.6}
$$

Splitting this equation into real and imaginary parts, we find:

$$(\lambda_2 + \lambda_3)\sin(\sigma_1 - \sigma_2)(\sin 2\sigma_1 \hat{w}_1^2 + \sin 2\sigma_2 \hat{w}_2^2) = 0, \quad \text{and} \tag{5.7a}$$

$$[(\lambda_2 + \lambda_3)(\cos 2\sigma_1 \hat{w}_1^2 + \cos 2\sigma_2 \hat{w}_2^2) + \lambda_7 \hat{w}_S^2]\sin(\sigma_1 - \sigma_2) = 0. \tag{5.7b}$$

Furthermore, eq. (5.6) is obviously satisfied for

$$\lambda_2 + \lambda_3 = 0, \quad \text{and } \lambda_7 = 0. \tag{5.8}$$

On the other hand, for

$$\sigma_2 = \sigma_1 \pm n\pi, \quad n \text{ integer}, \tag{5.9}$$

λ_2, λ_3 and λ_7 are not constrained by equation (5.6).

Finally, the real-case consistency condition $\hat{w}_1^2 = 3\hat{w}_2^2$ would in the complex case have to be supplemented with the above phase constraint (5.9).

We find the solutions given in table 2. The table is organised as follows. From left to right, the first column gives the name of the vacuum, the second gives the specification in the irreducible-representation framework (IRF), and the third gives its translation to the reducible-representation framework (RRF).

The need to introduce the parameter ϵ in table 2 results from the definitions given by eq. (5.1), where the $\hat{w}_i$, $i = 1, 2$ are chosen to be non-negative. Naively one might expect the number of constraints to be equal to the number of free parameters of the solution. This is not the case, as can be illustrated by considering eq. (C.7) in the limit $w_S = 0$.

In the complex case there is a richer structure of possible vacua and again we have solutions that are similar when specified in terms of ρ's but not in terms of w's. Furthermore, in some cases solutions which can be described in an elegant way in one of the frameworks do not look so elegant in the other or may fall into a particular case of a more general one already given in that framework. As an illustration of the first remark let us consider the solution $(\rho_1, \rho_2, \rho_3) = x(1, 1, e^{i\tau})$, this is a special case of C-III-a, with $y = x$, however the same solution after a permutation becomes a special case of C-III-h. On the other hand, the solution $(\rho_1, \rho_2, \rho_3) = x(1, e^{i\tau}, e^{-i\tau})$ is a special case of C-III-i, with solution C-I-a a special case of this one again.

In tables 3 and 4 we list the conditions on the potential parameters, in the irreducible-representation framework.

The vacuum C-III-c (see tables 2 and 3) falls in the category satisfying eq. (5.5a). Examples of the constraints (5.7) are to be found in the vacua C-IV-b and C-IV-e (see tables 2 and 4). The constraints (5.8) apply to the vacuum C-V (see tables 2 and 4).

The case C-III-c of table 2 is very interesting. It is a solution with $\hat{w}_S = 0$ and $\lambda_4 = 0$ with the additional constraint $\lambda_2 + \lambda_3 = 0$, allowing for a free relative phase between the vevs of h_1 and h_2. This solution can be specified in terms of two non-zero moduli and one single phase. Once we replace λ_2 in terms of λ_3, the Lagrangian is still left with two of the initial coefficients that multiplied terms that were sensitive to a relative phase between h_1 and h_2, to wit λ_3 and λ_7. However, the new term in λ_3 is now invariant under the separate rephasing of these two doublets, the same is not true for the term in λ_7. The fact that

	IRF (Irreducible Rep.)	RRF (Reducible Rep.)
	w_1, w_2, w_S	ρ_1, ρ_2, ρ_3
C-I-a	$\hat{w}_1, \pm i\hat{w}_1, 0$	$x, xe^{\pm\frac{2\pi i}{3}}, xe^{\mp\frac{2\pi i}{3}}$
C-III-a	$0, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$y, y, xe^{i\tau}$
C-III-b	$\pm i\hat{w}_1, 0, \hat{w}_S$	$x + iy, x - iy, x$
C-III-c	$\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0$	$xe^{i\rho} - \frac{y}{2}, -xe^{i\rho} - \frac{y}{2}, y$
C-III-d,e	$\pm i\hat{w}_1, \epsilon\hat{w}_2, \hat{w}_S$	$xe^{i\tau}, xe^{-i\tau}, y$
C-III-f	$\pm i\hat{w}_1, i\hat{w}_2, \hat{w}_S$	$re^{i\rho} \pm ix, re^{i\rho} \mp ix, \frac{3}{2}re^{-i\rho} - \frac{1}{2}re^{i\rho}$
C-III-g	$\pm i\hat{w}_1, -i\hat{w}_2, \hat{w}_S$	$re^{-i\rho} \pm ix, re^{-i\rho} \mp ix, \frac{3}{2}re^{i\rho} - \frac{1}{2}re^{-i\rho}$
C-III-h	$\sqrt{3}\hat{w}_2 e^{i\sigma_2}, \pm\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$xe^{i\tau}, y, y$
		$y, xe^{i\tau}, y$
C-III-i	$\sqrt{\frac{3(1+\tan^2\sigma_1)}{1+9\tan^2\sigma_1}}\hat{w}_2 e^{i\sigma_1},$	$x, ye^{i\tau}, ye^{-i\tau}$
	$\pm\hat{w}_2 e^{-i\arctan(3\tan\sigma_1)}, \hat{w}_S$	$ye^{i\tau}, x, ye^{-i\tau}$
C-IV-a*	$\hat{w}_1 e^{i\sigma_1}, 0, \hat{w}_S$	$re^{i\rho} + x, -re^{i\rho} + x, x$
C-IV-b	$\hat{w}_1, \pm i\hat{w}_2, \hat{w}_S$	$re^{i\rho} + x, -re^{-i\rho} + x, -re^{i\rho} + re^{-i\rho} + x$
C-IV-c	$\sqrt{1 + 2\cos^2\sigma_2}\,\hat{w}_2,$	$re^{i\rho} + r\sqrt{3(1 + 2\cos^2\rho)} + x,$
	$\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$re^{i\rho} - r\sqrt{3(1 + 2\cos^2\rho)} + x, -2re^{i\rho} + x$
C-IV-d*	$\hat{w}_1 e^{i\sigma_1}, \pm\hat{w}_2 e^{i\sigma_1}, \hat{w}_S$	$r_1 e^{i\rho} + x, (r_2 - r_1)e^{i\rho} + x, -r_2 e^{i\rho} + x$
C-IV-e	$\sqrt{-\frac{\sin 2\sigma_2}{\sin 2\sigma_1}}\,\hat{w}_2 e^{i\sigma_1},$	$re^{i\rho_2} + re^{i\rho_1}\xi + x, re^{i\rho_2} - re^{i\rho_1}\xi + x,$
	$\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$-2re^{i\rho_2} + x$
C-IV-f	$\sqrt{2 + \frac{\cos(\sigma_1 - 2\sigma_2)}{\cos\sigma_1}}\,\hat{w}_2 e^{i\sigma_1},$	$re^{i\rho_1} + re^{i\rho_2}\psi + x,$
	$\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$re^{i\rho_1} - re^{i\rho_2}\psi + x, -2re^{i\rho_1} + x$
C-V*	$\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$xe^{i\tau_1}, ye^{i\tau_2}, z$

Table 2. Complex vacua. Notation: $\epsilon = 1$ and -1 for C-III-d and C-III-e, respectively; $\xi = \sqrt{-3\sin 2\rho_1/\sin 2\rho_2}$, $\psi = \sqrt{[3 + 3\cos(\rho_2 - 2\rho_1)]/(2\cos\rho_2)}$. With the constraints of table 4 the vacua labelled with an asterisk (*) are in fact real.

the moduli of the two vevs are different will play an important rôle in the discussion of spontaneous CP violation. In this particular case, due to hermiticity of the scalar potential the only coefficient that could be chosen to be complex is λ_7, however this choice would not lead to explicit CP violation since a simultaneous rephasing of h_1 and h_2 would allow to remove the phase of λ_7, rendering all coefficients of the potential real. Another way of achieving the same result would be by rephasing h_S alone. Neither of these transformations alters the specifications of the vacuum corresponding to this case.

Cases C-IV-a, C-IV-d and C-V are listed in table 2 for completeness and to allow for an enlightening discussion. Once one takes into consideration the constraints given in table 4 they become real.

