Three proofs of the inequality $e < \left(1+\frac{1}{n}\right)^{n+0.5}$

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Introduction

The number e is one of the most indispensable numbers in mathematics. This number is also referred to as Euler's number or Napier's constant. Classically the number e can be defined as (see [1, 5, 7–9], and and references therein)

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}, \tag{1}$$

Note that we could also define the number e through the limit

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+0.5}. \tag{2}$$

Let us see the motivation behind the above result. The reader can observe that the limit (2) is modestly different than the classical limit (1). Let us approximate e from these two limits using $n = 1000$. From the classical limit, we get $e \approx 2.71692393$; which is accurate only to 3 decimal places. From the new limit (2), we get $e \approx 2.71828205$; which is e accurate to 6 decimal places. Thus, the new limit appears to be a big improvement over the classical result.

It is well known that for any value of $n > 1$ (see [1]),

$$e > \left(1+\frac{1}{n}\right)^{n}.$$

In this work, we present three proofs of the inequality:

$$e < \left(1+\frac{1}{n}\right)^{n+0.5}.$$

For deriving the inequality, we use the Taylor series expansion and the Hermite Hadamard inequality. Let us now present our first proof through the Taylor series expansion.

Proof through the Taylor series expansion

Proof: The Taylor series expansion of the function $\ln(1+x)$ around the point $x=0$ is given by the following alternating series (see [4, 6] or calculus book)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots; \quad -1 < x \leq 1. \tag{3}$$

Let us replace x by $1/n$ in the above series, and multiply both the sides by n:

$$n \ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2\,n} + \frac{1}{3\,n^2} - \frac{1}{4\,n^3} - \frac{1}{5\,n^4} + \cdots.$$

Replacing n by $2\,n$ and $-2\,n$ in the above series gives the following two series:

$$2n \ln\left(1+\frac{1}{2n}\right) = 1 - \frac{1}{4\,n} + \frac{1}{12\,n^2} - \frac{1}{32\,n^3} + \frac{1}{80\,n^4} + \cdots;$$
$$-2n \ln\left(1-\frac{1}{2n}\right) = 1 + \frac{1}{4\,n} + \frac{1}{12\,n^2} + \frac{1}{32\,n^3} + \frac{1}{80\,n^4} + \cdots.$$

Adding the above two series we get:

$$2n\left[\ln\left(1+\frac{1}{2n}\right) - \ln\left(1-\frac{1}{2n}\right)\right] = 2 + \frac{1}{6n^2} + \frac{1}{40n^4} + \cdots;$$
$$2n \ln\left(\frac{2n+1}{2n-1}\right) = 2 + \frac{1}{6n^2} + \frac{1}{40n^4} + \cdots.$$

Next we divide both sides by 2:

$$\ln\left(\frac{2n+1}{2n-1}\right)^n = 1 + \frac{1}{12n^2} + \frac{1}{80n^4} + \cdots.$$

Now replacing n by $n+0.5$ gives the following series:

$$\ln\left(\frac{2n+2}{2n}\right)^{n+0.5} = 1 + \frac{1}{12(n+0.5)^2} + \frac{1}{80(n+0.5)^4} + \cdots.$$

Therefore

$$\ln\left(1+\frac{1}{n}\right)^{n+0.5} > 1;$$

So

$$e < \left(1+\frac{1}{n}\right)^{n+0.5}. \tag{4}$$ ■

Now let us prove the above inequality through the Hermite Hadamard inequality [2].

Proof through the Hermite Hadamard inequality

If a function f is differentiable in the interval $[a, b]$ and its derivative is an increasing function on (a,b); then for all $x_1, x_2 \in [a, b]$ such that $x_1 \neq x_2$; the following inequality holds [2, 3]:

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\,\mathrm{d}x.$$

The above inequality is referred to as the Hermite Hadamard inequality.

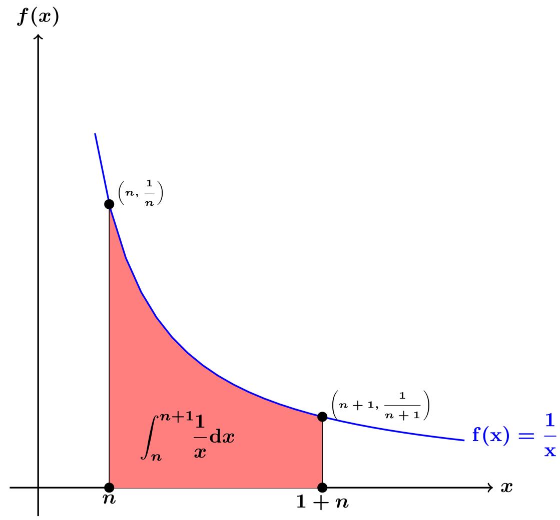

Figure 1: Graph of $f(x) = 1/x$. The shaded area is equal to $\ln(1 + {}^1\!/\!_n)$.