Vacuum	Constraints
C-I-a	$\mu_1^2 = -2\left(\lambda_1 - \lambda_2\right)\hat{w}_1^2$
C-III-a	$\mu_0^2 = -\frac{1}{2}\lambda_b\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\hat{w}_2^2 - \frac{1}{2}\left(\lambda_b - 8\cos^2\sigma_2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = \frac{4\cos\sigma_2\hat{w}_S}{\hat{w}_2}\lambda_7$
C-III-b	$\mu_0^2 = -\frac{1}{2}\lambda_b\hat{w}_1^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\hat{w}_1^2 - \frac{1}{2}\lambda_b\hat{w}_S^2,$ $\lambda_4 = 0$
C-III-c	$\mu_1^2 = -(\lambda_1 + \lambda_3)(\hat{w}_1^2 + \hat{w}_2^2),$ $\lambda_2 + \lambda_3 = 0, \lambda_4 = 0$
C-III-d,e	$\mu_0^2 = (\lambda_2 + \lambda_3)\frac{(\hat{w}_1^2 - \hat{w}_2^2)^2}{\hat{w}_S^2} - \epsilon\lambda_4\frac{(\hat{w}_1^2 - \hat{w}_2^2)(\hat{w}_1^2 - 3\hat{w}_2^2)}{4\hat{w}_2\hat{w}_S}$ $\quad -\frac{1}{2}\left(\lambda_5 + \lambda_6\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 - \lambda_2\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \epsilon\lambda_4\frac{\hat{w}_S(\hat{w}_1^2 - \hat{w}_2^2)}{4\hat{w}_2} - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_7 = \frac{\hat{w}_1^2 - \hat{w}_2^2}{\hat{w}_S^2}(\lambda_2 + \lambda_3) - \epsilon\frac{(\hat{w}_1^2 - 5\hat{w}_2^2)}{4\hat{w}_2\hat{w}_S}\lambda_4$
C-III-f,g	$\mu_0^2 = -\frac{1}{2}\lambda_b\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \frac{1}{2}\lambda_b\hat{w}_S^2, \lambda_4 = 0$
C-III-h	$\mu_0^2 = -2\lambda_b\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -4\left(\lambda_1 + \lambda_3\right)\hat{w}_2^2 - \frac{1}{2}\left(\lambda_b - 8\cos^2\sigma_2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = \mp\frac{2\cos\sigma_2\hat{w}_S}{\hat{w}_2}\lambda_7$
C-III-i	$\mu_0^2 = \frac{16\left(1 - 3\tan^2\sigma_1\right)^2}{(1 + 9\tan^2\sigma_1)^2}(\lambda_2 + \lambda_3)\frac{\hat{w}_2^4}{\hat{w}_S^2} \pm \frac{6\left(1 - \tan^2\sigma_1\right)(1 - 3\tan^2\sigma_1)}{(1 + 9\tan^2\sigma_1)^{\frac{3}{2}}}\lambda_4\frac{\hat{w}_2^3}{\hat{w}_S}$ $\quad -\frac{2(1 + 3\tan^2\sigma_1)}{1 + 9\tan^2\sigma_1}(\lambda_5 + \lambda_6)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\frac{4(1 + 3\tan^2\sigma_1)}{1 + 9\tan^2\sigma_1}(\lambda_1 - \lambda_2)\hat{w}_2^2 \mp \frac{\left(1 - 3\tan^2\sigma_1\right)}{2\sqrt{1 + 9\tan^2\sigma_1}}\lambda_4\hat{w}_2\hat{w}_S$ $\quad -\frac{1}{2}(\lambda_5 + \lambda_6)\hat{w}_S^2,$ $\lambda_7 = -\frac{4\left(1 - 3\tan^2\sigma_1\right)\hat{w}_2^2}{(1 + 9\tan^2\sigma_1)\hat{w}_S^2}(\lambda_2 + \lambda_3) \mp \frac{\left(5 - 3\tan^2\sigma_1\right)\hat{w}_2}{2\sqrt{1 + 9\tan^2\sigma_1}\hat{w}_S}\lambda_4$

Table 3. Constraints on complex vacua. Notation: $\epsilon = 1$ and -1 for C-III-d and C-III-e, respectively. Where two possible signs ($\pm$ or $\mp$) are given, they correspond to those of table 2. Here, λ_b is defined in eq. (4.5).

Solution C-IV-d is more general than solution C-IV-a and reduces to C-IV-a once we fix $w_2 = 0$, so it suffices to discuss C-IV-d. Both of these require $\lambda_4 = 0$ and $\lambda_7 = 0$, and as a result the potential acquires symmetry for the transformation of h_1, h_2 and h_S under a unitary transformation of the form $U = \text{diag}(e^{i\tau}, e^{i\tau}, 1)$ which allows to remove the phase σ_1 from the vacuum, making it real.

At first glance case C-V looks like the most general case, however we are assuming that it does not fall into any of the previous cases, so, as a result, full generality requires

Vacuum	Constraints
C-IV-a*	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_1^2 - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\hat{w}_1^2 - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0, \lambda_7 = 0$
C-IV-b	$\mu_0^2 = \left(\lambda_2 + \lambda_3\right)\frac{\left(\hat{w}_1^2 - \hat{w}_2^2\right)^2}{\hat{w}_S^2} - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 - \lambda_2\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0, \lambda_7 = -\frac{\left(\hat{w}_1^2 - \hat{w}_2^2\right)}{\hat{w}_S^2}\left(\lambda_2 + \lambda_3\right)$
C-IV-c	$\mu_0^2 = 2\cos^2\sigma_2\left(1 + \cos^2\sigma_2\right)\left(\lambda_2 + \lambda_3\right)\frac{\hat{w}_2^4}{\hat{w}_S^2}$ $- \left(1 + \cos^2\sigma_2\right)\left(\lambda_5 + \lambda_6\right)\hat{w}_2^2 - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left[2\left(1 + \cos^2\sigma_2\right)\lambda_1 - \left(2 + 3\cos^2\sigma_2\right)\lambda_2 - \cos^2\sigma_2\lambda_3\right]\hat{w}_2^2$ $- \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = -\frac{2\cos\sigma_2\hat{w}_2}{\hat{w}_S}\left(\lambda_2 + \lambda_3\right), \lambda_7 = \frac{\cos^2\sigma_2\hat{w}_2^2}{\hat{w}_S^2}\left(\lambda_2 + \lambda_3\right)$
C-IV-d*	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5 + \lambda_6\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0, \lambda_7 = 0$
C-IV-e	$\mu_0^2 = \frac{\sin^2(2(\sigma_1 - \sigma_2))}{\sin^2(2\sigma_1)}\left(\lambda_2 + \lambda_3\right)\frac{\hat{w}_2^4}{\hat{w}_S^2}$ $- \frac{1}{2}\left(1 - \frac{\sin 2\sigma_2}{\sin 2\sigma_1}\right)\left(\lambda_5 + \lambda_6\right)\hat{w}_2^2 - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left(1 - \frac{\sin 2\sigma_2}{\sin 2\sigma_1}\right)\left(\lambda_1 - \lambda_2\right)\hat{w}_2^2 - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0, \lambda_7 = -\frac{\sin(2(\sigma_1 - \sigma_2))\hat{w}_2^2}{\sin 2\sigma_1 \hat{w}_S^2}\left(\lambda_2 + \lambda_3\right)$
C-IV-f	$\mu_0^2 = -\frac{\left(\cos(\sigma_1 - 2\sigma_2) + 3\cos\sigma_1\right)\cos(\sigma_2 - \sigma_1)}{2\cos^2\sigma_1}\lambda_4 \frac{\hat{w}_2^3}{\hat{w}_S}$ $- \frac{\cos(\sigma_1 - 2\sigma_2) + 3\cos\sigma_1}{2\cos\sigma_1}\left(\lambda_5 + \lambda_6\right)\hat{w}_2^2 - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\frac{\cos(\sigma_1 - 2\sigma_2) + 3\cos\sigma_1}{\cos\sigma_1}\left(\lambda_1 + \lambda_3\right)\hat{w}_2^2$ $- \frac{3\cos 2\sigma_1 + 2\cos(2(\sigma_1 - \sigma_2)) + \cos 2\sigma_2 + 4}{4\cos(\sigma_1 - \sigma_2)\cos\sigma_1}\lambda_4 \hat{w}_2 \hat{w}_S - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_2 + \lambda_3 = -\frac{\cos\sigma_1 \hat{w}_S}{2\cos(\sigma_2 - \sigma_1)\hat{w}_2}\lambda_4, \lambda_7 = -\frac{\cos(\sigma_2 - \sigma_1)\hat{w}_2}{2\cos\sigma_1 \hat{w}_S}\lambda_4$
C-V*	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5 + \lambda_6\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\left(\hat{w}_1^2 + \hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5 + \lambda_6\right)\hat{w}_S^2,$ $\lambda_2 + \lambda_3 = 0, \lambda_4 = 0, \lambda_7 = 0$

Table 4. Constraints on complex vacua, continued. The vacua labelled with an asterisk (*) are in fact real.

$\lambda_2 + \lambda_3 = 0$, $\lambda_4 = 0$ and $\lambda_7 = 0$ and there is no term in the potential sensitive to independent rephasing of each of the h fields. As a result any phase in the vevs can be rotated away. Under these circumstances, it is equivalent to a real set of vacua.

There are, in particular, two possible complex vacua that have been discussed previously in the literature. One of them is:

$$\hat{w}e^{i\sigma}, \quad \hat{w}e^{-i\sigma}, \quad \hat{w}_S, \tag{5.10}$$

by Pakvasa and Sugawara [18]. We shall refer to this as the PS vacuum, assuming $\hat{w} \neq 0$ and $\hat{w}_S \neq 0$. There is also a solution given by Ivanov and Nishi [22]

$$\hat{w}e^{i\sigma}, \quad \hat{w}e^{i\sigma}, \quad \hat{w}_S, \tag{5.11}$$

which we shall refer to as the IN vacuum, assuming again $\hat{w} \neq 0$ and $\hat{w}_S \neq 0$. By imposing the minimisation conditions it can be checked that both of these solutions require λ_4 equal to zero, corresponding to the SO(2) symmetry of the potential given by eq. (2.12). It is clear that eq. (5.11) does not break this symmetry spontaneously.

The PS vacuum specified by eq. (5.10) is only consistent for the following choices:

$$\text{PS-a}: \quad \lambda_4 = 0, \quad \text{and} \quad \sigma = \pm\pi/2, \qquad\qquad \text{included in case C-III-f,g,} \tag{5.12}$$

$$\text{PS-b}: \quad \lambda_4 = 0, \quad \text{and } \lambda_7 = -2\cos 2\sigma \frac{\hat{w}^2}{\hat{w}_S^2}(\lambda_2 + \lambda_3), \quad \begin{array}{l} \text{included in case C-IV-e} \\ \text{with } \sigma_2 = -\sigma_1. \end{array} \tag{5.13}$$

As a special case of PS-b, we can have $\lambda_7 = 0$ and $\sigma = \pi/4$ or $3\pi/4$, with $(\lambda_2 + \lambda_3)$ unconstrained.

The IN vacuum specified by eq. (5.11) is only consistent if either of the following two sets of conditions is verified:

$$\text{IN-a}: \quad \lambda_4 = 0, \quad \text{and} \quad \sigma = \pm\pi/2, \quad \text{included in C-III-f,g,} \tag{5.14}$$

$$\text{IN-b}: \quad \lambda_4 = 0, \quad \text{and } \lambda_7 = 0, \qquad \text{included in case C-IV-d.} \tag{5.15}$$

As discussed above, the conditions listed under IN-b lead to a real vacuum since they allow for the common phase of w_1 and w_2 to be rotated away.

Special limits. Some of the vacua listed in table 2 can be seen as special limits of another, more general case. These include

- C-III-a and C-III-h are equivalent in terms of the RRF.

- C-III-b, for $\lambda_7 = 0$ becomes real and falls into C-IV-a.

- C-IV-a is contained in C-IV-d with $\hat{w}_2 = 0$.

- Solution C-IV-e reduces to C-IV-b for $\sigma_2 = \sigma_1 \pm \pi/2$, in the limit $\sigma_1 \to 0$.

- C-IV-c is contained in C-IV-f for $\sigma_1 = 0$.

5.2 The Reducible-Representation Framework (RRF)

Below follow further general comments on the different vacua, as well as some constraints on the parameters in the reducible-representation framework:

- Vacuum C-I-a requires

$$2\lambda + \gamma = x^2[2A + 2C + 2\overline{C} - D - 2E_1 - E_2 - E_3 + 2E_4]. \qquad (5.16)$$

- Vacuum C-III-h allows for a particular realisation in terms of the RRF with $y = x$, of the form

$$\text{C-RRF-a:} \qquad (\rho_1, \rho_2, \rho_3) = x(1, e^{i\tau}, e^{i\tau}). \qquad (5.17)$$

Here, we have applied an overall phase rotation to the first solution presented in table 2 for this vacuum, thus complying with the notation defined by eq. (3.9). Depending on where we put the phase, we may have two different translations in terms of the IRF: one of them in C-III-h and the other in C-III-a. This leads to three independent minimisation conditions (as specified by the label III), two "radial" ones from $\partial V/\partial v_1 = 0$ and $\partial V/\partial v_2 = 0$ and one "angular" one from $\partial V/\partial \tau = 0$. The two radial equations are quadratic in $\cos \tau$, whereas the third is linear. With x and τ as input, these equations allow to constrain three parameters of the potential. Alternatively, one may remove terms quadratic in $\cos \tau$ by forming linear combinations of the two radial equations. With the rescaling

$$\lambda' = \lambda/x^2, \quad \gamma' = \gamma/x^2, \qquad (5.18)$$

we find two radial equations

$$\cos \tau = \frac{-4D - 4E_1 + E_2 + E_3 - E_4 - 2\gamma'}{2(2D + E_4)}, \qquad (5.19a)$$

$$= \frac{-A - C - \overline{C} - D - 2E_1 - \gamma' + \lambda'}{E_2 + E_3 + E_4}, \qquad (5.19b)$$

and the angular equation

$$\cos \tau = \frac{-2E_1 - E_2 - E_3 - E_4 - 2\gamma'}{2(2D + E_4)}. \qquad (5.20)$$

Consistency of the first and the third expressions leads to

$$2D + E_1 - E_2 - E_3 = 0, \qquad (5.21)$$

which corresponds to

$$\lambda_4 = \sqrt{2}\lambda_7. \qquad (5.22)$$

Consistency of the second and third equation leads to:

$$(2A + 2C + 2\overline{C} - E_4)(2D + E_4) - 3E_1^2 - 2(2D + E_4)\lambda' - 2E_1\gamma' = 0. \qquad (5.23)$$

Invoking eq. (5.18), we may solve for x^2:

$$x^2 = \frac{2(2D + E_4)\lambda + 2E_1\gamma}{(2A + 2C + 2\overline{C} - E_4)(2D + E_4) - 3E_1^2}. \qquad (5.24)$$

The conditions

$$x^2 > 0 \quad \text{and} \quad |\cos\tau| \le 1 \qquad (5.25)$$

will further constrain the potential parameters for this particular vacuum.

As presented in table 2, Case C-III-h illustrates, once again, the fact that trivial permutations of the reducible triplet lead to different constraints for the IRF.

- The cases C-III-d, C-III-e and C-III-i, when presented in the RRF, in the limit $y = x$, become

$$\text{C-RRF-b:} \qquad (\rho_1, \rho_2, \rho_3) = x(1, e^{i\tau}, e^{-i\tau}), \qquad (5.26)$$

and permutations.[3] These three IRF cases merge into one RRF case.

There are three minimisation conditions, involving x^2 and $\cos\tau$. The constraint from the minimisation with respect to τ can be expressed as a $\cos\tau$-dependent relation among the quartic terms:

$$E_2 + E_3 = (4\cos\tau - 2)D + E_1 + 2(1 - \cos\tau)E_4, \qquad (5.27)$$

whereas the others can be solved for γ and λ. Making use of eq. (5.27), these take the form

$$\gamma = \frac{x^2}{2}[(2 - 8\cos^2\tau)D - 3E_1 + (1 - 4\cos\tau)E_4], \qquad (5.28)$$

$$\lambda = \frac{x^2}{2}[2(A + C + \overline{C}) - 4\cos\tau(1 - \cos\tau)D + (2\cos^2\tau + 2\cos\tau - 1)E_1$$
$$- (2\cos^2\tau - 2\cos\tau + 1)E_4]. \qquad (5.29)$$

The three constraints (minimisation conditions) of table 3 will for C-III-d, C-III-e, and C-III-i take forms equivalent to these eqs. (5.27)–(5.29). Equations (5.28) and (5.29) can be solved for x^2, but the two solutions impose a $\cos\tau$-dependent consistency condition on the coefficients of the potential, given by eq. (5.27).

- The vacua C-RRF-a and C-RRF-b have the same form as the PS and IN vacua of eqs. (5.10)–(5.15). However, it must be stressed that they refer to the fields of the reducible-representation framework.

6   Complex vacua vs real vacua

The complex vacua, which are specified by three moduli and two (relative) phases are found as solutions of five conditions, whereas the real vacua are found as solutions of three. The following questions then arise: can the complex vacua be seen as generalisations of the real

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[3]Vacuum C-I-a is of course a special case of this one, with $\tau = 2\pi/3$.

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Complex	Real "origin"
C-I-a	none
C-III-a	R-II-1a
C-III-b	none
C-III-c	R-I-2a,2b,2c, R-II-3
C-III-d,e	none
C-III-f	none
C-III-g	none
C-III-h	R-II-1b,1c
C-III-i	R-II-1b,1c
C-IV-a*	R-III
C-IV-b	none
C-IV-c	R-II-1b,1c
C-IV-d*	R-III
C-IV-e	none
C-IV-f	R-II-1b,1c
C-V*	R-III

Table 5. Transitions from complex to real vacua in the IRF. The vacua labelled with an asterisk (*) were shown to be real.

ones? Are the conditions on the moduli compatible with those for one or more of the real vacua? Are these more restrictive, or less restrictive?