Proof: Let us consider the function $f(x) = 1/x$ on the interval $[n, n+1]$. Figure 1 shows the graph. It may be seen that the derivative $f'(x) = {}^{-1}\!/\!_{x^2}$ is an increasing function in the interval $(n, n+1)$. Thus, the Hermite Hadamard inequality holds. Applying the Hermite Hadamard inequality to the function for $x_1 = n$ and $x_2 = n+1$ we get:

$$\begin{aligned} f\left(\frac{n+n+1}{2}\right) &< \frac{1}{n+1-n} \int_n^{n+1} f(x)\,\mathrm{d}x; \\ \frac{2}{2n+1} &< \ln\left(1+\frac{1}{n}\right); \\ \frac{1}{n+0.5} &< \ln\left(1+\frac{1}{n}\right); \\ 1 &< \ln\left(1+\frac{1}{n}\right)^{n+0.5}; \\ e &< \left(1+\frac{1}{n}\right)^{n+0.5}. \end{aligned} \tag{5}$$

■

The Third Proof

For $n > 0$, we define function $\mathcal{F}(n)$ by the equation $(1+1/n)^{n+\mathcal{F}(n)} = e$. Solving this equation for $\mathcal{F}(n)$, we find that

$$\mathcal{F}(n) = \frac{1}{\ln\left(1 + {}^{1}\!/\!{}_{n}\right)} - n. \tag{6}$$

Now let us first show that $\mathcal{F}(n)$ is a monotonically increasing function. That is; for all $n \geq 1$, $\mathcal{F}'(n) > 0$. The derivative of this function is

$$\mathcal{F}'(n) = \frac{1}{\left(\ln\left(1 + {}^{1}\!/\!{}_{n}\right)\right)^2 n^2 \left(1 + {}^{1}\!/\!{}_{n}\right)} - 1. \tag{7}$$

To show the positivity of $\mathcal{F}'(n)$, let us consider the following functions:

$$f(x) = \ln(1+x);$$
$$g(x) = \frac{x}{\sqrt{1+x}}.$$

The difference between the first derivatives of the above two functions is

$$g'(x) - f'(x) = \frac{1}{2}\,\frac{x + 2 - 2\sqrt{1+x}}{(1+x)^{3/2}}.$$

Since $(x+2) > 2\sqrt{1+x}$ for all $x > 1$.

$$g'(x) - f'(x) > 0;$$

and therefore

$$\ln(1+x) < \frac{x}{\sqrt{1+x}}.$$

Now substituting $x = 1/n$ in the above inequality and squaring both the sides will show that

$$\frac{1}{n^2\,\left(1 + {}^{1}\!/\!{}_{n}\right)\,\left(\ln\left(1 + {}^{1}\!/\!{}_{n}\right)\right)^2} > 1.$$

From equation (7) and the above inequality, we see that $\mathcal{F}'(n) > 0$.

Therefore the function (6) is strictly increasing. To show that the function is bounded from above, let us find the limit

$$\lim_{n\to\infty} \frac{1}{\ln\left(1 + \dfrac{1}{n}\right)} - n = \lim_{n\to\infty} \frac{1 - n\,\ln\left(1 + \dfrac{1}{n}\right)}{\ln\left(1 + \dfrac{1}{n}\right)}. \tag{8}$$

Substituting the power series of $\ln(1+{}^1\!/\!{}_n) = {}^1\!/\!{}_n - {}^1\!/\!{}_{2\,n^2} + {}^1\!/\!{}_{3\,n^3} - \cdots$;

$$\lim_{n\to\infty} \frac{1 - n\,\ln\left(1+\dfrac{1}{n}\right)}{\ln\left(1+\dfrac{1}{n}\right)} = \lim_{n\to\infty} \frac{1 - n\,\left[{}^1\!/\!{}_n - {}^1\!/\!{}_{2\,n^2} + {}^1\!/\!{}_{3\,n^3} - \cdots\right]}{\left[{}^1\!/\!{}_n - {}^1\!/\!{}_{2\,n^2} + {}^1\!/\!{}_{3\,n^3} - \cdots\right]};$$
$$= 0.5.$$

Since the function $\mathcal{F}(n)$ is strictly increasing function, and $\lim_{n\to\infty} \mathcal{F}(n) = 0.5$, we can conclude that $\mathcal{F}(n) < 0.5$, and therefore

$$e = \left(1+\frac{1}{n}\right)^{n+\mathcal{F}(n)} < \left(1+\frac{1}{n}\right)^{n+0.5}.$$

The facts

$$e = \left(1+\frac{1}{n}\right)^{n+\mathcal{F}n} \qquad \text{and} \qquad \lim_{n\to\infty} \mathcal{F}(n) = 0.5;$$

suggests that, among approximations of the form $e \approx (1+1/n)^{n+a}$, the best approximations for large n is achieved by using $a = 0.5$. Furthermore, if $a < 0.5$, then for sufficiently large n we will have $\mathcal{F}(n) > a$, and therefore

$$\left(1+\frac{1}{n}\right)^{n+a} < \left(1+\frac{1}{n}\right)^{n+\mathcal{F}(n)} = e.$$

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