In order to discuss how a complex vacuum may be related to a real one, let us introduce some notations. Let us denote by $\mathcal{C}(\text{C-X-y})$ the set of constraints (such as given in tables 3 and 4) satisfied by a particular complex vacuum. Likewise, we let the real vacuum R-X′-y′ satisfy the constraints $\mathcal{C}(\text{R-X′-y′})$ (see table 1). Then, we may consider a real vacuum R-X′-y′ the "origin" of a particular complex vacuum C-X-y if the following two conditions are satisfied:

- the C-X-y specification, in an existing real limit for the vacuum (there may be an ambiguity of sign) coincides with that of R-X′-y′, and

- the constraints are compatible,

$$\mathcal{C}(\text{C-X-y}) \subset \mathcal{C}(\text{R-X′-y′}). \tag{6.1}$$

The latter condition is important due to the fact that the transition to a real vacuum is not always possible.

We list in table 5 the real vacua satisfying these two requirements.

In the following subsection, we study a particular example, how a complex vacuum is related to a real one.

6.1 Transition from Vacua R-II-1b,1c to Vacuum C-III-h

An important difference between the constraints of the vacua R-II-1b,1c and C-III-h is that in the former case, the potential parameters λ_4 and λ_7 are free, i.e., they are uncorrelated. For the vacuum C-III-h, on the other hand, they are correlated as (see table 3)

$$\lambda_4 = \mp \frac{2\cos\sigma_2 \hat{w}_S}{\hat{w}_2}\lambda_7. \tag{6.2}$$

Modulo positivity and other physical constraints, the whole λ_4–λ_7-plane is available for the real vacuua R-II-1b,1c, whereas only the lines defined by eq. (6.2) are available for an "extension" to a complex vacuum C-III-h. This holds even for infinitesimal phases, i.e., $\cos\tau_2 \to \pm 1$.

The following question arises: under what conditions is the complex vacuum deeper? It turns out that the difference can be expressed as being proportional to λ_4 or λ_7:

$$\Delta V \equiv V(\text{R-II-1b,1c}) - V(\text{C-III-h}) = -4\lambda_7\hat{w}_2^2\hat{w}_S^2(1 \mp \cos\sigma_2)^2. \tag{6.3}$$

The question of relating complex vacua to real ones is relevant for the discussion of global minima [28, 29] as well as to understand the possible correlations of different parameters of the potential.

7 The case of $\lambda_4 = 0$

As mentioned in section 2.3, in the case of $\lambda_4 = 0$ the potential has an additional, continuous SO(2) symmetry. This case was dismissed by Derman [20], as being "un-natural". This was due to the fact that this condition, when expressed in terms of the parameters of the potential written by Derman, given by eqs. (2.9), acquires the form given by eq. (4.10), which is not instructive and the resulting symmetry is not apparent. Spontaneous breaking of this SO(2) symmetry leads to massless particles. In this case, one way to promote this to a viable model is to break this symmetry softly, by adding a term to the bilinear part of the potential:

$$V = V_2 + V_2' + V_4, \tag{7.1}$$

with V_2 and V_4 as defined by equations (2.10), and

$$V_2' = \frac{1}{2}\nu^2(h_2^\dagger h_1 + h_1^\dagger h_2). \tag{7.2}$$

The minimisation conditions (3.3)–(3.5) will now become

$$\frac{\partial V}{\partial w_S^*} = \frac{1}{2}w_S\mu_0^2 + \frac{1}{4}w_S(|w_1|^2 + |w_2|^2)(\lambda_5 + \lambda_6)$$
$$+ \frac{1}{4}w_S^*(w_1^2 + w_2^2)\lambda_7 + \frac{1}{2}w_S^*w_S^2\lambda_8 = 0, \tag{7.3}$$

$$\frac{\partial V}{\partial w_1^*} = \frac{1}{2}w_1\mu_1^2 + \frac{1}{2}w_2\nu^2 + \frac{1}{2}w_1(|w_1|^2 + |w_2|^2)\lambda_1 + \frac{1}{2}w_2(w_1^*w_2 - w_1w_2^*)\lambda_2$$
$$+ \frac{1}{2}w_1^*(w_1^2 + w_2^2)\lambda_3 + \frac{1}{4}w_1|w_S|^2(\lambda_5 + \lambda_6) + \frac{1}{2}w_1^*w_S^2\lambda_7 = 0, \tag{7.4}$$

$$\frac{\partial V}{\partial w_2^*} = \frac{1}{2}w_2\mu_1^2 + \frac{1}{2}w_1\nu^2 + \frac{1}{2}w_2(|w_1|^2 + |w_2|^2)\lambda_1 - \frac{1}{2}w_1(w_1^*w_2 - w_1w_2^*)\lambda_2$$
$$+ \frac{1}{2}w_2^*(w_1^2 + w_2^2)\lambda_3 + \frac{1}{4}w_2|w_S|^2(\lambda_5 + \lambda_6) + \frac{1}{2}w_2^*w_S^2\lambda_7 = 0. \tag{7.5}$$

With these new conditions there will be some changes in the solutions. In particular, the new term will bring new sources of CP violation, and spontaneous CP violation may be easier to achieve. Notice that such a term also softly breaks some types of discrete symmetries of the S_3 doublet h_1 and h_2 that might otherwise be present. This feature was exploited long ago in the context of two-Higgs-doublet models [30]. Soft breaking of the S_3 symmetry of the scalar potential has been applied in [31] in order to obtain a special relation among the vevs of the three doublets that would allow to account for the observed charged lepton masses.

An important implication of the type of vacuum solution and of the corresponding allowed region of parameter space is the resulting different possible spectra for the physical scalars.

8 Spontaneous CP violation

The S_3-symmetric potential offers a very rich phenomenology, and can accommodate a variety of physical situations, as outlined in sections 4 and 5, where we classified the different vacua.

We assumed, for simplicity, that all parameters of the potential are real. Therefore our discussion is done in the framework of explicit CP conservation. This raises the question of whether or not CP can be violated spontaneously. For that purpose we can inspect the list of complex solutions presented in table 2. CP can only be spontaneously violated if the Lagrangian is invariant under CP and if at the same time there is no transformation that can be identified with a CP transformation, leaving both the Lagrangian and the vacuum invariant. The idea of spontaneous CP violation was first proposed by T.D. Lee [3] in the context of two Higgs doublets. In the context of the SM, with a single Higgs doublet, a CP transformation of the scalar doublet amounts to its complex conjugation and the scalar sector cannot violate CP. In models with several Higgs doublets complex conjugation may be combined with a unitary transformation acting on the set of doublets, since this transformation leaves the kinetic energy term of the Lagrangian invariant. In this case the most general CP transformation is given by:

$$\Phi_i \xrightarrow{\text{CP}} U_{ij}\Phi_j^*\,, \tag{8.1}$$

with U an arbitrary unitary matrix.[4] This equation together with the assumption that the vacuum is CP invariant:

$$\text{CP}|0\rangle = |0\rangle\,, \tag{8.2}$$

leads to the following condition [32]:

$$U_{ij}\langle 0|\Phi_j|0\rangle^* = \langle 0|\Phi_i|0\rangle\,, \tag{8.3}$$

implying that there is spontaneous CP violation if none of the CP symmetries allowed by the Lagrangian satisfy this equation. For real vevs this condition is obviously verified.

---

[4]Some authors refer to this transformation as a "generalised" CP transformation. This is somewhat misleading since it suggests that there is also a "non-generalised" CP transformation.

If the Lagrangian has a discrete symmetry one must take it into consideration before drawing conclusions. In the discussion that follows we do not take the Yukawa sector into consideration. We now comment on each one of the cases presented in table 2 concerning the possibility of having spontaneous CP violation:

- The case C-I-a is a familiar one that has been discussed long ago in the framework of the reducible representation [32]. It was pointed out that it has complex vacuum expectation values with calculable non-trivial phases, assuming geometrical values, entirely determined by the symmetry of the scalar potential. These phases cannot be rotated away and yet they do not lead to spontaneous CP violation, since there is a matrix U satisfying the constraint of eq. (8.3), namely:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (8.4)$$

which is at the same time a symmetry of the potential.

In terms of the irreducible-representation framework we can write this solution as $(\pm i\hat{w}_1, \hat{w}_1, 0)$ and the matrix U satisfying the constraint (8.3) becomes $U = \mathrm{diag}(-1, 1, 1)$ making use of the symmetry of the potential for $h_1 \to -h_1$. It was shown [32] that solutions with calculable phases whose values are independent of the coupling constants of the scalar potential do not necessarily conserve CP. Characteristic features of such solutions in models with several Higgs doublets as well as the interplay between symmetries and geometrical CP violation have been analysed by several authors [33–39].

- Case C-III-a allows for a nontrivial phase which can be determined as a function of λ_4, and λ_7, as shown in table 3. This solution violates CP spontaneously.

- Reasoning analogous to that for C-I-a can be applied to cases C-III-b, C-III-d, C-III-e where again the matrix U given above, $U = \mathrm{diag}(-1, 1, 1)$, satisfies eq. (8.3) in terms of the irreducible representation framework. On the other hand, cases C-III-f and C-III-g require $\lambda_4 = 0$ and therefore the potential acquires an additional SO(2) symmetry. In these cases U can be chosen as $U = \mathrm{diag}(-1, -1, 1)$. Case C-IV-b also requires $\lambda_4 = 0$, as a result the potential is also symmetric under $h_2 \to -h_2$ and one can choose $U = \mathrm{diag}(1, -1, 1)$.

- Case C-III-c is a very interesting one. On one hand it requires $\lambda_4 = 0$ and on the other hand $\lambda_2 + \lambda_3 = 0$. The most general S_3 symmetric potential has four terms that are sensitive to the relative phases among h_1, h_2 and h_S, which are those multiplied by the coefficients λ_2, λ_3, λ_4 and λ_7. However, with these restrictions only the term in λ_7 will be sensitive to relative phases and is the only one that could be chosen to be complex. However, in such a case λ_7 could be made real by a simultaneous rephasing of h_1 and h_2 with the same phase or else of h_S independently. Clearly, under such transformations the three vevs still remain in case C-III-c. This case

does not allow for explicit CP violation by the scalar potential. Making use of an overall phase transformation this solution can be written as $(\hat{w}_1 e^{i\sigma}, \hat{w}_2 e^{-i\sigma}, 0)$. It is clear that for $\hat{w}_1 = \hat{w}_2$, CP would be conserved with U interchanging the two first vevs. For $\hat{w}_1 \neq \hat{w}_2$ CP can be spontaneously violated by the potential. There is therefore a region in parameter space where the scalar potential does not violate CP explicitly and yet CP can be violated spontaneously. The phase σ is not constrained by the minimisation conditions but it parametrises triple and quartic Higgs couplings. In realistic models, Yukawa couplings also have to be specified, and here again, in general, this phase will play a rôle. This very special situation arises from the fact that $\hat{w}_S = 0$ thus eliminating all terms in the minimisation conditions proportional to λ_7. This raises the question of what fixes this phase.

- C-III-f,g are discussed above together with C-III-b, CP is not spontaneously violated.

- It is clear from this discussion that, in general, C-III-h can violate CP and as we can see from table 3 the phase is determined by parameters of the potential.

- For C-III-i we can verify eq. (8.3) in the reducible representation framework with U acting as a permutation between the two vevs with modulus y and there is no spontaneous CP violation.

- Solution C-IV-a is in fact real, as discussed in section 5, since it requires λ_4 and λ_7 to be zero and therefore CP is not violated.

- C-IV-b is discussed above together with C-III-b, CP is not spontaneously violated.

- For C-IV-c the reasoning is similar to the one in case C-III-c; as a result eq. (8.3) is not verified and CP can be violated.

- For C-IV-d again λ_4 and λ_7 must be zero and the same reasoning followed for C-IV-a leads to the conclusion that CP is not violated.

- Solution C-IV-e requires $\lambda_4 = 0$, the extra symmetry acquired by the potential is not sufficient for eq. (8.3) to be obeyed and CP can be violated.

- For C-IV-f there is no symmetry of the potential allowing to verify eq. (8.3) and therefore CP can be violated.

- Solution C-V looks like the most general case but the constraints imposed on the parameters of the potential make it equivalent to a real set of vacua as discussed in section 5, so that there is no spontaneous CP violation.

The PS vacuum specified by eq. (5.10) requires $\lambda_4 = 0$ and therefore there is symmetry under the interchange of the components of the S_3 doublet. As a result, it is possible to verify eq. (8.3) and CP is conserved, even when it falls into case C-IV-e, because it then requires $\sigma_2 = -\sigma_1$.

Vacuum	λ_4	SCPV	Vacuum	λ_4	SCPV	Vacuum	λ_4	SCPV
C-I-a	X	no	C-III-f,g	0	no	C-IV-c	X	yes
C-III-a	X	yes	C-III-h	X	yes	C-IV-d	0	no
C-III-b	0	no	C-III-i	X	no	C-IV-e	0	yes
C-III-c	0	yes	C-IV-a	0	no	C-IV-f	X	yes
C-III-d,e	X	no	C-IV-b	0	no	C-V	0	no

Table 6. Spontaneous CP violation.

The IN vacuum specified by eq. (5.11) also requires $\lambda_4 = 0$. It is clear from the previous discussion that this solution does not lead to spontaneous CP violation. In fact, from eqs. (5.14) and (5.15) it is clear that the allowed region of parameter space where this solution minimises the potential is such that either the phase σ can be rotated away and therefore is not physical (IN-b) or it is fixed as $\pm\pi/2$ (IN-a) falling into one of the cases C-III-f or C-III-g which were already discussed above.

We summarise these cases in table 6. In this table, we also indicate whether or not λ_4 is equal to zero. There are cases where CP can be violated, also with $\lambda_4 = 0$. The only case where spontaneous CP violation is possible in principle, and there is no real limit (see section 6), is C-IV-e.

9   Dark matter

Multi-Higgs models may provide viable Dark Matter candidates in the form of one or more inert scalars. This idea was first proposed in the context of two-Higgs-doublet models. The extra doublet is odd under an unbroken Z_2 symmetry and as a result the lightest member is stable [15, 16]. Different implementations of this idea have been discussed in the literature in different contexts [40–54].

Inert dark matter has also been studied in the context of three-Higgs-doublet models without an S_3 symmetry [55–57]. Models with three Higgs doublets have a richer phenomenology than those with only two. A strong motivation for such an extension is the possibility of having CP violation in the scalar sector [55, 56]. In these models dark matter is also stabilised via a Z_2 symmetry.

Dark matter has been proposed within S_3-symmetric models, exploiting fields that get a vanishing vev. Models of this kind include those where the singlet plays the rôle of the SM Higgs, whereas the S_3 doublet provides dark matter [58–60]. An alternative way to embed dark matter could be to have the S_3 singlet as inert in a solution where it has a zero vev, such as C-III-c. For the C-III-c solution a specific example would correspond to the following S_3 representation assignments for the quarks:

$$Q_{i\mathrm{L}} : \ (2,1), \quad u_{i\mathrm{R}} : \ (2,1), \quad d_{i\mathrm{R}} : \ (2,1). \tag{9.1}$$

In order to prevent h_S from coupling to the quarks we would need an additional Z_2 symmetry under which $h_S \to -h_S$ and all other fields remain invariant. The form for the fermion

mass matrices would be

$$
\begin{pmatrix}
-aw_1 & aw_2 & bw_1 \\
aw_2 & aw_1 & bw_2 \\
cw_1 & cw_2 & 0
\end{pmatrix},
\tag{9.2}
$$

with a different set of coefficients (a, b, c) for the up and down quark sector. Since solution C-III-c requires λ_4 to vanish, no term in the potential will break this Z_2 symmetry. The C-III-c vacuum breaks spontaneously the SO(2) symmetry obtained from having $\lambda_4 = 0$, therefore, one way to obtain a realistic scalar spectrum is to include the additional soft breaking term given by eq. (7.2). This has significant consequences for the solution of the new minimisation conditions. The vacuum transforms into:

$$
(\hat{w}e^{i\sigma}, \hat{w}, 0),
\tag{9.3}
$$

with

$$
\cos\sigma = -\frac{1}{4}\nu^2\frac{1}{\hat{w}^2}\frac{1}{\lambda_2 + \lambda_3},
\tag{9.4}
$$

for a well defined region of parameter space. This was obtained by requiring $\hat{w}_S = 0$ and $\lambda_4 = 0$, but relaxing the condition $\lambda_2 + \lambda_3 = 0$. This solution violates CP spontaneously since the potential no longer has symmetry under the interchange of h_1 and h_2. At this stage, this should be seen as a toy model. A full analysis of possible realistic implementations generating the observed fermion masses and mixing is beyond the scope of this paper.

10 Concluding remarks

The S_3-symmetric potential, with three doublets, is specified in terms of 10 parameters. It can accommodate 2 charged Higgs pairs and 5 neutral ones. If their masses were to be specified freely, one would need 7 parameters, leaving $10 - 7 = 3$ "free". On the other hand, if we consider an arbitrary vacuum, then 5 minimisation conditions have to be satisfied, determining 5 parameters. This mis-match illustrates that the spectrum can not be chosen freely, it will be constrained. Alternatively, one might pick a vacuum for which not all 5 minimisation conditions are independent. This would be the case, for example, when one vev vanishes.

Residual symmetries of the potential after spontaneous symmetry breaking play a very important rôle in constraining the possibility of having spontaneous CP violation [21, 22, 64]. In ref. [21] it is proved that real vacua of S_3 symmetric 3HDM always preserve an S_2 symmetry, whilst constraints on complex-valued minima are much less severe and there are complex minima which totally break the S_3 symmetry.

The transition from real to complex vacua is not trivial. Our work is done in the context of explicit CP conservation. Table 1 illustrates a point that had already been emphasized in ref. [21], which is that for real vacua, in the reducible representation framework, without imposing a condition equivalent to $\lambda_4 = 0$, the only allowed solutions with all three vevs different from each other are $(x, -x, 0)$ and their permutations. This is the reason why real vacua always preserve an S_2 symmetry. Complex vacua, on the other hand, can evade this restriction as can be seen, for example, from solution C-IV-f.

In this work we focused our attention on the study of the scalar potential. The first necessary step to render such models realistic is to specify how the fermions transform under S_3 and how to generate a realistic CKM matrix [61, 62] (see also ref. [63]).

Acknowledgments

A Converting between the two frameworks

The potentials in the reducible-representation framework, eqs. (2.9), and the irreducible-representation framework, eqs. (2.10), are related as follows:

$$\begin{pmatrix} \mu_0^2 \\ \mu_1^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix}, \tag{A.1}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 4 & 4 & 1 & 1 & -4 & 1 & -2 & 1 \\ 3[0 & 0 & -1 & 1 & 0 & 1 & 0 & -1] \\ 2 & -1 & 2 & 2 & -2 & -1 & 2 & -1 \\ \sqrt{2}[4 & -2 & -2 & -2 & -1 & 1 & 1 & 1] \\ 2[4 & 4 & -2 & -2 & 2 & -2 & 1 & -2] \\ 2[4 & -2 & 4 & -2 & 2 & 1 & -2 & -2] \\ 4 & -2 & -2 & 4 & 2 & -2 & -2 & 1 \\ 4[1 & 1 & 1 & 1 & 2 & 1 & 1 & 1] \end{pmatrix} \begin{pmatrix} A \\ C \\ \overline{C} \\ D \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}, \tag{A.2}$$

with the inverse

$$\begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \mu_0^2 \\ \mu_1^2 \end{pmatrix}, \tag{A.3}$$

$$\begin{pmatrix} A \\ C \\ \overline{C} \\ D \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 4 & 0 & 4 & 4\sqrt{2} & 2 & 2 & 4 & 1 \\ 2[4 & 0 & -2 & -2\sqrt{2} & 2 & -1 & -2 & 1] \\ 2[1 & -3 & 4 & -2\sqrt{2} & -1 & 2 & -2 & 1] \\ 2[1 & 3 & 4 & -2\sqrt{2} & -1 & -1 & 4 & 1] \\ 2[-4 & 0 & -4 & -\sqrt{2} & 1 & 1 & 2 & 2] \\ 2[2 & 6 & -4 & 2\sqrt{2} & -2 & 1 & -4 & 2] \\ 2[-4 & 0 & 8 & 2\sqrt{2} & 1 & -2 & -4 & 2] \\ 4[1 & -3 & -2 & \sqrt{2} & -1 & -1 & 1 & 1] \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}. \tag{A.4}$$

B Positivity

Das and Dey [24] have discussed *necessary* positivity conditions. Here, we discuss *necessary* and *sufficient* positivity conditions, following the approach of refs. [55, 65]. In the general case, these are rather involved. However, for the case of $\lambda_4 = 0$, they can be expressed in explicit form.

B.1 General formulation

We start by rewriting the Higgs SU(2) doublets as:

$$h_i = ||h_i||\hat{h}_i, \quad i = 1, 2, S, \tag{B.1}$$

where $||h_i||$ are the norms of the spinors, and $\hat{h}_i$ are unit spinors. We let the norms of eq. (B.1) be parametrised as follows:

$$||h_1|| = r\cos\gamma\sin\theta, \qquad ||h_2|| = r\sin\gamma\sin\theta, \qquad ||h_S|| = r\cos\theta. \tag{B.2}$$

The complex product between two different unit spinors will be a complex number with modulus less than or equal to unity, i.e.

$$\hat{h}_2^\dagger \cdot \hat{h}_1 = \rho_3 e^{i\theta_3}, \qquad \hat{h}_S^\dagger \cdot \hat{h}_2 = \rho_1 e^{i\theta_1}, \qquad \hat{h}_1^\dagger \cdot \hat{h}_S = \rho_2 e^{i\theta_2}. \tag{B.3}$$

Using this parametrisation,[5] we can write:

$$
\begin{aligned}
h_1^\dagger h_1 &= r^2\cos^2\gamma\sin^2\theta, & h_2^\dagger h_2 &= r^2\sin^2\gamma\sin^2\theta, & h_S^\dagger h_S &= r^2\cos^2\theta, \\
h_2^\dagger h_1 &= r^2\cos\gamma\sin\gamma\sin^2\theta\rho_3 e^{i\theta_3}, & h_1^\dagger h_2 &= r^2\cos\gamma\sin\gamma\sin^2\theta\rho_3 e^{-i\theta_3}, \\
h_S^\dagger h_2 &= r^2\sin\gamma\sin\theta\cos\theta\rho_1 e^{i\theta_1}, & h_2^\dagger h_S &= r^2\sin\gamma\sin\theta\cos\theta\rho_1 e^{-i\theta_1}, \\
h_1^\dagger h_S &= r^2\cos\gamma\sin\theta\cos\theta\rho_2 e^{i\theta_2}, & h_S^\dagger h_1 &= r^2\cos\gamma\sin\theta\cos\theta\rho_2 e^{-i\theta_2},
\end{aligned}
\tag{B.4}
$$

where $r \geq 0$, $\gamma \in [0, \pi/2]$, $\theta \in [0, \pi/2]$, $\rho_i \in [0, 1]$ and $\theta_i \in [0, 2\pi\rangle$.

The potential can now be written as

$$V = r^4 V_4 + r^2 V_2, \tag{B.5}$$

with only the quartic, V_4, part relevant for positivity:

$$V_4 = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4 + \lambda_5 A_5 + \lambda_6 A_6 + \lambda_7 A_7 + \lambda_8 A_8, \tag{B.6}$$

---

[5]Note that this parametrisation is unrelated to that used in the body of the paper. In particular, ρ_1, ρ_2 and ρ_3 do not here refer to the vacuum expectation values.

where

$$A_1 = \sin^4 \theta, \tag{B.7}$$

$$A_2 = -4\rho_3^2 \sin^2 \theta_3 \sin^2 \gamma \cos^2 \gamma \sin^4 \theta, \tag{B.8}$$

$$A_3 = \left[\cos^4 \gamma - 2\cos^2 \gamma \sin^2 \gamma + \sin^4 \gamma + 2\rho_3^2(1 + \cos 2\theta_3)\cos^2 \gamma \sin^2 \gamma\right] \sin^4 \theta, \tag{B.9}$$

$$A_4 = 2\left[2\rho_2\rho_3 \cos\theta_2 \cos\theta_3 \cos^2 \gamma + \rho_1 \cos\theta_1(\cos^2 \gamma - \sin^2 \gamma)\right] \sin\gamma \cos\theta \sin^3 \theta, \tag{B.10}$$

$$A_5 = \cos^2 \theta \sin^2 \theta, \tag{B.11}$$

$$A_6 = (\rho_1^2 \sin^2 \gamma + \rho_2^2 \cos^2 \gamma)\cos^2 \theta \sin^2 \theta, \tag{B.12}$$

$$A_7 = 2(\rho_1^2 \cos 2\theta_1 \sin^2 \gamma + \rho_2^2 \cos 2\theta_2 \cos^2 \gamma)\cos^2 \theta \sin^2 \theta, \tag{B.13}$$

$$A_8 = \cos^4 \theta. \tag{B.14}$$

The positivity condition is then

$$V_4 \geq 0, \quad \text{for all } \theta, \gamma, \rho_1, \rho_2, \rho_3, \theta_1, \theta_2, \theta_3. \tag{B.15}$$

An alternative formulation of the positivity conditions has been given in terms of bilinears [66].

B.2 The necessary conditions of Das and Dey

In eq. (4) of Das and Dey [24], they have listed seven necessary (but not sufficient) conditions for positivity. They can be reproduced by looking at the boundaries in θ-γ space.

B.2.1 $\theta = 0$

$$V_4(\theta = 0) > 0 \Rightarrow \lambda_8 > 0, \tag{B.16}$$

which is eq. (4b) of Das and Dey.

B.2.2 $\gamma = 0$

$$V_4(\gamma = 0) > 0 \Rightarrow (\lambda_1 + \lambda_3)\sin^4 \theta + \left[\lambda_5 + \rho_2^2(\lambda_6 + 2\lambda_7 \cos 2\theta_2)\right]\sin^2 \theta \cos^2 \theta + \lambda_8 \cos^4 \theta > 0.$$

Minimising this with respect to θ_2 we get

$$(\lambda_1 + \lambda_3)\sin^4 \theta + \left[\lambda_5 + \rho_2^2(\lambda_6 - 2|\lambda_7|)\right]\sin^2 \theta \cos^2 \theta + \lambda_8 \cos^4 \theta > 0.$$

Minimising this with respect to ρ_2 we get

$$(\lambda_1 + \lambda_3)\sin^4 \theta + \left[\lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|)\right]\sin^2 \theta \cos^2 \theta + \lambda_8 \cos^4 \theta > 0.$$

This can be treated in the same way as was done in the 2HDM [65], and is equivalent to the three conditions:

$$\lambda_1 + \lambda_3 > 0, \quad \lambda_8 > 0, \quad \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{(\lambda_1 + \lambda_3)\lambda_8}. \tag{B.17}$$

These are equivalent to eqs. (4b), (4c), (4e) and (4f) of Das and Dey.

B.2.3 $\theta = \pi/2$

$$V_4(\theta = \pi/2) > 0 \Rightarrow (\lambda_1 + \lambda_3)\cos^4\gamma$$
$$+ 2\{(\lambda_1 - \lambda_3) + \rho_3^2[(\lambda_3 - \lambda_2) + (\lambda_2 + \lambda_3)\cos 2\theta_3]\}\cos^2\gamma\sin^2\gamma$$
$$+ (\lambda_1 + \lambda_3)\sin^4\gamma > 0\,.$$

Minimising this with respect to θ_3 we get

$$(\lambda_1 + \lambda_3)\cos^4\gamma + 2\{(\lambda_1 - \lambda_3) + \rho_3^2[(\lambda_3 - \lambda_2) - |\lambda_2 + \lambda_3|]\}\cos^2\gamma\sin^2\gamma$$
$$+ (\lambda_1 + \lambda_3)\sin^4\gamma > 0,$$

or

$$(\lambda_1 + \lambda_3)\cos^4\gamma + 2\left[(\lambda_1 - \lambda_3) + 2\rho_3^2\min(-\lambda_2, \lambda_3)\right]\cos^2\gamma\sin^2\gamma$$
$$+ (\lambda_1 + \lambda_3)\sin^4\gamma > 0.$$

Minimising this with respect to ρ_3 we get

$$(\lambda_1 + \lambda_3)\cos^4\gamma + 2\left[(\lambda_1 - \lambda_3) + 2\min(0, -\lambda_2, \lambda_3)\right]\cos^2\gamma\sin^2\gamma$$
$$+ (\lambda_1 + \lambda_3)\sin^4\gamma > 0.$$

This can be treated in the same way as in ref. [65], and is equivalent to the two conditions:

$$\lambda_1 + \lambda_3 > 0, \quad \lambda_1 - \lambda_3 + 2\min(0, -\lambda_2, \lambda_3) > -|\lambda_1 + \lambda_3|. \tag{B.18}$$

These are equivalent to

$$\lambda_1 + \lambda_3 > 0, \quad \lambda_1 > 0, \quad \lambda_1 - \lambda_2 > 0. \tag{B.19}$$

The combination of these three inequalities is equivalent to the combination of (4a), (4c) and (4d) of Das and Dey.

B.2.4 $\gamma = \pi/2$

$$V_4(\gamma = \pi/2) > 0 \Rightarrow (\lambda_1 + \lambda_3)\sin^4\theta - 2\lambda_4\rho_1\cos\theta_1\cos\theta\sin^3\theta$$
$$+ (\lambda_5 + \rho_1^2(\lambda_6 + 2\lambda_7\cos 2\theta_1))\cos^2\theta\sin^2\theta + \lambda_8\cos^4\theta > 0\,.$$

The λ_4-term of this expression complicates matters due to the factor $\cos\theta\sin^3\theta$. This has not been completely solved. We can, however, reproduce (4g) of Das and Dey by putting $\rho_1 = 1$, $\theta = \pi/4$ and $\theta_1 = 0$ and $\theta_1 = \pi$, respectively. This gives us

$$\lambda_1 + \lambda_3 - 2\lambda_4 + \lambda_5 + \lambda_6 + 2\lambda_7 + \lambda_8 > 0 \quad \text{and} \quad \lambda_1 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + 2\lambda_7 + \lambda_8 > 0,$$

which combine into eq. (4g) of Das and Dey.

B.3 Positivity for models with $\lambda_4 = 0$

If we put $\lambda_4 = 0$, we get

$$
\begin{aligned}
V_4 = \big[&(\lambda_1 + \lambda_3)(\cos^4 \gamma + \sin^4 \gamma) \\
&+ 2(\lambda_1 - \lambda_3 + \rho_3^2(\lambda_3 - \lambda_2 + (\lambda_2 + \lambda_3)\cos 2\theta_3))\cos^2 \gamma \sin^2 \gamma\big] \sin^4 \theta \\
&+ \big[\lambda_5 + \rho_2^2(\lambda_6 + 2\lambda_7 \cos 2\theta_2)\cos^2 \gamma + \rho_1^2(\lambda_6 + 2\lambda_7 \cos 2\theta_1)\sin^2 \gamma\big] \sin^2 \theta \cos^2 \theta \\
&+ \lambda_8 \cos^4 \theta > 0.
\end{aligned}
\tag{B.20}
$$

We minimise this with respect to θ_1, θ_2 and θ_3 to get

$$
\begin{aligned}
V_4 = \big[&(\lambda_1 + \lambda_3)(\cos^4 \gamma + \sin^4 \gamma) \\
&+ 2(\lambda_1 - \lambda_3 + 2\rho_3^2 \min(-\lambda_2, \lambda_3))\cos^2 \gamma \sin^2 \gamma\big] \sin^4 \theta \\
&+ \big[\lambda_5 + \rho_2^2(\lambda_6 - 2|\lambda_7|)\cos^2 \gamma + \rho_1^2(\lambda_6 - 2|\lambda_7|)\sin^2 \gamma\big] \sin^2 \theta \cos^2 \theta \\
&+ \lambda_8 \cos^4 \theta > 0.
\end{aligned}
\tag{B.21}
$$

Next, we minimise this with respect to ρ_1, ρ_2 and ρ_3 to get

$$
\begin{aligned}
V_4 = \big[&(\lambda_1 + \lambda_3)(\cos^4 \gamma + \sin^4 \gamma) \\
&+ 2(\lambda_1 - \lambda_3 + \min(0, -2\lambda_2, 2\lambda_3))\cos^2 \gamma \sin^2 \gamma\big] \sin^4 \theta \\
&+ \big[\lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|)\big] \sin^2 \theta \cos^2 \theta \\
&+ \lambda_8 \cos^4 \theta > 0.
\end{aligned}
\tag{B.22}
$$

First we consider the boundaries in $\gamma\theta$-plane:

B.3.1 $\theta = 0$

Like in the previous section, this leads to

$$
\lambda_8 > 0.
\tag{B.23}
$$

B.3.2 $\gamma = 0$

Like in the previous section, this leads to

$$
\lambda_1 + \lambda_3 > 0, \quad \lambda_8 > 0, \quad \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{(\lambda_1 + \lambda_3)\lambda_8}.
\tag{B.24}
$$

B.3.3 $\theta = \pi/2$

Like in the previous section, this leads to

$$
\lambda_1 + \lambda_3 > 0, \quad \lambda_1 > 0, \quad \lambda_1 - \lambda_2 > 0.
\tag{B.25}
$$

B.3.4 $\gamma = \pi/2$

$$
(\lambda_1 + \lambda_3)\sin^4 \theta + (\lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|))\cos^2 \theta \sin^2 \theta + \lambda_8 \cos^4 \theta > 0,
$$

Like in the previous section, this leads to

$$
\lambda_1 + \lambda_3 > 0, \quad \lambda_8 > 0, \quad \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{(\lambda_1 + \lambda_3)\lambda_8}.
\tag{B.26}
$$

B.3.5 Interior points in the θ-γ plane

Minimising (B.22) with respect to γ we find (surprisingly) that the only possibility for an interior minimum occurs when $\gamma = \pi/4$. Substituting this value of γ into (B.22) we get

$$
\begin{aligned}
V_4 = {} & [\lambda_1 + \min(0, -\lambda_2, \lambda_3)] \sin^4 \theta \\
& + [\lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|)] \sin^2 \theta \cos^2 \theta + \lambda_8 \cos^4 \theta > 0.
\end{aligned} \tag{B.27}
$$

Like in the previous section, this leads to

$$
\lambda_1 + \min(0, -\lambda_2, \lambda_3) > 0, \quad \lambda_8 > 0, \quad \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{(\lambda_1 + \min(0, -\lambda_2, \lambda_3))\lambda_8}. \tag{B.28}
$$

or explicitly

$$
\begin{aligned}
& \lambda_1 > 0, \quad \lambda_1 - \lambda_2 > 0, \quad \lambda_1 + \lambda_3 > 0, \quad \lambda_8 > 0, \\
& \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{\lambda_1 \lambda_8}, \\
& \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{(\lambda_1 - \lambda_2)\lambda_8}, \\
& \lambda_5 + \min(0, \lambda_6 - 2|\lambda_7|) > -2\sqrt{(\lambda_1 + \lambda_3)\lambda_8}.
\end{aligned} \tag{B.29}
$$

C Minimisation conditions in terms of moduli and phases

Here, we present explicit results for the derivatives of the potential (the minimisation conditions) in terms of moduli of the vevs, and their phases.

C.1 Conditions in the reducible-representation framework

In the notation of equation (3.6),

$$
(\rho_1, \rho_2, \rho_3) = (v_1 e^{i\tau_1}, v_2 e^{i\tau_2}, v_3 e^{i\tau_3}), \tag{C.1}
$$

the derivatives with respect to moduli and phases can be written as

$$
\begin{aligned}
\frac{\partial V}{\partial v_1} = {} & -\lambda v_1 + \frac{1}{2}\gamma \left[v_2 \cos(\tau_2 - \tau_1) + v_3 \cos(\tau_1 - \tau_3) \right] + A v_1^3 + \frac{1}{2}(C + \overline{C}) v_1 (v_2^2 + v_3^2) \\
& + \frac{1}{2} D v_1 \left[v_2^2 \cos(2\tau_2 - 2\tau_1) + v_3^2 \cos(2\tau_1 - 2\tau_3) \right] \\
& + \frac{1}{4} E_1 \left[(3v_1^2 v_2 + v_2^3) \cos(\tau_2 - \tau_1) + (3v_1^2 v_3 + v_3^3) \cos(\tau_1 - \tau_3) \right] \\
& + \frac{1}{4}(E_2 + E_3) v_2 v_3 \left[2v_1 \cos(\tau_3 - \tau_2) + v_2 \cos(\tau_1 - \tau_3) + v_3 \cos(\tau_2 - \tau_1) \right] \\
& + \frac{1}{4} E_4 v_2 v_3 \left[2v_1 \cos(\tau_2 + \tau_3 - 2\tau_1) + v_2 \cos(2\tau_2 - \tau_1 - \tau_3) + v_3 \cos(\tau_1 + \tau_2 - 2\tau_3) \right] = 0,
\end{aligned} \tag{C.2}
$$

$$
\begin{aligned}
\frac{\partial V}{\partial \tau_1} = {} & \frac{1}{2}\gamma v_1 \left[v_2 \sin(\tau_2 - \tau_1) - v_3 \sin(\tau_1 - \tau_3) \right] \\
& + \frac{1}{2} D v_1^2 \left[v_2^2 \sin(2\tau_2 - 2\tau_1) - v_3^2 \sin(2\tau_1 - 2\tau_3) \right] \\
& + \frac{1}{4} E_1 v_1 \left[(v_2^3 + v_1^2 v_2) \sin(\tau_2 - \tau_1) - (v_3^3 + v_3 v_1^2) \sin(\tau_1 - \tau_3) \right] \\
& - \frac{1}{4}(E_2 + E_3) v_1 v_2 v_3 \left[v_2 \sin(\tau_1 - \tau_3) - v_3 \sin(\tau_2 - \tau_1) \right] \\
& + \frac{1}{4} E_4 v_1 v_2 v_3 \left[2v_1 \sin(\tau_2 + \tau_3 - 2\tau_1) - v_2 \sin(\tau_1 - 2\tau_2 + \tau_3) - v_3 \sin(\tau_1 + \tau_2 - 2\tau_3) \right] = 0,
\end{aligned} \tag{C.3}
$$

with $\partial V/\partial v_2$, $\partial V/\partial v_3$, $\partial V/\partial \tau_2$, and $\partial V/\partial \tau_3$ given by cyclic permutations of the indices 1, 2 and 3.

C.2 Conditions in the irreducible-representation framework

We choose w_S real and parametrise the complex w_1 and w_2 in terms of moduli and phases:

$$(w_1, w_2, w_S) = (\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S). \tag{C.4}$$

Then the derivatives with respect to moduli can be written as

$$\left(\frac{\partial V}{\partial \hat{w}_1}\right)_0 = \mu_1^2 \hat{w}_1 + \lambda_1 \hat{w}_1 (\hat{w}_1^2 + \hat{w}_2^2) + \lambda_2 \hat{w}_1 \hat{w}_2^2 [\cos(2\sigma_1 - 2\sigma_2) - 1] + \lambda_3 \hat{w}_1 [\hat{w}_1^2 + \hat{w}_2^2 \cos(2\sigma_1 - 2\sigma_2)]$$

$$+ \lambda_4 \hat{w}_1 \hat{w}_2 \hat{w}_S [\cos(2\sigma_1 - \sigma_2) + 2\cos\sigma_2] + \frac{1}{2}(\lambda_5 + \lambda_6)\hat{w}_1 \hat{w}_S^2 + \lambda_7 \hat{w}_1 \hat{w}_S^2 \cos 2\sigma_1, \tag{C.5}$$

$$\left(\frac{\partial V}{\partial \hat{w}_2}\right)_0 = \mu_1^2 \hat{w}_2 + \lambda_1 \hat{w}_2 (\hat{w}_1^2 + \hat{w}_2^2) + \lambda_2 \hat{w}_1^2 \hat{w}_2 [\cos(2\sigma_1 - 2\sigma_2) - 1] + \lambda_3 \hat{w}_2 [\hat{w}_1^2 \cos(2\sigma_1 - 2\sigma_2) + \hat{w}_2^2]$$

$$+ \frac{\lambda_4}{2} \hat{w}_S [\hat{w}_1^2 \cos(2\sigma_1 - \sigma_2) + (2\hat{w}_1^2 - 3\hat{w}_2^2)\cos\sigma_2] + \frac{1}{2}(\lambda_5 + \lambda_6)\hat{w}_2 \hat{w}_S^2 + \lambda_7 \hat{w}_2 \hat{w}_S^2 \cos 2\sigma_2, \tag{C.6}$$

$$\left(\frac{\partial V}{\partial \hat{w}_S}\right)_0 = \mu_0^2 \hat{w}_S + \frac{\lambda_4}{2} \hat{w}_2 [\hat{w}_1^2 \cos(2\sigma_1 - \sigma_2) + (2\hat{w}_1^2 - \hat{w}_2^2)\cos\sigma_2] + \frac{1}{2}(\lambda_5 + \lambda_6)(\hat{w}_1^2 + \hat{w}_2^2)\hat{w}_S$$

$$+ \lambda_7 \hat{w}_S [\hat{w}_1^2 \cos 2\sigma_1 + \hat{w}_2^2 \cos 2\sigma_2] + \lambda_8 \hat{w}_S^3, \tag{C.7}$$

and those with respect to angles as

$$\left(\frac{\partial V}{\partial \sigma_1}\right)_0 = -(\lambda_2 + \lambda_3)\hat{w}_1^2 \hat{w}_2^2 \sin(2\sigma_1 - 2\sigma_2) - \lambda_4 \hat{w}_1^2 \hat{w}_2 \hat{w}_S \sin(2\sigma_1 - \sigma_2) - \lambda_7 \hat{w}_1^2 \hat{w}_S^2 \sin 2\sigma_1, \tag{C.8}$$

$$\left(\frac{\partial V}{\partial \sigma_2}\right)_0 = (\lambda_2 + \lambda_3)\hat{w}_1^2 \hat{w}_2^2 \sin(2\sigma_1 - 2\sigma_2) + \frac{\lambda_4}{2} \hat{w}_2 \hat{w}_S [\hat{w}_1^2 \sin(2\sigma_1 - \sigma_2) - (2\hat{w}_1^2 - \hat{w}_2^2)\sin\sigma_2]$$

$$- \lambda_7 \hat{w}_2^2 \hat{w}_S^2 \sin 2\sigma_2. \tag{C.9}$$

When we add the soft term discussed in section 7, these derivatives get modified as follows:

$$\frac{\partial V}{\partial \hat{w}_1} = \left(\frac{\partial V}{\partial \hat{w}_1}\right)_0 + \frac{1}{2}\nu^2 \hat{w}_2 \cos(\sigma_1 - \sigma_2), \tag{C.10}$$

$$\frac{\partial V}{\partial \hat{w}_2} = \left(\frac{\partial V}{\partial \hat{w}_2}\right)_0 + \frac{1}{2}\nu^2 \hat{w}_1 \cos(\sigma_1 - \sigma_2), \tag{C.11}$$

$$\frac{\partial V}{\partial \hat{w}_S} = \left(\frac{\partial V}{\partial \hat{w}_S}\right)_0, \tag{C.12}$$

$$\frac{\partial V}{\partial \sigma_1} = \left(\frac{\partial V}{\partial \sigma_1}\right)_0 - \frac{1}{2}\nu^2 \hat{w}_1 \hat{w}_2 \sin(\sigma_1 - \sigma_2), \tag{C.13}$$

$$\frac{\partial V}{\partial \sigma_2} = \left(\frac{\partial V}{\partial \sigma_2}\right)_0 + \frac{1}{2}\nu^2 \hat{w}_1 \hat{w}_2 \sin(\sigma_1 - \sigma_2). \tag{C.14}$$

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